Chapter 5

Some Famous Discrete Distributions

1. Uniform
2. Bernoulli
3. Binomial
4. Negative Binomial
5. Geometric
6. Hypergeometric
7. Poisson

5.2: Discrete Uniform Distribution:

If the discrete random variable X assumes the values x1, x2, …, xk with equal probabilities, then X has the discrete uniform distribution given by:

$$f(x) = P(X = x) = f(x;k) = \begin{cases} \dfrac{1}{k} \; ; \; x = x_1, x_2, \cdots, x_k \\ 0 \; ; \; elsewhere \end{cases}$$

Note:

· f(x)=f(x;k)=P(X=x)

. k is called the parameter of the distribution which is the no. of outcomes of the experiment.

. All sthe ample points are equally Likely to occurrence.

Theorem 5.1:

If the discrete random variable X has a discrete uniform distribution with parameter k, then the mean and the variance of X are:

$$E(X) = \mu = \frac{\sum_{i=1}^{k} x_i}{k} \qquad Var(X) = \sigma^2 = \frac{\sum_{i=1}^{k} (x_i - \mu)^2}{k}$$

Example 5.2:

- **Experiment: tossing a balanced die.**
- **Sample space: S={1,2,3,4,5,6}**
- **Each sample point of S occurs with the same probability 1/6.**
- **Let the r. v. X= the number observed when tossing a balanced die.**

The probability distribution of X is:

$$f(x) = P(X = x) = f(x;6) = \begin{cases} \frac{1}{6} \; ; \; x = 1, 2, \cdots, 6 \\ 0 \; ; \; elsewhere \end{cases}$$

$P(1 < X < 4)$

$P(X <= 3)$

$P(3 < X < 6)$

$$\text{E(X)} = \mu = \frac{\sum_{i=1}^{k} x_i}{k} = \frac{1+2+3+4+5+6}{6} = 3.5$$

$$\text{Var(X)} = \sigma^2 = \frac{\sum_{i=1}^{k} (x_i - \mu)^2}{k} = \frac{\sum_{i=1}^{k} (x_i - 3.5)^2}{6}$$

$$= \frac{(1-3.5)^2 + (2-3.5)^2 + \cdots + (6-3.5)^2}{6} = \frac{35}{12}$$

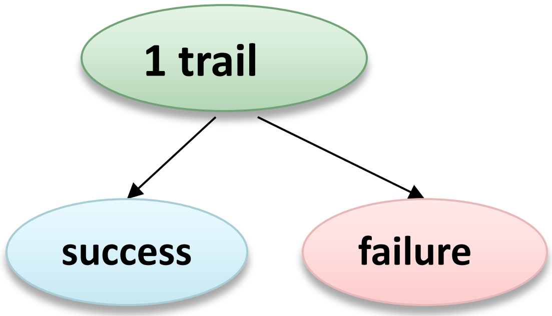
5.3 Bernoulli Distribution:

Bernoulli Trial:

- Bernoulli trial is an experiment with only two possible outcomes.
- The two possible outcomes are labeled:
  success (*s*) and failure (*f*)
- The probability of success is P(*s*)=*p* and the probability of failure is P(*f*)= $q = 1-p$.
- **Examples:**

1. Tossing a coin (success=H, failure=T, and *p*=P(H))
2. Inspecting an item (success=defective, failure=non- defective, and *p*=P(defective))

Bernoulli distribution has the following function:



$$f(x) = P(X = x) = \begin{cases} p^x \; q^{1-x} \; ; \; x = 0, 1 \\ 0 \; ; \quad otherwise \end{cases}$$

The mean and the variance of the Bernoulli distribution are:

$$\mu = \boldsymbol{p}$$
$$\sigma^2 = \boldsymbol{p}\,\mathbf{q}$$

H.W. Proof that.

Example (Bernoulli): Toss a coin and let S denote head. Find p, mean and variance.

$$p(X=S)=p(X=head)=p(H)=\frac{1}{2}$$

$$q=1-p=\frac{1}{2}$$

$$\mu=p=\frac{1}{2}$$

$$\delta^2=pq=\frac{1}{4}$$

5.3 Binomial Distribution:

Binomial Random Variable: Consider the random variable :

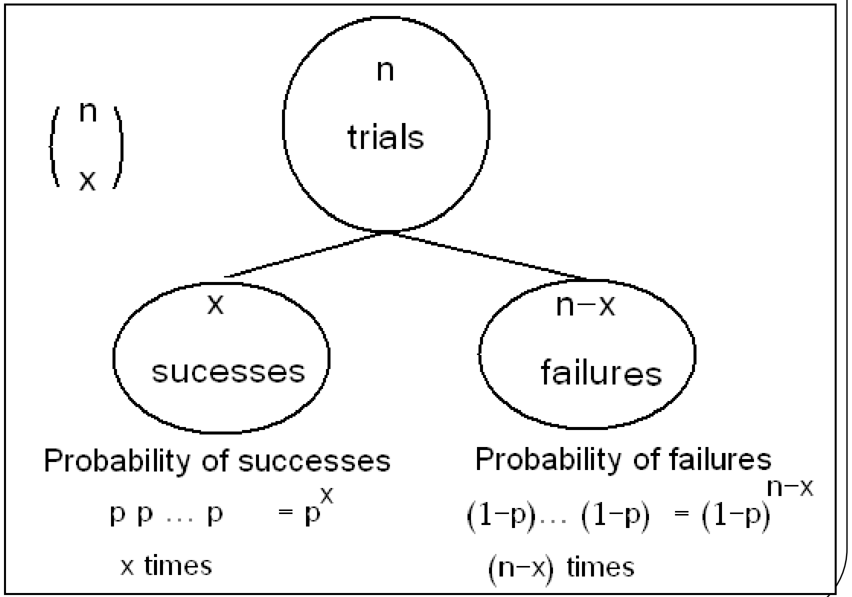
X = The number of successes in the n trials in a Bernoulli process

The random variable X has a binomial distribution with parameters n (number of trials) and p (probability of success), and we write:

$$X \sim \text{Binomial}(n,p) \quad \text{or} \quad X \sim b(x;n,p)$$

The probability distribution of X is given by:

$$f(x)=P(X=x)=\begin{cases}\binom{n}{x} p^x (1-p)^{n-x} \; ; & x=0,1,2,\ldots,n \\ 0 \; ; & \textit{otherwise}\end{cases}$$



where

$$\binom{n}{x}=\frac{n!}{x!(n-x)!}$$

$$x!=x(x-1)(x-2)....(1)$$

We can write the probability distribution of X as a table as follows.

x	f(x)=P(X=x)=b(x;n,p)
0	$\binom{n}{0}p^0(1-p)^{n-0}=(1-p)^n$
1	$\binom{n}{1}p^1(1-p)^{n-1}$
2	$\binom{\mathrm{n}}{2}\mathrm{p}^2(1-\mathrm{p})^{\mathrm{n}-2}$
$\vdots$	$\vdots$
$n-1$	$\binom{n}{n-1}p^{n-1}(1-p)^1$
n	$\binom{n}{n}p^n(1-p)^0=p^n$
Total	1.00

Example:

Suppose that 25% of the products of a manufacturing process are defective. Three items are selected at random, and classified as defective (D) or non-defective (N). Find the probability distribution of **the number of defective items.**

Solution:

·Experiment: selecting 3 items at random, inspected, and classified as (D) or (N).

·The sample space is

S={DDD,DDN,DND,DNN,NDD,NDN,NND,NNN}

· Let X = the number of defective items in the sample

· We need to find the probability distribution of X.

(1) First Solution: (Read)

Outcome	Probability	X
NNN	$\frac{3}{4}\times\frac{3}{4}\times\frac{3}{4}=\frac{27}{64}$	**0**
NND	$\frac{3}{4}\times\frac{3}{4}\times\frac{1}{4}=\frac{9}{64}$	**1**
NDN	$\frac{3}{4}\times\frac{1}{4}\times\frac{3}{4}=\frac{9}{64}$	**1**
NDD	$\frac{3}{4}\times\frac{1}{4}\times\frac{1}{4}=\frac{3}{64}$	**2**
DNN	$\frac{1}{4}\times\frac{3}{4}\times\frac{3}{4}=\frac{9}{64}$	**1**
DND	$\frac{1}{4}\times\frac{3}{4}\times\frac{1}{4}=\frac{3}{64}$	**2**
DDN	$\frac{1}{4}\times\frac{1}{4}\times\frac{3}{4}=\frac{3}{64}$	**2**
DDD	$\frac{1}{4}\times\frac{1}{4}\times\frac{1}{4}=\frac{1}{64}$	**3**

Then, the probability distribution of X is

x	f(x)=P(X=x)
0	$\frac{27}{64}$
1	$\frac{9}{64}+\frac{9}{64}+\frac{9}{64}=\frac{27}{64}$
2	$\frac{3}{64}+\frac{3}{64}+\frac{3}{64}=\frac{9}{64}$
3	$\frac{1}{64}$

(2) Second Solution:

Bernoulli trial is the process of inspecting the item. The results are success=D or failure=N, with probability of success P(*s*)=25/100=1/4=0.25.

The experiments is a Bernoulli process with:

- **number of trials: n=3**
- **Probability of success: p=1/4=0.25**
- **X ~ Binomial(n,p)=Binomial(3,1/4)**
- **The probability distribution of X is given by:**

$$f(x)=P(X=x)=b(x;3,\frac{1}{4})=\begin{cases}\binom{3}{x}(\frac{1}{4})^{x}\,(\frac{3}{4})^{3-x}; & x=0,1,2,3\\ 0\ ; & otherwise\end{cases}$$

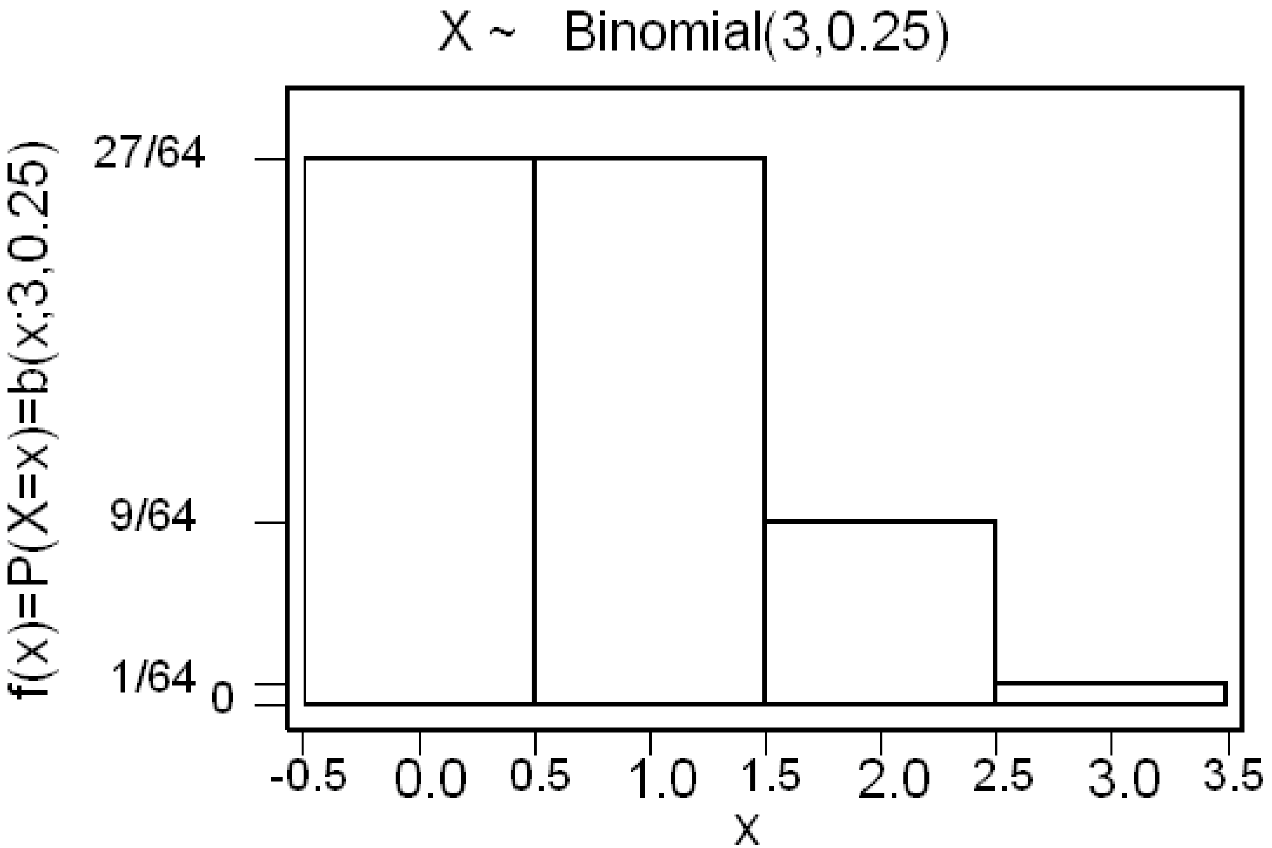
$$f(0)=P(X=0)=b(0;3,\frac{1}{4})=\binom{3}{0}(\frac{1}{4})^{0}\,(\frac{3}{4})^{3}=\frac{27}{64}$$

$$f(2)=P(X=2)=b(2;3,\frac{1}{4})=\binom{3}{2}(\frac{1}{4})^{2}\,(\frac{3}{4})^{1}=\frac{9}{64}$$

$$f(3)=P(X=3)=b(3;3,\frac{1}{4})=\binom{3}{3}(\frac{1}{4})^{3}\,(\frac{3}{4})^{0}=\frac{1}{64}$$

The probability distribution of X is

x	f(x)=P(X=x) =b(x;3,1/4)
0	27/64
1	27/64
2	9/64
3	1/64



Theorem 5.2:

The mean and the variance of the binomial distribution b(x;*n*,*p*) are:

$$\mu = n\,p$$

$$\sigma^2 = n\,p\,(1 - p)$$

<u>H.W. : Proof that.</u>

Example:

In the previous example, find the expected value (mean) and the variance of the number of defective items.

Solution:

- X = number of defective items
- We need to find $E(X)=\mu$ and $Var(X)=\sigma^2$
- We found that $X \sim$ Binomial(n,p)=Binomial(3,1/4)
- .n=3 and p=1/4

The expected number of defective items is

$$E(X)=\mu = n\,p = (3)\,(1/4) = 3/4 = 0.75$$

The variance of the number of defective items is

$$Var(X)=\sigma^2 = n\,p\,(1-p) = (3)\,(1/4)\,(3/4) = 9/16 = 0.5625$$

$$
\begin{aligned}
E[X] &= \sum_{x=0}^{n} x\, P(X=n) \\
&= \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=0}^{n} \frac{x\, n!\, p^x (1-p)^{n-x}}{x!(n-x)!} \\
&= \sum_{x=1}^{n} \frac{x\, n!\, p^x (1-p)^{n-x}}{x!(n-x)!} && \text{Set summation from x=1 since when x=0, the expression = 0} \\
&= np \sum_{x=1}^{n} \frac{(n-1)!\, p^{x-1} (1-p)^{n-x}}{(x-1)!(n-x)!} && \text{Factor out an np and cancel an x} \\
&= np \sum_{y=0}^{n-1} \frac{(n-1)!\, p^{y} (1-p)^{n-y-1}}{(y)!(n-y-1)!} && \text{Change of variable y=x-1} \\
&= np \underbrace{\sum_{y=0}^{m} \binom{m}{y} p^y (1-p)^{m-y}}_{\text{Binomial pdf for y successes in m trials = 1}} && \text{Another change of variable m=n-1} \\
&= np
\end{aligned}
$$

$$
\begin{aligned}
V(X) &= E[(X-\mu)^2] \\
&= E[X^2] - \mu^2 \\
&= \sum_{x=0}^{n} x^2 P(X=n) \\
&= \sum_{x=0}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x}
\end{aligned}
$$

Again, we have a problem in that it is going to be hard to get rid of the x^2.

$$
\begin{aligned}
E[X(X-1)] &= E[X^2] - E[X] \\
&= \sum_{x=0}^{n} x(x-1)\binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=0}^{n} \frac{x(x-1)\, n!\, p^x (1-p)^{n-x}}{x!(n-x)!} \\
&= \sum_{x=2}^{n} \frac{x(x-1)\, n!\, p^x (1-p)^{n-x}}{x!(n-x)!} && \text{Set summation from x=2 since when x=0 and x=1, the expression = 0} \\
&= \sum_{x=2}^{n} \frac{n!\, p^x (1-p)^{n-x}}{(x-2)!(n-x)!} && \text{Cancel x(x-1)} \\
&= n(n-1) \sum_{x=2}^{n} \frac{(n-2)!\, p^x (1-p)^{n-x}}{(x-2)!(n-x)!} && \text{Factor out n(n-1)} \\
&= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!\, p^{x-2} (1-p)^{n-x}}{(x-2)!(n-x)!} && \text{Factor out } p^2 \\
&= n(n-1)p^2 \sum_{y=0}^{n-2} \frac{(n-2)!\, p^{y} (1-p)^{n-y-2}}{(y)!(n-y-2)!} && \text{Change of variable y = x - 2} \\
&= n(n-1)p^2 \underbrace{\sum_{y=0}^{m} \frac{m!\, p^{y} (1-p)^{m-y}}{y!(m-y)!}}_{\text{Binomial pdf for y successes in m trials = 1}} && \text{Another change of variable m=n-2} \\
&= n(n-1)p^2
\end{aligned}
$$

Then,

$$
\begin{aligned}
V(X) &= E[(X-\mu)^2] \\
&= E[X^2] - \mu^2 \\
&= E[X(X-1)] + E[X] - \mu^2 \\
&= n(n-1)p^2 + np - n^2p^2 \\
&= n^2p^2 - np^2 + np - n^2p^2 \\
&= np(1-p)
\end{aligned}
$$

Example:

In the previous example, find the following probabilities:

(1) The probability of getting at least two defective items.

(2) The probability of getting at most two defective items.

Solution:

$X \sim \text{Binomial}(3,1/4)$

$$f(x) = P(X = x) = b(x;3,\frac{1}{4}) = \begin{cases} \binom{3}{x} (\frac{1}{4})^x (\frac{3}{4})^{3-x} & for \;\; x = 0, 1, 2, 3 \\ 0 & otherwise \end{cases}$$

x	.f(x)=P(X=x)=b(x;3,1/4)
0	27/64
1	27/64
2	9/64
3	1/64

(1) The probability of getting at least two defective items:

$$P(X\geq 2)=P(X=2)+P(X=3)= f(2)+f(3)= \frac{9}{64}+\frac{1}{64}=\frac{10}{64}$$

(2) The probability of getting at most two defective item:

$$P(X\leq 2) = P(X=0)+P(X=1)+P(X=2)$$

$$= f(0)+f(1)+f(2) = \frac{27}{64}+\frac{27}{64}+\frac{9}{64}=\frac{63}{64}$$

or

$$P(X\leq 2)= 1–P(X>2) = 1–P(X=3) = 1– f(3) = 1-\frac{1}{64}=\frac{63}{64}$$

Example 5.4: **Reading assignment**

Example 5.5: **Reading assignment**

Example 5.6: **Reading assignment**

Cumulative Distribution Function

Cumulative Distribution Function

The formula for the binomial cumulative probability function is

$$F(x,p,n)=\sum_{i=0}^{x}\binom{n}{i}(p)^{i}(1-p)^{(n-i)}$$

MGF(Binomial)

$$M_X(t)=\sum_{x=0}^{n}e^{tx}\binom{n}{x}p^{x}(1-p)^{n-x}$$

$$=\sum_{x=0}^{n}(pe^{t})^{x}\binom{n}{x}(1-p)^{n-x}$$

The binomial formula gives

$$\sum_{x=0}^{n}\binom{n}{x}u^{x}v^{n-x}=(u+v)^{n}.$$

Hence, letting $u=pe^{t}$ *and* $v=1-p$, *we have*

$$M_X(t)=[pe^{t}+(1-p)]^{n}.$$

<u>Exercise: Find the first 4 moments by using mgf</u>

Notations on the Binomial Distribution

The Binomial process is A sequence of Bernoulli trials under the following conditions

1-Each trial results in one of two possible, mutually exclusive, outcomes. One of the possible outcomes is denoted (arbitrarily) as a success, and the other is denoted a failure.
2-The probability of a success, denoted by **p**, remains constant from trial to trial. The probability of a failure, **1-p**, is denoted by **q.**
3-The trials are independent, that is the outcome of any particular trial is not affected by the outcome of any other trial

Note: The parameters of binomial are n (no. of trials), p (probability of success)

Exercises

Example 5.5:

The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that

i at least 10 survive,

ii from 3 to 8 survive, and

iii exactly 5 survive?

Solution

$$P(X \geq 10) = 1 - P(x < 10) = 1 - \sum_{x=0}^{9} b(x; 15, 0.4) = 1 - 0.9662 = 0.0338$$

$$P(3 \leq X \leq 8) = \sum_{x=3}^{8} b(x; 15, 0.4) = \sum_{x=0}^{8} b(x; 15, 0.4) - \sum_{x=0}^{2} b(x; 15, 0.4) = 0.9050 - 0.0271 = 0.8779$$

$$P(X = 5) = b(5; 15, 0.4) = \sum_{x=0}^{5} b(x; 15, 0.4) - \sum_{x=0}^{4} b(x; 15, 0.4) = 0.4032 - 0.2173 = 0.1859$$

5.4 In a certain city district the need for money to buy drugs is stated as the: reason for 75% of all thefts (السرقات). Find the probability that among the next 5 theft cases reported in this district,
(a) exactly 2 resulted from the need for money to buy drugs;
(b) at most 3 resulled from the need for money to buy drugs.

Solution

5.4 For $n = 5$ and $p = 3/4$, we have

(a) $P(X = 2) = \binom{5}{2}(3/4)^2(1/4)^3 = 0.0879,$

(b) $P(X \leq 3) = \sum_{x=0}^{3} b(x; 5, 3/4) = 1 - P(X = 4) - P(X = 5)$
$= 1 - \binom{5}{4}(3/4)^4(1/4)^1 - \binom{5}{5}(3/4)^5(1/4)^0 = 0.3672.$

5.5 According to Chemical Engineermi/ Progress (Nov. 1990), approximately 30% of all pipework failures in chemical plants are caused by operator error.
(a) What is the probability that out of the next 20 pipework failures at least 10 are due to operator error?
(b) What is the probability that no more than 4 out of 20 such failures are due to operator error?
(c) Suppose, for a particular plant, that, out of the random sample of 20 such failures, exactly 5 are operational errors. Do you feel that the 30% figure stated above applies to this plant? Comment

Solution 5.5 We are considering a $b(x; 20, 0.3)$.

(a) $P(X \geq 10) = 1 - P(X \leq 9) = 1 - 0.9520 = 0.0480$.

(b) $P(X \leq 4) = 0.2375$.

(c) $P(X = 5) = 0.1789$. This probability is not very small so this is not a rare event. Therefore, $P = 0.30$ is reasonable.

5.7 One prominent physician claims that 70% of those with lung cancer are chain smokers. If his assertion is correct,
(a) find the: probability that of 10 such patients recently admitted to a hospital, fewer than half are chain smokers:
(b) find the probability that of 20 such patients recently admitted to a hospital, fewer than half are chain smokers

Solution 5.7 $p = 0.7$.

(a) For $n = 10$, $P(X < 5) = P(X \leq 4) = 0.0474$.

(b) For $n = 20$, $P(X < 10) = P(X \leq 9) = 0.0171$.

5.9 In testing a certain kind of truck tire over a rugged terrain, it is found that 25% of the trucks fail to complete the test run without a blowout. Of the next 15 trucks tested, find the probability that
(a) from 3 to 6 have blowouts;
(b) fewer than 4 have blowouts:
(c) more than 5 have blowouts.

Solution

5.9 For $n = 15$ and $p = 0.25$, we have

(a) $P(3 \leq X \leq 6) = P(X \leq 6) - P(X \leq 2) = 0.9434 - 0.2361 = 0.7073.$

(b) $P(X < 4) = P(X \leq 3) = 0.4613.$

(c) $P(X > 5) = 1 - P(X \leq 5) = 1 - 0.8516 = 0.1484.$

5.11 The probability that a patient recovers from a delicate heart operation is 0.9. What is the probability that exactly 5 of the next 7 patients having this operation survive?

Solution

5.11 From Table A.1 with $n = 7$ and $p = 0.9$, we have
$P(X = 5) = P(X \leq 5) - P(X \leq 4) = 0.1497 - 0.0257 = 0.1240.$

Some Famous Discrete Distributions

Geometric Distributions and Negative Binomial

Geometric Distributions

The geometric R.V. has the following characteristics:

1-The outcome of each trial is Bernoulli, that is either a success(S) or failure(F)

2-The Probability of success is constant P(S)=p, P(F)=q=1-p

3-The trials are repeated until 'one' successes occur.

For example:

- A coin is tossed until a head is obtained.
- From now we count the no. of days until we get a rainy day.
- In a box there are some red balls and some white balls, we draw the balls randomly unless say 'a red ball ' red balls is obtained.

4 - The **no. of trials** are not fixed they are **variable**.

The random variable X : **number of the trial on which the first success occurs**

Probability Mass Function

The no. of trials 'X' are at least 1 to get the first success,

$$g(x,p) = p\,q^{x-1}, \quad 0 \le p \le 1, \;\; x = 1,2,3, \ldots..$$

The geometric distribution has 1 parameters; 'p' the probability of success

The mean and variance values of Geometric Distribution

Theorem 5.4: The mean and variance values of Geometric Distribution are

$$\mu = \frac{1}{p}, \qquad \sigma^2 = \frac{1-p}{p^2}$$

The Proof :

The expected value :

$$E(X) = \sum_{i=1}^{\infty} i \times q^{i-1} p = \sum_{i=1}^{\infty} i \times q^{i-1} p = p \sum_{i=1}^{\infty} i \times q^{i-1}$$

$$= p(q + 2q^2 + 3q^3 + \ldots) = p(1-q)^{-2} = \frac{p}{p^2} = \frac{1}{p}$$

The variance :

$$E(X^2) = \sum_{i=1}^{\infty} i^2 \times q^{i-1} p = \sum_{i=1}^{\infty} (i-1+1)^2 \times q^{i-1} p$$

$$= p \sum_{i=1}^{\infty} \{(i-1)^2 + 2(i-1) + 1) \times q^{i-1}$$

$$= p \sum_{i=1}^{\infty} (i-1)^2 \times q^{i-1} + 2p \sum_{i=1}^{\infty} (i-1) \times q^{i-1} + p \sum_{i=1}^{\infty} q^{i-1}$$

$$= \sum_{j=0}^{\infty} j^2 p \times q^j + 2p \sum_{j=0}^{\infty} j \times q^j + 1$$

$$= q \sum_{j=1}^{\infty} j^2 p \times q^j + 2q \sum_{j=1}^{\infty} j \times pq^{j-1} + 1$$

$$= qE(X^2) + 2qE(X) + 1$$

$$\Rightarrow (1-q)E(X^2) = \frac{2q}{p} + 1$$

$$\Rightarrow pE(X^2) = \frac{2q}{p} + 1 = \frac{2q+p}{p} = \frac{1+q}{p}$$

$$\Rightarrow E(X^2) = \frac{1+q}{p^2} \quad \text{giving variance}$$

$$\text{V(X)} = \frac{1+q}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{q}{p^2} = \frac{1-p}{p^2}$$

$E(X) = \sum_{x=0}^{\infty} x * P(X = x)$

$\sum_{x=0}^{\infty} x * p * q^{x}$ change the index, the first terms of the sum is zero so we can start the index from 1

$\sum_{x=1}^{\infty} x * p * q^{(x-1)}$

$p * \sum_{x=1}^{\infty} x * q^{(x-1)}$ factor out a p

note that the arguement in the sum is a derivative:

$\partial q^x / \partial q = x q^{(x-1)}$

$p * \sum_{x=1}^{\infty} \partial[q^{(x-1)}] / \partial q$

it is not a trivial task to interchange a sumation and differentiation but it is valid in this case.

$p * \partial/\partial q \sum_{x=1}^{\infty} q^{(x-1)}$ the sum is a geometric sum missing the first term and will evaluate to $(1 / (1 - q) - 1)$

$p * \partial/\partial q * (1 / (1 - q) - 1)$ take the derivative

$p * (1 / (1 - q)^2)$ write in terms of p

$p * (1/ (1 - (1-p)^2)) = p / p^2 = 1/p$

Cumulative Distribution Function CDF

$F(x) = P(X \leq x) = P(X = 1) + P(X = 2) + \ldots.. P(X = x)$

$= p + q\,p + q^2\,p + \ldots.. \; q^{x-1}\,p$

Sum of n terms is $S_n = \dfrac{a(1-r^n)}{1 - r}$

Let p = a = first term

q = ratio r

x = n.

$$\frac{p\,(1 - q^x)}{1 - q}$$

Since 1 - q = p this becomes

$P(X \leq x) = 1 - q^x$

Moment Generating function(Geometric Distribution)

$$M_X(t) = \sum_{x=1}^{\infty} e^{tx} P(X = x) = \sum_{x=1}^{\infty} pe^{t(x-1+1)} q^{x-1}$$

$$= pe^t \sum_{x=1}^{\infty} \; e^{t(x-1)} q^{x-1} = pe^t \sum_{x=1}^{\infty} \; \{e^t q\}^{x-1}$$

$$= pe^t (1 + \; e^t q + \; (qe^t)^2 + \; (qe^t)^2 + \ldots)$$

$$= pe^t (1 - qe^t)^{-1} = \frac{pe^t}{(1 - qe^t)}$$

Exercise: find the mean and variance of geometric distribution using the m.g.f. of it

Example 5.18: In a certain manufacturing process it is known that, on the average, 1 in every 100 items is defective. What is the probability that the fifth item inspected is the **first defective item found?**

$$g(x;p) = \quad g(5;0.01) = 0.01x0.99^4 = 0.0096$$

The Negative Binomial

- This distribution is used when the independent trials ’x’ are repeated to get ‘r’ successes. The number of these trials are the values of random variable.
- For example a coin is tossed until k= 2 heads are obtained.
- Or from today we count the no. of days until we get k= three rainy days.
- In a box there are some red balls and some white balls, we draw the balls randomly unless say k=5 red balls are obtained.

The random variable X is , the number of the trial on which the k^{th} success occurs.

Negative binomial distribution: If repeated independent trials can result in a success with probability ***p*** and a failure with probability ***q=1-p***, then the probability distribution of the random variable X, which is the number of the trial on which the k^{th} success occurs. is given by:

$$b^*(x;k,p) = \binom{x-1}{k-1} p^k q^{x-k}, \quad x = k, k+1, k+2, \ldots$$

The no. of trials ’X’ are at least “k” to get the “k” success,

Note : In the negative binomial distribution k=1, then we will get geometric distribution . I n other word, it is a generalization of Geometric Distribution.

The moment generating function of the negative binomial distribution Mean and Variance of Negative Binomial

We will derive the m.g.f of the distribution then we will use it to derive the mean and variance

The moment generating function:

$$M_X(t) = E(e^{tX}) = \sum_{x=r}^{\infty} \binom{x-1}{r-1} e^{tx} \times q^{x-r} p^r$$

$$= \sum_{x=r}^{\infty} \binom{x-1}{r-1} e^{t(x-r)} \times e^{rt} \times q^{x-r} p^r = p^r e^{rt} \sum_{x=r}^{\infty} \binom{x-1}{r-1} (qe^t)^{x-r}$$

$$= (pe^t)^r (1-qe^t)^{-r} = \left(\frac{pe^t}{(1-qe^t)} \right)^r$$

The mean :

$$\mu = E(X) = \frac{d}{dt} \left(\frac{pe^t}{(1-qe^t)} \right)^r \Big|_{t=0} = r \left(\frac{pe^t}{(1-qe^t)} \right)^{r-1} \frac{d}{dt} pe^t (1-qe^t)^{-1} \Big|_{t=0}$$

$$= \left\{ r \left(\frac{pe^t}{1-qe^t} \right)^{r-1} \times (pe^t (1-qe^t)^{-1} + pqe^t (1-qe^t)^{-2} \right\}_{t=0}$$

$$= r \left(\frac{p}{(1-q)} \right)^{r-1} (p(1-q)^{-1} + pq(1-q)^{-2}) \Big|_{t=0}$$

$$= r(1 + \frac{pq}{p^2}) = \frac{rp(p+q)}{p^2} = \frac{r}{p}$$

The variance :

V. Proof that

$$\sigma^2 = \frac{rq}{p}$$

Exc: 5.54 Find the probability that a person flipping a coin gets
(a) the third head on the seventh flip;
(b) the first head on the fourth flip.

Solution

(a) Using the negative binomial distribution, we get $b^*(7; 3, 1/2) = \binom{6}{2}(1/2)^7 = 0.1172.$

(b) From the geometric distribution, we have $g(4; 1/2) = (1/2)(1/2)^3 = 1/16.$

5.57 The probability that a student pilot (طيار) passes (يعبر) the written test for a private pilot's license is 0.7. Find the probability that the student will pass the test (النجاح في الاختبار هو مرة واحدة أكيد)
(a) on the third try;
(b) before the fourth try.

Solution

5.57 Using the geometric distribution

(a) $P(X = 3) = g(3; 0.7) = (0.7)(0.3)^2 = 0.0630.$

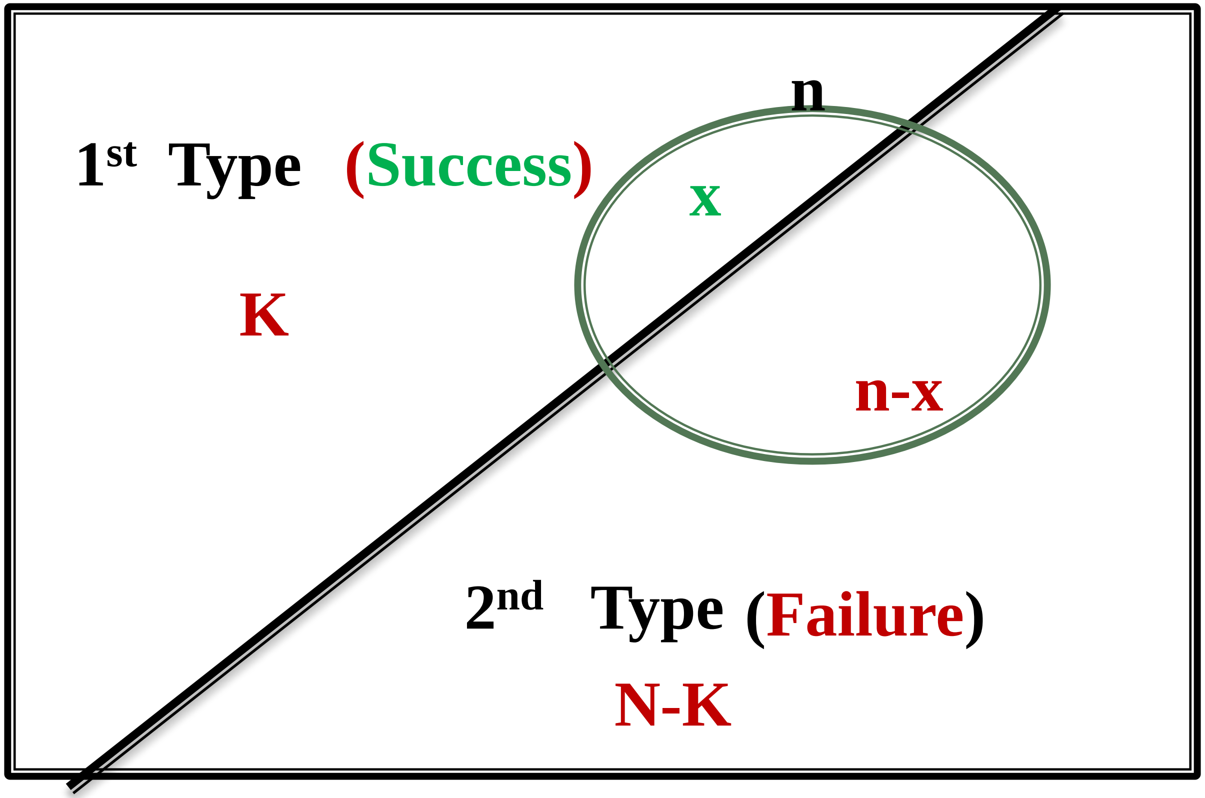
(b) $P(X < 4) = \sum_{x=1}^{3} g(x; 0.7) = \sum_{x=1}^{3} (0.7)(0.3)^{x-1} = 0.9730.$

Some Famous Discrete Distributions

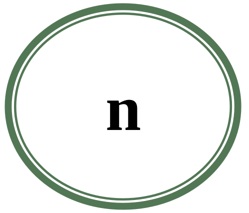
Hypergeometric Distribution

5.4 Hypergeometric Distribution :

N (Population size)



We select n elements randomly



X- number of elements of 1st type in the sample
Or
-number of successes in the sample

- Suppose there is a population with 2 types of elements:

1-st Type = success

2-nd Type = failure

- N= population size

  K= number of elements of the 1-st type

  $N-K$ = number of elements of the 2-nd type

- We select a sample of n elements at random from the population
- Let X = number of elements of 1-st type (number of successes) in the sample
- We need to find the probability distribution of X.

There are to two methods of selection:

1. selection with replacement
2. selection without replacement

(1) If we select the elements of the sample at random and with replacement, then

X ~ Binomial(n,p); where $p = \frac{K}{N}$

(2) Now, suppose we select the elements of the sample at random and without replacement. When the selection is made without replacement, the random variable X has a hyper geometric distribution with parameters N, n, and K. and we write

X ~ h(x;N,n,K). where

$$f(x) = P(X = x) = h(x; N, n, K) = \begin{cases} \dfrac{\binom{K}{x} \times \binom{N-K}{n-x}}{\binom{N}{n}}; & x = 0, 1, 2, \cdots, \min(n, K) \\ 0; \; otherwise \end{cases}$$

The assumptions leading to hypergeometric distribution are:

1-A fixed sample (trials) of size 'n' is drawn from a population of size N,

2-The population(outcome of trials) has two outcomes Success (S) and Failure(F).

3-Since the sampling is without replacement, the successive trials are dependent.

4-The trials are repeated a fixed no. of times. (sample size fixed)

5-the probability of success changes from trial to trial . (The successive trials are dependent)

Exercises

A box contains 6 blue and 4 red balls. An experiment is performed a ball is chosen and its color not observed. Find the prob, that after 5 trials, 3 blue balls will have been chosen when
(a): the balls are replaced,
(b): the balls not replaced

(a): when the balls are replaced the next trial becomes independent of what was drawn in the previous trial, as again the no. of red balls are 4 and blue are 6. Hence prob. Of blue ball remains constant and binomial is used:

Solution

$$(a): n = \text{the no. of balls chosen} = 5; p = P(S) = p(b) = \frac{b}{b+r} = \frac{6}{10} = \frac{2}{5}; p(\text{not blue}) = q = P(F) = \frac{3}{5}$$

$$n = 5, P(X=3) = \binom{5}{3}(\frac{2}{5})^3(\frac{3}{5})^2 =$$

(b): without replacement each trial depends on whether the previous was a red or not the successive trials are dependent. Hence HyperGeometric Distribution applies:

$$\mathrm{N = 10, K = 6, N - k = 4, n = 5,}$$

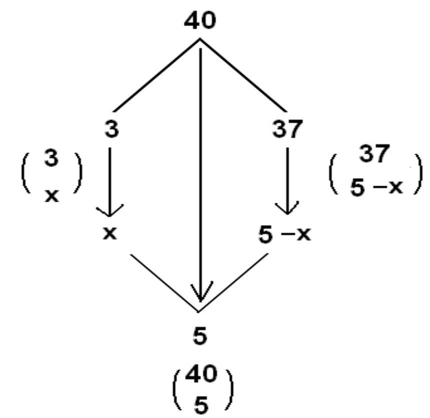
$$\mathrm{P(X=3)} = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} = \frac{\binom{6}{3}\binom{4}{2}}{\binom{10}{5}} = \frac{10}{21}$$

Example 5.11: Reading assignment

Example 5.9:

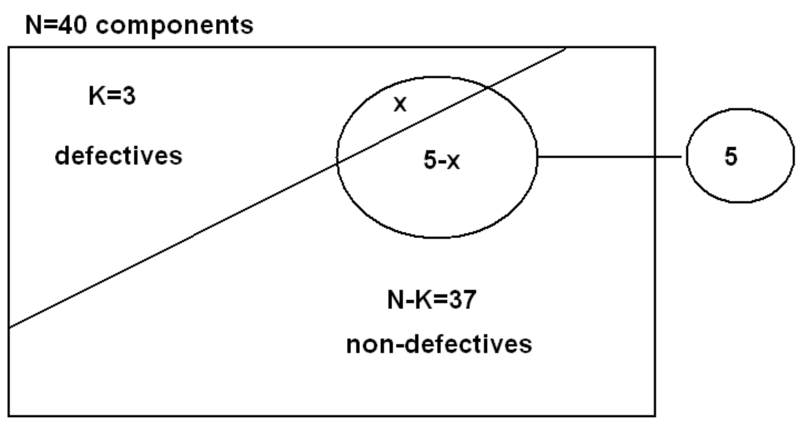
Lots of 40 components each are called acceptable if they contain no more than 3 defectives. The procedure for sampling the lot is to select 5 components at random (without replacement) and to reject the lot if a defective is found. What is the probability that exactly one defective is found in the sample if there are 3 defectives in the entire lot.

Solution:



- Let X= number of defectives in the sample
- N=40, K=3, and n=5
- X has a hypergeometric distribution with parameters N=40, n=5, and K=3.
- X~h(x;N,n,K)=h(x;40,5,3).
- The probability distribution of X is given by:

$$f(x) = P(X = x) = h(x;40,5,3) = \begin{cases} \dfrac{\binom{3}{x} \times \binom{37}{5-x}}{\binom{40}{5}}; & x = 0, 1, 2, \cdots, 5 \\ 0; \; otherwise & \end{cases}$$



But the values of X must satisfy:

$0 \le x \le K$ and $n - N + K \le x \le n \iff 0 \le x \le 3$ and $-42 \le x \le 5$

Therefore, the probability distribution of X is given by:

$$f(x) = P(X = x) = h(x;40,5,3) = \begin{cases} \dfrac{\binom{3}{x} \times \binom{37}{5-x}}{\binom{40}{5}}; & x = 0, 1, 2,3 \\ 0; \; otherwise \end{cases}$$

Now, the probability that exactly one defective is found in the sample is

.f(1)=P(X=1)=h(1;40,5,3)= $\dfrac{\binom{3}{1} \times \binom{37}{5-1}}{\binom{40}{5}} = \dfrac{\binom{3}{1} \times \binom{37}{4}}{\binom{40}{5}} = 0.3011$

Mean and Variance

Theorem 5.3:

The mean and variance of the hypergeometric distribution $h(x; N, n, k)$ are

$$\mu = \frac{nk}{N} \; and \; \sigma^2 = \frac{N-n}{n-1} * n * \frac{k}{N} * \left(1 - \frac{k}{N}\right)$$

H.W. Proof that (Hint: use the moment generating function.)

3-Hyper Geometric: Mean and Variance by Kth Moment: If X is distributed as $\approx HG(x;n,m,N)$

$$E(X^k)=\sum_{x=0}^{n} x^k \frac{\binom{m}{x}\binom{N-m}{n-x}}{\binom{N}{n}}=\frac{mn}{N}\sum_{x=0}^{n} x^{k-1}\frac{x\binom{m}{x}\binom{N-m}{n-x}}{\binom{N}{n}}$$

$$=\sum_{x=0}^{n} x^{k-1}\frac{x\left(\frac{m!}{x!(m-x)!}\right)\binom{N-m}{n-x}}{\left(\frac{N!}{n!(N-n)!}\right)}=\sum_{x=1}^{n} x^{k-1}\frac{\left(\frac{m(m-1)!}{(x-1)!(m-x)!}\right)\binom{N-m}{n-x}}{\frac{N}{n}\left(\frac{(N-1)!}{(n-1)!(N-n)!}\right)}$$

$$=\frac{mn}{N}\sum_{x=1}^{n} x^{k-1}\frac{\binom{m-1}{x-1}\binom{N-m}{n-x}}{\binom{N-1}{n-1}}=\frac{mn}{N}\sum_{y=0}^{n-1}(y+1)^{k-1}\frac{\binom{m-1}{y}\binom{N-m}{n-1-y}}{\binom{N-1}{n-1}}$$

$=\frac{mn}{N}E(Y+1)^{k-1}$, Where Y is a Negative binomial NG$(y;\, n\text{-}1, m-1, N-1)$.................................... (1)

Using $k=1$ in (1); $E(X)=\frac{mn}{N}E(1)=\frac{mn}{N}$ ....(2)

Using $k=2$, in (1): $E(X^2)=\frac{mn}{N}[E(Y)+1]=\frac{mn}{N}\{\frac{(m-1)(n-1)}{N-1}+1\}=\frac{n(n-1)\,m(m-1)}{N(N-1)}+\frac{mn}{N}$

$$V(X)=E(X^2)-(EX)^2=\frac{n(n-1)m(m-1)}{N(N-1)}+\frac{mn}{N}-\left(\frac{mn}{N}\right)^2=\frac{mn}{N}\left\{\frac{mn-m-n+1}{N-1}+1-\frac{mn}{N}\right\}$$

$$=\frac{mn}{N}\left\{\frac{mnN-mN-nN+N+N^2-N-mnN+mn}{N(N-1)}\right\}=\frac{mn}{N}\left\{\frac{-mN-nN+N^2+mn}{N(N-1)}\right\}$$

$=\frac{mn}{N}\left\{\frac{N^2-nN-mN+mn}{N(N-1)}\right\}=n\,\frac{m}{N}\{\frac{(N-m)(N-n)}{N(N-1)}\}$..(3)

If we let $p=\frac{m}{N}$ = probability of success, $q=1-p=\frac{m-n}{N}$ then (2), (3) reduce to: $\mu=np$ and $\sigma^2=npq\ (\frac{N-n}{N-1})$....(4)

while $(\frac{N-n}{N-1})$ is called Correction Factor for Finite population: $f\,p\,c=(\frac{N-n}{N-1})$

Relationship to the Binomial Distribution

A binomial distribution can be used to approximate the hypergeometric distribution when n is small, compared to N. In fact, as a rule of thumb the approximation is good when $n/N \leq 0.05$

$$h(X; N, n, K) \approx b(X; n, k/N).$$

Note:

If n is small compared to N and K, then there will be almost no difference between selection without replacement and selection with replacement

$$\left(\frac{K}{N} \approx \frac{K-1}{N-1} \approx \cdots \approx \frac{K-n+1}{N-n+1}\right).$$

5.40 It is estimated that 4000 of the 10,000 voting residents of a town are against a new sales tax. If 15 eligible voters are selected at random and asked their opinion, what is the probability that at most 7 favor the new tax?

Solution

5.40 The binomial approximation of the hypergeometric with $p = 1 - 4000/10000 = 0.6$ gives a probability of $\sum_{x=0}^{7} b(x; 15, 0.6) = 0.2131$.

Example 5.12(مكرر): **Lots of 40 components each are called unacceptable if they contain as many as 3 defective or more.**

- **The procedure for sampling the lot is to select 5 components at random and to reject the lot if a defective is found. What is the probability that exactly 1 defective is found in the sample if there are 3 defectives in the entire lot?**

Solution

Using hypergeometric distribution with $n = 5, N = 40, k = 3$ **and** $\underline{x = 1}$;

$$x \sim h(x;40,5,3) = \frac{\binom{3}{x}\binom{37}{n-x}}{\binom{40}{5}}, \boldsymbol{h(1,40,5,3)} = \frac{\binom{3}{1}\binom{37}{4}}{\binom{40}{5}} = 0.3011$$

5.33

A random committee of size 3 is selected from 4 doctors and 2 nurses. Write a formula for the probability distribution of the random variable X ***representing the number of doctors on the committee***. Find $P(2 \le X \le 3)$.

Solution

$$5.33 \quad h(x;6,3,4) = \frac{\binom{4}{x}\binom{2}{3-x}}{\binom{6}{3}}, \text{ for } x = 1,2,3.$$

$$P(2 \le X \le 3) = h(2;6,3,4) + h(3;6,3,4) = \frac{4}{5}.$$

5.36

A manufacturing company uses an acceptance scheme on production items before they are shipped. The plan is a two-stage one. Boxes of **25** are readied for shipment and a sample of **3** is tested for defectives. If any defectives are found, the entire box is sent back for 100% screening. If no defectives are found, the box is shipped.

(a) What is the probability that a box containing 3 defectives will be shipped?

(b) What is the probability that a box containing only 1 defective will be sent back for screening?

Solution

$$P(X=0) = h(0{,}25{,}3{,}3) = \frac{\binom{3}{0}\binom{22}{3-0}}{\binom{25}{3}} = \frac{3!\,22!\,3!\;\;22!}{0!\,3!\,3!\,19!\,25!} = \frac{22*21*20}{25*24*23} = \frac{9240}{13800} = \frac{77}{115}$$

$$P(X=1) = h(1{,}25{,}3{,}1) = \frac{\binom{1}{1}\binom{24}{3-1}}{\binom{25}{3}} = \frac{24!\;\;3!\;\;22!}{2!\;\;22!\;\;25!} = \frac{3}{25}$$

5.40 (مكرر)

It is estimated that 4000 of the 10,000 voting residents of a town are against a new sales tax. If 15 eligible voters are selected at random and asked their opinion, what is the probability that at most 7 favor the new tax?

Solution

5.40 The binomial approximation of the hypergeometric with $p = 1 - 4000/10000 = 0.6$ gives a probability of $\sum_{x=0}^{7} b(x; 15, 0.6) = 0.2131$.

5.45

A foreign student club lists as its members 2 Canadians, 3 Japanese, 5 Italians, and 2 Germans. If a committee of 4 is selected at random, find the probability that

(a) all nationalities are represented;

(b) all nationalities except the Italians are represented

Solution

(a) The extension of the hypergeometric distribution gives a probability $\frac{\binom{2}{1}\binom{3}{1}\binom{5}{1}\binom{2}{1}}{\binom{12}{4}} = \frac{4}{33}$.

(b) Using the extension of the hypergeometric distribution, we have

$$\frac{\binom{2}{1}\binom{3}{1}\binom{2}{2}}{\binom{12}{4}} + \frac{\binom{2}{2}\binom{3}{1}\binom{2}{1}}{\binom{12}{4}} + \frac{\binom{2}{1}\binom{3}{2}\binom{2}{1}}{\binom{12}{4}} = \frac{8}{165}.$$

Some Famous Discrete Distributions

Poisson Distribution:

5.6 Poisson Distribution:

Poisson experiment: is an experiment yielding numerical values of a random variable that **count** the number of outcomes occurring in a **given time interval or a specified region denoted by *t*.**

The R.V. X is: The number of outcomes occurring in a given time interval or a specified region denoted by *t*.

- **Examples**:
  - ~~**the number of telephone** calls **per hour** received by an office~~
  - the **number of postponed** baseball games due to rain
  - the **number of field mice per acre**
  - the **number of typing error per page**

- **Properties of Poisson Process**: **Read**
  - The number of outcomes in one time interval or specified region is independent of the number that occurs in any other disjoint time interval or region of space.
  - The probability that a single outcome will occur during a very short time interval or in a small region is proportional to the length of the time interval or the size of the region and does not depend on the number of outcomes occurring outside this time interval or region.
  - The probability that more than one outcome will occur in such a short time interval or fall in such a small region is negligible.

- **Let λ be the average (mean) number of outcomes per unit time or unit region (t=1).**

The average (mean) number of outcomes (mean of X) in the time interval or region t is:

$$\mu = \lambda\, t$$

The random variable X is called a Poisson random variable with parameter μ ($\mu=\lambda t$), and we write X~Poisson(μ), if its probability distribution is given by:

$$f(x) = P(X = x) = p(x;\mu) = \begin{cases} \dfrac{e^{-\mu}\mu^{x}}{x!} & ; \quad x = 0, 1, 2, 3, \ldots \\ 0 & ; \quad otherwise \end{cases}$$

Theorem 5.5:

The mean and the variance of the Poisson distribution Poisson(x;μ) are:

$$\mu = \lambda t$$

$$\sigma^2 = \mu = \lambda t$$

Note:

- **λ is the average (mean) of the distribution in the unit time (t=1).**
- **If X=The number of calls received in a month (unit time t=1 month) and X~Poisson(λ), then:**

(i) Y = number of calls received in a year.

Y ~ Poisson (μ); $\mu=12\lambda$ (t=12)

(ii) W = number of calls received in a day.

W ~ Poisson (μ); $\mu=\lambda/30$ (t=1/30)

Example 5.16: Reading Assignment

Example 5.17: Reading Assignment

Exercise: Prove that mean and variance of Poisson Random Variable are equal

$$\mu = E(X) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^{x}}{x!} = \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^{x}}{(x-1)!}$$

$$= \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} = e^{-\lambda}(1 + \lambda + \lambda^2 + \lambda^3 + \ldots..)$$

$$= \lambda e^{-\lambda} e^{+\lambda} = \lambda e^{0} = \lambda \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$E(X^2) = E(X(X-1) + E(X) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$E\{X(X-1)\} = \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda}\lambda^{x}}{x!} = \sum_{x=2}^{\infty} \frac{e^{-\lambda}\lambda^{x}}{(x-2)!}$$

$$= \lambda^2 e^{-\lambda} \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} = e^{-\lambda}\lambda^2(1 + \lambda + \lambda^2 + \lambda^3 + \ldots..)$$

$$= \lambda^2 e^{-\lambda} e^{+\lambda} = \lambda^2 e^{0} = \lambda^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

From (1), (2), (3)

$$E(X^2) = \lambda^2 + \lambda \quad \text{and} \quad \delta^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Derivation of the Poisson Distribution Moment Generating Function

Wed, 03/07/2012 - 14:21 — Ben

This post shows how to derive the Poisson distribution MGF.

To start off with, consider the poisson pdf: $\frac{\lambda^k e^{-\lambda}}{k!}$, where $k \in \mathbb{N} > 0$ and $\lambda > 0 \in \mathbb{R}$.

As is standard the moment generating function for a distribution $X \sim Pois(\lambda)$ is defined as:

$$\begin{aligned} M_X &= E[e^{tX}] \\ &= \sum_{k=0}^{\infty} e^{tk} \frac{\lambda^k e^{-\lambda}}{k!} \\ &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} \end{aligned}$$

From the definition we have of the expansion of the McLaurin series:
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, the continuation of the moment generating function is:

$$\begin{aligned} M_X &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} \\ &= e^{-\lambda} e^{\lambda e^t} \\ &= e^{\lambda(e^t - 1)} \end{aligned}$$

which is the common form of the MGF for the Poisson distribution.

Example:

Suppose that the number of typing errors per page has a Poisson distribution with average 6 typing errors.

(1) What is the probability that in a given page:

(i) The number of typing errors will be 7?

(ii) The number of typing errors will at least 2?

(2) What is the probability that in 2 pages there will be 10 typing errors?

(3) What is the probability that in a half page there will be no typing errors?

Solution: **X = number of typing errors per page.**

X ~ Poisson (6) **(t=1, λ=6, μ=λt=6)**

$$f(x) = P(X = x) = p(x;6) = \frac{e^{-6}6^x}{x!}\ ; \quad x = 0, 1, 2, \ldots$$

(i) $$f(7) = P(X = 7) = p(7;6) = \frac{e^{-6}6^7}{7!} = 0.13768$$

(ii) **P(X≥2) = P(X=2)+ P(X=3)+ . . . =** $\sum_{x=2}^{\infty} P(X = x)$

P(X≥2) = 1– P(X<2) = 1 – [P(X=0)+ P(X=1)]

$$= 1 - [f(0) + f(1)] = 1 - \left[\frac{e^{-6}6^0}{0!} + \frac{e^{-6}6^1}{1!}\right]$$

= 1 – [0.00248+0.01487]

= 1 – 0.01735 = 0.982650

(2) **X = number of typing errors in 2 pages**

X ~ Poisson(12) **(t=2, λ=6, μ=λt=12)**

$$f(x) = P(X = x) = p(x;12) = \frac{e^{-12}12^x}{x!} : \quad x = 0, 1, 2\ldots$$

$$f(10) = P(X = 10) = \frac{e^{-12}12^{10}}{10} = 0.1048$$

(3) X = number of typing errors in a half page.

X ~ Poisson (3) **(t=1/2, λ=6, $\mu=\lambda t$=6/2=3)**

$$f(x) = P(X = x) = p(x;3) = \frac{e^{-3}\,3^x}{x!} : \quad x = 0\,,1\,,2\ldots$$

$$f(0) = P(X = 0) = \frac{e^{-3}(3)^0}{0!} = 0.0497871$$

Example 5.20: read: During a laboratory experiment the average number of radioactive particles (الجزيئات المشعة) passing through a counter in **1 millisecond is** 4. What is the probability that 6 particles enter the counter in a given millisecond?

$$p(6;4) = \frac{e^{-4}(4)^6}{6!} \quad 0.1042$$

Example 5.21: **Ten** is the average number of oil tankers arriving (ناقلات النفط صهاريج) **each day** at a certain port (ميناء)city. Find the prob. that

- The facilities at the port can handle at most 15 tankers per day.
- What is the probability that on a given day tankers have to be turned away?

$$P(X > 15) = 1 - P(X \leq 15) = 1 - \sum_{x=0}^{15} p(x;10) = 1 - 0.9513 = 0.0487$$

Poisson approximation for binomial distribution:

Theorem 5.6:

Let X be a binomial random variable with probability distribution b(x;n,p). If n→∞, p→0, and μ=np remains constant, then the binomial distribution b(x;n,p) can approximated by Poisson distribution p(x;μ). For large **n** and small **p** we have:

b(x;n,p) ≈ Poisson(μ) (μ=np) OR $\binom{n}{x} p^x (1-p)^{n-x} \approx \frac{e^{-\mu}\mu^x}{x!}; x = 0,1,\cdots,n; \quad (\mu = np)$

Example 5.22: In a certain industrial facility accidents occur infrequently. It is known that the probability of an accident on any given day is 0.005 and accidents are independent of each other.

(a) What, is the probability that in any given period of 400 days there will be an accident on one day?

(b) What is the probability that there are at most three days with an accident?

Solution:

Let. X be a binomial random variable with n = 400 and p = 0.005. Thus np = 2.

Using the Poisson approximation,

(a) P(X = 1) = e -2 * 2 = 0.271. and

(b) P{X ≤ 3) = e - 2 * 20/ 0! + e - 2 * 2+ e - 2 * 22/ 2! + e - 2 * 23/ 3! = 0.857. .

Example 5.23 (Exercise)

In a manufacturing process where glass products arc produced, defects or bubbles occur, occasionally rendering the piece undesirable: for marketing. It is known that, on average, 1 in every 1000 of these items produced has one or more bubbles. What is the probability that a random sample of 8000 will yield fewer than 7 items

ssessing bubbles?