Solution of the Second Midterm Exam for MATH 244, Semester 2, 1433H

I) Choose the correct answer:

a) Which of the following sets of vectors form a basis of $\mathbb{R}^2$?

Answer: $\{(1,1),(3,1)\}$.

Reason: 1) The set $S = \{(2,1),(1,-1),(0,2)\}$ is linearly dependent because $\dim S = 3 > 2 = \dim \mathbb{R}^2$.

2) The set $S = \{(0,1),(0,-3)\}$ is linearly dependent because $(0,-3) = -3(0,1)$.

b) If $u = (-2,-3,4,-6)$, $v = (4,1,6,16)$ and $w = (8,-13,0,20)$, then the vector $x \in \mathbb{R}^4$ for which $5x - 2v + 3u = 2(w - 5x)$ is

Answer: $x = (2,-1,0,6)$.

Reason:

$$
\begin{aligned}
& 5x - 2v + 3u = 2(w - 5x) \\
\Longrightarrow\ & 5x - 2v + 3u = 2w - 10x \\
\Longrightarrow\ & 15x = 2w + 2v - 3u \\
\Longrightarrow\ & x = \frac{1}{15}(2w + 2v - 3u) \\
\Longrightarrow\ & x = \frac{1}{15}(2(8,-13,0,20) + 2(4,1,6,16) - 3(-2,-3,4,-6)) \\
\Longrightarrow\ & x = \frac{1}{15}((16,-26,0,40) + (8,2,12,32) + (6,9,-12,18)) \\
\Longrightarrow\ & x = \frac{1}{15}(30,-15,0,90) \\
\Longrightarrow\ & x = (2,-1,0,6).
\end{aligned}
$$

c) If $A = \begin{bmatrix} -3 & 4 \\ 2 & 0 \end{bmatrix}$, then the set of matrices $\{A, B\}$ is linearly independent if

Answer: $B = \begin{bmatrix} 3 & 4 \\ -2 & 0 \end{bmatrix}$.

Reason: 1) $B = \begin{bmatrix} 3 & -4 \\ -2 & 0 \end{bmatrix}$ is a scalar multiple of A, namely, $B = -A$.

2) Any set containing the zero vector $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is linearly dependent.

d) If $\|\alpha(-2,2,-1)\| = 3$, then the values that α can take are

Answer: $\{-1, 1\}$.

Reason:

$$\begin{gathered} \|\alpha(-2,2,-1)\| = 3 \\ \Longrightarrow |\alpha| \, \|(-2,2,-1)\| = 3 \\ \Longrightarrow |\alpha| \sqrt{(-2)^2 + 2^2 + (-1)^2} = 3 \\ \Longrightarrow |\alpha| \sqrt{4+4+1} = 3 \\ \Longrightarrow 3|\alpha| = 3 \\ \Longrightarrow |\alpha| = 1 \\ \Longrightarrow \alpha = \{-1, 1\}. \end{gathered}$$

e) If u and v are vectors in $\mathbb{R}^n$, then $d(2u+v, u)$ equals

Answer: $\|u+v\|$

Reason:

$$\begin{aligned} d(2u+v, u) &= \|2u+v-u\| \\ &= \|u+v\|. \end{aligned}$$

II) Decide if the following statements are true (T) of false (F). Justify your answer.

a) For all u, $v \in \mathbb{R}^n$, $\|u+v\| = \|u\| + \|v\|$.

Answer: F

Example: Take $u = (1,1)$, $v = (-1,1)$ in $\mathbb{R}^2$, then

$$\begin{aligned} \|u+v\| &= \|(1,1)+(-1,1)\| \\ &= \|(0,2)\| \\ &= \sqrt{0^2+2^2} \\ &= 2. \end{aligned}$$

But

$$\begin{aligned} \|u\|+\|v\| &= \|(1,1)\|+\|(-1,1)\| \\ &= \sqrt{1^2+1^2}+\sqrt{(-1)^2+1^2} \\ &= \sqrt{2}+\sqrt{2} \\ &= 2\sqrt{2}. \end{aligned}$$

Clearly, for our choice

$$\|u+v\| \neq \|u\|+\|v\|.$$

b) The coordinate vector $(P(X))_S$ of $P(X) = 2 - X + 3X^2$ with respect to the basis $S = \{1+X, 1-X, X^2\}$ of $\mathcal{P}_2(X)$ is $(2,-1,3)$.

Answer: F

Reason: Note that

$$\begin{aligned} 2(1+X) - 1(1-X) + 3X^2 &= 2 + 2X - 1 + X + 3X^2 \\ &= 1 + 3X + 3X^2 \\ &\neq P(X). \end{aligned}$$

c) The column vectors of the matrix $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ are linearly independent.

Answer: T

Reason: The column vectors $\mathbf{c}_1 = \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}$ and $\mathbf{c}_2 = \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix}$ are linearly independent if and only if $\det(A) \neq 0$. Now,

$$\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1 \neq 0.$$

d) If $\|u+v\| = 5$ and $\|u-v\| = 3$, then $u.v = 6$.

Answer: F

Reason:

$$\begin{aligned} u.v &= \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2 \\ &= \frac{1}{4}5^2 - \frac{1}{4}3^2 \\ &= 4 \neq 6. \end{aligned}$$

e) Let $V = \mathbb{R}$. If the addition and multiplication on V are defined as $a + b = a^b$ and $k(a) = ka$, for all $a, b \in V$ and all scalars $k \in \mathbb{R}$, then V is a vector space.

Answer: F

Reason: Axiom 2 in DEFINITION 1 (page 172) is not satisfied. For example, if we take $a = 1$, $b = 2$, then

$$\begin{aligned} a+b &= 1+2 \\ &= 1^2 \\ &= 1, \end{aligned}$$

but

$$\begin{aligned} b+a &= 2+1 \\ &= 2^1 \\ &= 2. \end{aligned}$$

So, for our choice

$$a + b \neq b + a.$$

III) Determine the value of a such that the matrices

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & -4 \\ a & -2 \end{bmatrix}$$

are linearly independent.

Answer: The above matrices are linearly independent if and only if the vector equation

$$k_1 \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + k_2 \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + k_3 \begin{bmatrix} 1 & -4 \\ a & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{1}$$

has only the trivial solution. The vector equation (1) is equivalent to

$$\begin{bmatrix} k_1 & 2k_1 \\ 0 & k_1 \end{bmatrix} + \begin{bmatrix} k_2 & 0 \\ k_2 & 0 \end{bmatrix} + \begin{bmatrix} k_3 & -4k_3 \\ ak_3 & -2k_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

or

$$\begin{bmatrix} k_1 + k_2 + k_3 & 2k_1 - 4k_3 \\ k_2 + ak_3 & k_1 - 2k_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which gives the system

$$\begin{aligned} k_1 + k_2 + k_3 &= 0 \\ 2k_1 - 4k_3 &= 0 \\ k_2 + ak_3 &= 0 \\ k_1 - 2k_3 &= 0 \end{aligned} \tag{2}$$

The augmented matrix for system (2) is given by

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 0 & -4 & 0 \\ 0 & 1 & a & 0 \\ 1 & 0 & -2 & 0 \end{bmatrix} \underset{\sim}{-2R_1+R_2 \to R_2, -R_1+R_4 \to R_4}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -2 & -6 & 0 \\ 0 & 1 & a & 0 \\ 0 & -1 & -3 & 0 \end{bmatrix} \underset{\sim}{\tfrac{1}{2}R_2+R_3 \to R_3, -\tfrac{1}{2}R_2+R_4 \to R_4}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -2 & -6 & 0 \\ 0 & 0 & a-3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, system (2) has only the trivial solution if and only if $a - 3 \neq 0$. Therefore, for all $a \in \mathbb{R} - \{3\}$, the given matrices are linearly independent.

IV) Determine whether

$$V = \{(x, y) : x, y \in \mathbb{R}\}$$

is a vector space, when addition and multiplication on V are defined by

$$(x, y) + \left(x^{'}, y^{'}\right) = \left(xx^{'}, yy^{'}\right), \qquad \forall (x, y), \left(x^{'}, y^{'}\right) \in V$$

respectively

$$k(x, y) = (kx, ky), \qquad \forall k \in \mathbb{R}, \forall (x, y) \in V.$$

Answer: V is not a vector space because axiom 7 in DEFINITION 1 (page 172) does not hold. Take, for example, $k = -1$, $(x, y) = (1, 2)$ and $\left(x^{'}, y^{'}\right) = (2, 3)$, then

$$\begin{aligned} k\left[(x, y) + \left(x^{'}, y^{'}\right)\right] &= -1\left[(1, 2) + (2, 3)\right] \\ &= -1(2, 6) \\ &= (-2, -6) \end{aligned}$$

But

$$\begin{aligned} k(x, y) + k\left(x^{'}, y^{'}\right) &= -1(1, 2) - 1(2, 3) \\ &= (-1, -2) + (-2, -3) \\ &= (2, 6) \end{aligned}$$

That is, for our choice

$$k\left[(x, y) + \left(x^{'}, y^{'}\right)\right] \neq k(x, y) + k\left(x^{'}, y^{'}\right).$$

V) Show that $S = \{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$, if $v_1 = (1, 1, 0)$, $v_2 = (1, 1, 1)$ and $v_3 = (0, 1, -1)$.

Answer: The number of vectors in S is equal to $\dim\left(\mathbb{R}^3\right)$. Therefore, to prove that S is a basis for $\mathbb{R}^3$ we only need to prove that S is a linearly independent set or that S spans $\mathbb{R}^3$.

The set S is linearly independent if and only if the matrix whose column vectors are v_1, v_2 and v_3 has a nonzero determinant. Now,

$$\begin{aligned}\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{vmatrix} &= \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} - \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} \\ &= -2-(-1) \\ &= -1 \neq 0.\end{aligned}$$

VI) Prove that $W = \left\{(a,b,c) \in \mathbb{R}^3 : b = 5a, c = 0\right\}$ is a subspace of $\mathbb{R}^3$.

A vector in W can be written in the form $(a, 5a, 0)$ where $a \in \mathbb{R}$.

(1) The vector $(0,0,0) \in \mathbb{R}^3$, hence $W \neq \phi$.

(2) Let $(a, 5a, 0)$ and $(b, 5b, 0)$ be two vectors in W, then

$$\begin{aligned}(a,5a,0)+(b,5b,0) &= (a+b, 5a+5b, 0) \\ &= (a+b, 5(a+b), 0) \\ &= (c, 5c, 0)\end{aligned}$$

where $c = a + b \in \mathbb{R}$. This shows that $(a, 5a, 0) + (b, 5b, 0) \in W$.

(3) Let k be any scalar, and $(a, 5a, 0)$ be a vector in W, then

$$\begin{aligned}k(a,5a,0) &= (ka, 5ka, 0) \\ &= (ka, 5(ka), 0) \\ &= (d, 5d, 0)\end{aligned}$$

where $d = ka \in \mathbb{R}$. Thus, $k(a, 5a, 0) \in W$.

Steps (1), (2) and (3) prove that W is a subspace of $\mathbb{R}^3$.