SIAM J. NUMER. ANAL.
Vol. 44, No. 2, pp. 826–850

SPECTRAL DISCRETIZATION OF THE VORTICITY, VELOCITY, AND PRESSURE FORMULATION OF THE STOKES PROBLEM*

CHRISTINE BERNARDI† AND NEJMEDDINE CHORFI‡

Abstract. We consider the Stokes problem in a square or a cube provided with nonstandard boundary conditions which involve the normal component of the velocity and the tangential components of the vorticity. We write a variational formulation of this problem with three independent unknowns: the vorticity, the velocity, and the pressure. Next, we propose a discretization by spectral methods which relies on this formulation and, since it leads to an inf-sup condition on the pressure in a natural way, we prove optimal error estimates for the three unknowns. We present numerical experiments which are in perfect coherence with the analysis.

Key words. Stokes problem, vorticity-velocity-pressure formulation, spectral method

AMS subject classifications. 65N35, 35Q30

DOI. 10.1137/050622687

1. Introduction.

We consider the Stokes problem in a two- or three-dimensional bounded domain when provided with boundary conditions on the normal component of the velocity and the vorticity in dimension 2 and on the normal component of the velocity and the tangential components of the vorticity in dimension 3. The well-posedness of this problem is first proved in the pioneering paper [6]; however, the formulation that is considered in this work deals with the velocity and the pressure as only unknowns and requires the convexity or some regularity of the domain. As first proposed in [18] and [26] (see also [19] and [1]), this problem admits an equivalent variational formulation where the unknowns are the vorticity, the velocity, and the pressure. This formulation involves the domains of the divergence and curl operators, as first suggested in [25]. We also refer the reader to [21] for a different formulation where the unknowns are the vorticity, the vector potential, and the pressure and to [20] for a comparison between different formulations. The aim of this paper is to propose and analyze a discrete problem which relies on the vorticity, velocity, and pressure formulation and is constructed by spectral methods.

Indeed, it seems that the numerical analysis of discretizations relying on this formulation has been performed only for finite element methods; see [26], [2], and [11]. We refer the reader to [8] for the analysis of a spectral discretization of the same problem relying on the velocity and pressure formulation. However, the formulation that we consider here leads naturally to a more accurate approximation of the pressure. One of the difficulties in the discretization consists in handling both the two- and three-dimensional cases. Indeed, the vorticity is a scalar function in dimension 2 and can be approximated in a standard polynomial space while it is a vector field in dimension 3: This requires the introduction of appropriate polynomial spaces which are the spectral analogues of Nédélec's finite elements; see [24]. The discretization that we propose takes into account these considerations, and its numerical analysis leads to optimal

*Received by the editors January 14, 2005; accepted for publication (in revised form) November 2, 2005; published electronically April 12, 2006.
http://www.siam.org/journals/sinum/44-2/62268.html

†Laboratoire Jacques-Louis Lions, C.N.R.S. & Université Pierre et Marie Curie, B.C. 187, 4 place Jussieu, 75252 Paris Cedex 05, France (bernardi@ann.jussieu.fr).
‡Département de Mathématiques, Faculté des Sciences de Tunis, Campus Universitaire, 1060 Tunis, Tunisie (nejmeddine.chorfi@fst.rnu.tn).

error estimates on the three unknowns. This is the main advantage of this formulation since, in most usual spectral discretizations of the Stokes problem, a lack of optimality appears in the estimate concerning the pressure; see [13, sects. 24–26], and [14] for a possible but less natural improvement. We present numerical experiments which confirm the optimality of the discretization and its efficiency for the Stokes problem provided with this type of boundary condition, both in the two- and three-dimensional situations.

The extension of this study to the case of the nonlinear Navier–Stokes equations is presently under consideration. The main difficulty here is the choice of variational spaces in order to preserve the compactness of the nonlinear term. We also intend to treat more complex geometries by using a spectral element discretization.

An outline of the paper is as follows.

- In section 2, we write the variational formulation of the problem in the case of homogeneous boundary conditions.
- Section 3 is devoted to the description of the spectral discrete problem. We also prove its well-posedness.
- Optimal error estimates are derived in section 4.
- The extension to the case of nonhomogeneous boundary conditions on the velocity is explained in section 5.
- In section 6, we present some numerical experiments which turn out to be in good agreement with the analysis.

2. The velocity, vorticity, and pressure formulation. Let Ω be a bounded connected domain in $\mathbb{R}^d$, $d = 2$ or 3, with a Lipschitz-continuous boundary $\partial\Omega$. We assume for simplicity that Ω is simply connected and has a connected boundary. The generic point in Ω is denoted by $\boldsymbol{x} = (x, y)$ or $\boldsymbol{x} = (x, y, z)$ according to the dimension d. We introduce the unit outward normal vector $\boldsymbol{n}$ to Ω on $\partial\Omega$ and consider the Stokes problem

$$\begin{cases} -\nu\,\Delta\boldsymbol{u} + \mathbf{grad}\,p = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 & \text{on } \partial\Omega, \\ \gamma_t(\mathbf{curl}\,\boldsymbol{u}) = \mathbf{0} & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

To make precise the sense of the operator γ_t, we recall the following.

- In dimension $d = 2$, for any vector field $\boldsymbol{v}$ with components v_x and v_y, $\mathbf{curl}\,\boldsymbol{v}$ stands for the scalar function $\partial_x v_y - \partial_y v_x$, so that the operator γ_t is the trace operator on $\partial\Omega$.
- In dimension $d = 3$, for any vector field $\boldsymbol{v}$ with components v_x, v_y, and v_z, $\mathbf{curl}\,\boldsymbol{v}$ stands for the vector field with components $\partial_y v_z - \partial_z v_y$, $\partial_z v_x - \partial_x v_z$, and $\partial_x v_y - \partial_y v_x$, and the operator γ_t is the tangential trace operator on $\partial\Omega$, defined by $\gamma_t(\boldsymbol{w}) = \boldsymbol{w} \times \boldsymbol{n}$.

Of course, the operator γ_t is only defined on smooth enough functions as will be made precise later on.

In system (2.1), the unknowns are the velocity $\boldsymbol{u}$ and the pressure p, while the data $\boldsymbol{f}$ represents a density of body forces. The viscosity ν is a positive constant. To go further, we introduce the vorticity $\boldsymbol{\omega} = \mathbf{curl}\,\boldsymbol{u}$ and observe that system (2.1) is

fully equivalent to

$$\begin{cases} \nu\,\mathbf{curl}\,\boldsymbol{\omega} + \mathbf{grad}\,p = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{\omega} = \mathbf{curl}\,\boldsymbol{u} & \text{in } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 & \text{on } \partial\Omega, \\ \gamma_t(\boldsymbol{\omega}) = \mathbf{0} & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Note that the operator **curl** in the first line of this system coincides with the previous one in dimension $d = 3$ while, in dimension $d = 2$, it is applied to scalar functions $\boldsymbol{\varphi}$: $\mathbf{curl}\,\boldsymbol{\varphi}$ here denotes the vector field with components $\partial_y\boldsymbol{\varphi}$ and $-\partial_x\boldsymbol{\varphi}$.

In order to write the variational formulation of problem (2.2), we consider the full scale of Sobolev spaces $H^s(\Omega)$. As usual, we denote by $L^2_0(\Omega)$ the space of functions in $L^2(\Omega)$ with a null integral on Ω. Let also $\mathcal{D}(\Omega)$ be the space of infinitely differentiable functions with a compact support in Ω. We introduce the domain $H(\operatorname{div},\Omega)$ of the divergence operator, namely

$$H(\operatorname{div},\Omega) = \big\{\boldsymbol{v} \in L^2(\Omega)^d;\ \operatorname{div}\boldsymbol{v} \in L^2(\Omega)\big\}. \tag{2.3}$$

A consequence of the Stokes formula, valid for smooth enough vector fields $\boldsymbol{v}$ and scalar function φ,

$$\int_\Omega (\operatorname{div}\boldsymbol{v})(\boldsymbol{x})\varphi(\boldsymbol{x})\,d\boldsymbol{x} = -\int_\Omega \boldsymbol{v}(\boldsymbol{x})\cdot(\mathbf{grad}\,\varphi)(\boldsymbol{x})\,d\boldsymbol{x} + \int_{\partial\Omega} (\boldsymbol{v}\cdot\boldsymbol{n})(\boldsymbol{\tau})\varphi(\boldsymbol{\tau})\,d\boldsymbol{\tau}, \tag{2.4}$$

is that the normal trace operator $\boldsymbol{v} \mapsto \boldsymbol{v}\cdot\boldsymbol{n}$ can be defined from $H(\operatorname{div},\Omega)$ into $H^{-\frac{1}{2}}(\partial\Omega)$. This leads us to introduce its kernel

$$H_0(\operatorname{div},\Omega) = \big\{\boldsymbol{v} \in H(\operatorname{div},\Omega);\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\big\}. \tag{2.5}$$

Similarly, we introduce the domain of the **curl** operator

$$H(\mathbf{curl},\Omega) = \big\{\boldsymbol{\vartheta} \in L^2(\Omega)^{\frac{d(d-1)}{2}};\ \mathbf{curl}\,\boldsymbol{\vartheta} \in L^2(\Omega)^d\big\}. \tag{2.6}$$

The Stokes formula here reads, for smooth enough functions $\boldsymbol{\vartheta}$ in $L^2(\Omega)^{\frac{d(d-1)}{2}}$ and $\boldsymbol{v}$ in $L^2(\Omega)^d$,

$$\int_\Omega (\mathbf{curl}\,\boldsymbol{\vartheta})(\boldsymbol{x})\cdot\boldsymbol{v}(\boldsymbol{x})\,d\boldsymbol{x} = \int_\Omega \boldsymbol{\vartheta}(\boldsymbol{x})\cdot(\mathbf{curl}\,\boldsymbol{v})(\boldsymbol{x})\,d\boldsymbol{x} - \int_{\partial\Omega} \gamma_t(\boldsymbol{\vartheta})(\boldsymbol{\tau})\cdot\tilde{\gamma}_t(\boldsymbol{v})(\boldsymbol{\tau})\,d\boldsymbol{\tau}, \tag{2.7}$$

where $\tilde{\gamma}_t(\boldsymbol{v})$ is equal to $\boldsymbol{v}$ in dimension $d = 3$ and to $v_y n_x - v_x n_y$ in dimension $d = 2$. This allows us to define the kernel

$$H_0(\mathbf{curl},\Omega) = \big\{\boldsymbol{\vartheta} \in H(\mathbf{curl},\Omega);\ \gamma_t(\boldsymbol{\vartheta}) = \mathbf{0} \text{ on } \partial\Omega\big\}. \tag{2.8}$$

Remark 2.1. Note that the spaces $H(\mathbf{curl},\Omega)$ and $H_0(\mathbf{curl},\Omega)$ coincide with the spaces $H^1(\Omega)$ and $H^1_0(\Omega)$ in dimension $d = 2$, so that their approximation relies on more standard discrete spaces than in dimension $d = 3$.

We now consider the following variational problem:

Find $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ *in* $H_0(\mathbf{curl}, \Omega) \times H_0(\mathrm{div}, \Omega) \times L^2_0(\Omega)$ *such that*

$$\begin{aligned} &\forall \boldsymbol{v} \in H_0(\mathrm{div}, \Omega), \quad && a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v}) + b(\boldsymbol{v}, p) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle, \\ &\forall q \in L^2_0(\Omega), && b(\boldsymbol{u}, q) = 0, \\ &\forall \boldsymbol{\vartheta} \in H_0(\mathbf{curl}, \Omega), && c(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{\vartheta}) = 0, \end{aligned} \tag{2.9}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H_0(\mathrm{div}, \Omega)$ and its dual space. The bilinear forms $a(\cdot, \cdot; \cdot)$, $b(\cdot, \cdot)$, and $c(\cdot, \cdot; \cdot)$ are defined by

$$\begin{aligned} a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v}) &= \nu \int_\Omega (\mathbf{curl}\, \boldsymbol{\omega})(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x}, \quad b(\boldsymbol{v}, q) = -\int_\Omega (\mathrm{div}\, \boldsymbol{v})(\boldsymbol{x}) q(\boldsymbol{x})\, d\boldsymbol{x}, \\ c(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{\varphi}) &= \int_\Omega \boldsymbol{\omega}(\boldsymbol{x}) \cdot \boldsymbol{\varphi}(\boldsymbol{x})\, d\boldsymbol{x} - \int_\Omega \boldsymbol{u}(\boldsymbol{x}) \cdot (\mathbf{curl}\, \boldsymbol{\varphi})(\boldsymbol{x})\, d\boldsymbol{x}. \end{aligned} \tag{2.10}$$

It can be noted that the boundary conditions that appear in (2.2) are treated as essential ones in (2.9). So a direct consequence of the density of $\mathcal{D}(\Omega)^d$ in $H_0(\mathrm{div}, \Omega)$ and of $\mathcal{D}(\Omega)^{\frac{d(d-1)}{2}}$ in $H_0(\mathbf{curl}, \Omega)$ (see [22, Chap. I, sect. 2]) is the following statement.

PROPOSITION 2.2. *Problems* (2.2) *and* (2.9) *are equivalent, in the sense that any triple* $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ *in* $H(\mathbf{curl}, \Omega) \times H(\mathrm{div}, \Omega) \times L^2_0(\Omega)$ *is a solution of problem* (2.2) *if and only if it is a solution of problem* (2.9).

We briefly recall from [26] (see also [11, sect. 3]) the main arguments for the analysis of problem (2.9), in view of their discrete analogues. Let V be the kernel

$$V = \big\{ \boldsymbol{v} \in H_0(\mathrm{div}, \Omega);\ \forall q \in L^2_0(\Omega),\ b(\boldsymbol{v}, q) = 0 \big\}. \tag{2.11}$$

Since the divergence of any function in $H_0(\mathrm{div}, \Omega)$ belongs to $L^2_0(\Omega)$, it is readily checked that V coincides with the space of divergence-free functions in $H_0(\mathrm{div}, \Omega)$. We also introduce the kernel

$$\mathcal{W} = \big\{ (\boldsymbol{\vartheta}, \boldsymbol{v}) \in H_0(\mathbf{curl}, \Omega) \times V;\ \forall \boldsymbol{\varphi} \in H_0(\mathbf{curl}, \Omega),\ c(\boldsymbol{\vartheta}, \boldsymbol{v}; \boldsymbol{\varphi}) = 0 \big\}. \tag{2.12}$$

As can easily be derived from the previously quoted density result, $\mathcal{W}$ coincides with the space of pairs $(\boldsymbol{\vartheta}, \boldsymbol{v})$ in $H_0(\mathbf{curl}, \Omega) \times V$ such that $\boldsymbol{\vartheta}$ is equal to $\mathbf{curl}\, \boldsymbol{v}$ in the distribution sense. Moreover, it follows from the continuity properties of the forms $b(\cdot, \cdot)$ and $c(\cdot, \cdot; \cdot)$ that both V and $\mathcal{W}$ are Hilbert spaces.

We observe that, for any solution $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ of problem (2.9), the pair $(\boldsymbol{\omega}, \boldsymbol{u})$ is a solution of the following reduced problem:

Find $(\boldsymbol{\omega}, \boldsymbol{u})$ *in* $\mathcal{W}$ *such that*

$$\forall \boldsymbol{v} \in V, \quad a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle. \tag{2.13}$$

So we first investigate its well-posedness.

LEMMA 2.3. *There exists a positive constant* α *such that the form* $a(\cdot, \cdot; \cdot)$ *satisfies*

$$\begin{aligned} &\forall \boldsymbol{v} \in V \setminus \{0\}, \quad \sup_{(\boldsymbol{\omega}, \boldsymbol{u}) \in \mathcal{W}} a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v}) > 0, \\ &\forall (\boldsymbol{\omega}, \boldsymbol{u}) \in \mathcal{W}, \quad \sup_{\boldsymbol{v} \in V} \frac{a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v})}{\|\boldsymbol{v}\|_{L^2(\Omega)^d}} \geq \alpha \left(\|\boldsymbol{\omega}\|_{H(\mathbf{curl}, \Omega)} + \|\boldsymbol{u}\|_{L^2(\Omega)^d} \right). \end{aligned} \tag{2.14}$$

Proof. It is performed in two steps, only in the case $d = 3$ for brevity.

(1) Let $\boldsymbol{v}$ be a function in V such that $a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v})$ cancels for all $(\boldsymbol{\omega}, \boldsymbol{u})$ in $\mathcal{W}$. Since Ω is simply connected, and $\boldsymbol{v}$ is divergence-free and satisfies $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$, it follows from [3, Thm. 3.17] that there exists a divergence-free function $\boldsymbol{\psi}$ in $H_0(\mathbf{curl}, \Omega)$ such that $\boldsymbol{v}$ is equal to $\mathbf{curl}\,\boldsymbol{\psi}$. Similarly, since Ω has a connected boundary and $\boldsymbol{\psi}$ is divergence-free, it follows from [3, Thm. 3.12] that there exists a function $\boldsymbol{z}$ in V such that $\boldsymbol{\psi}$ is equal to $\mathbf{curl}\,\boldsymbol{z}$. So the pair $(\boldsymbol{\psi}, \boldsymbol{z})$ belongs to $\mathcal{W}$. Taking $(\boldsymbol{\omega}, \boldsymbol{u})$ equal to $(\boldsymbol{\psi}, \boldsymbol{z})$ thus yields that $\boldsymbol{v}$ is zero, whence the first part of (2.14).

(2) For any $(\boldsymbol{\omega}, \boldsymbol{u})$ in $\mathcal{W}$, we observe that the function $\boldsymbol{v} = \mathbf{curl}\,\boldsymbol{\omega} + \boldsymbol{u}$ belongs to V. With this choice, we have

$$a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v}) = \nu\, \|\mathbf{curl}\,\boldsymbol{\omega}\|^2_{L^2(\Omega)^d} + \nu \int_\Omega (\mathbf{curl}\,\boldsymbol{\omega})(\boldsymbol{x}) \cdot \boldsymbol{u}(\boldsymbol{x})\, d\boldsymbol{x}.$$

Since $\boldsymbol{\omega}$ is equal to $\mathbf{curl}\,\boldsymbol{u}$, we obtain by integrating by parts

$$a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v}) = \nu\, \|\mathbf{curl}\,\boldsymbol{\omega}\|^2_{L^2(\Omega)^d} + \frac{\nu}{2}\, \|\boldsymbol{\omega}\|^2_{L^2(\Omega)^{\frac{d(d-1)}{2}}} + \frac{\nu}{2}\, \|\mathbf{curl}\,\boldsymbol{u}\|^2_{L^2(\Omega)^{\frac{d(d-1)}{2}}}.$$

Next, using [3, Cor. 3.16] yields that, since Ω is simply connected,

$$\forall \boldsymbol{w} \in V, \quad \|\boldsymbol{w}\|_{L^2(\Omega)^d} \le c\, \|\mathbf{curl}\,\boldsymbol{w}\|_{L^2(\Omega)^{\frac{d(d-1)}{2}}}. \tag{2.15}$$

Inserting this inequality applied to $\boldsymbol{u}$ into the previous line gives

$$a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v}) \ge \frac{\nu}{2}\, \|\boldsymbol{\omega}\|^2_{H(\mathbf{curl},\Omega)} + \frac{\nu}{2c^2}\, \|\boldsymbol{u}\|^2_{L^2(\Omega)^d}.$$

This, combined with the bound

$$\|\boldsymbol{v}\|_{L^2(\Omega)^d} \le \sqrt{2}\, \big(\|\mathbf{curl}\,\boldsymbol{\omega}\|^2_{L^2(\Omega)^d} + \|\boldsymbol{u}\|^2_{L^2(\Omega)^d}\big)^{\frac{1}{2}},$$

leads to the inf-sup condition in (2.14).

The next result is now a direct consequence of (2.14); see [22, Chap. I, Lem. 4.1].

COROLLARY 2.4. *For any data $\boldsymbol{f}$ in the dual space of $H_0(\mathrm{div}, \Omega)$, problem* (2.13) *has a unique solution $(\boldsymbol{\omega}, \boldsymbol{u})$ in $\mathcal{W}$. Moreover, this solution satisfies:*

$$\|\boldsymbol{\omega}\|_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u}\|_{L^2(\Omega)^d} \le c\, \|\boldsymbol{f}\|_{H_0(\mathrm{div},\Omega)'}. \tag{2.16}$$

We recall the inf-sup condition, which is easily derived by taking $\boldsymbol{v}$ equal to $\mathbf{grad}\,\mu$, where μ is the unique solution in $H^1(\Omega) \cap L^2_0(\Omega)$ of the Laplace equation with data q and zero Neumann boundary condition:

$$\forall q \in L^2_0(\Omega), \qquad \sup_{\boldsymbol{v} \in H_0(\mathrm{div},\Omega)} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{H(\mathrm{div},\Omega)}} \ge \beta\, \|q\|_{L^2(\Omega)}, \tag{2.17}$$

where β is a positive constant. We are now in a position to prove the main result of this section.

THEOREM 2.5. *For any data $\boldsymbol{f}$ in the dual space of $H_0(\mathrm{div}, \Omega)$, problem* (2.9) *has a unique solution $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ in $H_0(\mathbf{curl}, \Omega) \times H_0(\mathrm{div}, \Omega) \times L^2_0(\Omega)$. Moreover, this solution satisfies:*

$$\|\boldsymbol{\omega}\|_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u}\|_{H(\mathrm{div},\Omega)} + \|p\|_{L^2(\Omega)} \le c\, \|\boldsymbol{f}\|_{H_0(\mathrm{div},\Omega)'}. \tag{2.18}$$

Proof. We prove separately the existence and the uniqueness.

(1) With any data $\boldsymbol{f}$ in $H_0(\mathrm{div},\Omega)'$, we associate the unique solution $(\boldsymbol{\omega},\boldsymbol{u})$ of problem (2.13) by applying Corollary 2.4. It follows from the definition of V and $\mathcal{W}$ that the second and third lines in (2.9) are satisfied by this solution. Moreover, since the norms $\|\cdot\|_{L^2(\Omega)^d}$ and $\|\cdot\|_{H(\mathrm{div},\Omega)}$ coincide on V, this solution satisfies the first part of (2.18). On the other hand, the pressure p must now satisfy

$$\forall \boldsymbol{v}\in H_0(\mathrm{div},\Omega),\quad b(\boldsymbol{v},p)=\langle \boldsymbol{f},\boldsymbol{v}\rangle - a(\boldsymbol{\omega},\boldsymbol{u};\boldsymbol{v}).$$

Since the right-hand side of the previous line vanishes for all $\boldsymbol{v}$ in V (see (2.13)), the existence of a solution p of this equation in $L_0^2(\Omega)$, together with the second part of (2.18), is a consequence of condition (2.17); again see [22, Chap. I, Lem. 4.1].

(2) Let $(\boldsymbol{\omega},\boldsymbol{u},p)$ be a solution of (2.9) with data $\boldsymbol{f}$ equal to zero. Then $(\boldsymbol{\omega},\boldsymbol{u})$ is a solution of (2.13) with $\boldsymbol{f}=\boldsymbol{0}$, so that it is zero thanks to Corollary 2.4. Then the pressure p satisfies

$$\forall \boldsymbol{v}\in H_0(\mathrm{div},\Omega),\quad b(\boldsymbol{v},p)=0,$$

so that it is zero due to the inf-sup condition (2.17). This yields the uniqueness of the solution of (2.9).

We refer the reader to [10] for the characterization of the dual space of $H_0(\mathrm{div},\Omega)$ to which the data $\boldsymbol{f}$ must belong. To conclude, we state some regularity properties of the solution of problem (2.9) which can easily be derived from [3, sect. 2], [16], and [17].

PROPOSITION 2.6. *The mapping $\boldsymbol{f}\mapsto(\boldsymbol{\omega},\boldsymbol{u},p)$, where $(\boldsymbol{\omega},\boldsymbol{u},p)$ is the solution of problem* (2.9) *with data $\boldsymbol{f}$, is continuous from $H^{\max\{0,s-1\}}(\Omega)^d$ into $H^s(\Omega)^{\frac{d(d-1)}{2}}\times H^s(\Omega)^d\times H^s(\Omega)$ for*

(i) *all $s\le\frac12$ in the general case,*
(ii) *all $s\le 1$ when Ω is convex,*
(iii) *all $s<\frac{\pi}{\omega}$ in dimension $d=2$ when Ω is a polygon with largest angle equal to ω.*

Moreover, when the data $\boldsymbol{f}$ belongs to $L^2(\Omega)^d$, the pressure p belongs to $H^1(\Omega)$, together with the vorticity $\boldsymbol{\omega}$ in dimension $d=2$.

We refer the reader to [16] and [17] for more details about the previous statement. These properties seem weaker than the corresponding ones for the Stokes problem with Dirichlet boundary conditions on the velocity. But they are the appropriate ones for proving the convergence of the discretization.

3. The spectral discrete problem. From now on, we assume that Ω is the square or cube $]-1,1[^d$, $d=2$ or 3. The discrete spaces are constructed from the finite elements proposed by Nédélec on cubic three-dimensional meshes; see [24, sect. 2]. In order to describe them and for any triple (ℓ,m,n) of nonnegative integers, we introduce

- in dimension $d=2$, the space $\mathbb{P}_{\ell,m}(\Omega)$ of restrictions to Ω of polynomials with degree $\le\ell$ with respect to x and $\le m$ with respect to y,
- in dimension $d=3$, the space $\mathbb{P}_{\ell,m,n}(\Omega)$ of restrictions to Ω of polynomials with degree $\le\ell$ with respect to x, $\le m$ with respect to y, and $\le n$ with respect to z.

When ℓ and m are equal to n, these spaces are simply denoted by $\mathbb{P}_n(\Omega)$.

Let N be an integer ≥ 2. The space $\mathbb{D}_N$ which approximates $H_0(\mathrm{div},\Omega)$ is defined

by

(3.1)
$$\mathbb{D}_N = H_0(\mathrm{div}, \Omega) \cap \begin{cases} \mathbb{P}_{N,N-1}(\Omega) \times \mathbb{P}_{N-1,N}(\Omega) & \text{if } d = 2, \\ \mathbb{P}_{N,N-1,N-1}(\Omega) \times \mathbb{P}_{N-1,N,N-1}(\Omega) \times \mathbb{P}_{N-1,N-1,N}(\Omega) & \\ & \text{if } d = 3. \end{cases}$$

In contrast, the space $\mathbb{C}_N$ which approximates $H_0(\mathbf{curl}, \Omega)$ is rather different according to the dimension, for the reasons explained in Remark 2.1; it is defined by

(3.2)
$$\mathbb{C}_N = \begin{cases} H_0^1(\Omega) \cap \mathbb{P}_N(\Omega) & \text{if } d = 2, \\ H_0(\mathbf{curl}, \Omega) \cap \big(\mathbb{P}_{N-1,N,N}(\Omega) \times \mathbb{P}_{N,N-1,N}(\Omega) \times \mathbb{P}_{N,N,N-1}(\Omega)\big) & \text{if } d = 3. \end{cases}$$

Finally, for the approximation of $L_0^2(\Omega)$, we consider the space $\mathbb{M}_N$:

$$\mathbb{M}_N = L_0^2(\Omega) \cap \mathbb{P}_{N-1}(\Omega). \tag{3.3}$$

Setting $\xi_0 = -1$ and $\xi_N = 1$, we introduce the $N-1$ nodes ξ_j, $1 \le j \le N-1$, and the $N+1$ weights ρ_j, $0 \le j \le N$, of the Gauss–Lobatto quadrature formula. Denoting by $\mathbb{P}_n(-1,1)$ the space of restrictions to $[-1,1]$ of polynomials with degree $\le n$, we recall that the following equality holds:

$$\forall \Phi \in \mathbb{P}_{2N-1}(-1,1), \quad \int_{-1}^{1} \Phi(\zeta)\, d\zeta = \sum_{j=0}^{N} \Phi(\xi_j)\, \rho_j. \tag{3.4}$$

We also recall [13, form. (13.20)] the property, which is useful in what follows,

$$\forall \varphi_N \in \mathbb{P}_N(-1,1), \quad \|\varphi_N\|^2_{L^2(-1,1)} \le \sum_{j=0}^{N} \varphi_N^2(\xi_j)\, \rho_j \le 3\, \|\varphi_N\|^2_{L^2(-1,1)}. \tag{3.5}$$

Relying on this formula, we introduce the discrete product, defined on continuous functions u and v by

$$(u, v)_N = \begin{cases} \sum_{i=0}^{N} \sum_{j=0}^{N} u(\xi_i, \xi_j) v(\xi_i, \xi_j)\, \rho_i \rho_j & \text{if } d = 2, \\ \sum_{i=0}^{N} \sum_{j=0}^{N} \sum_{k=0}^{N} u(\xi_i, \xi_j, \xi_k) v(\xi_i, \xi_j, \xi_k)\, \rho_i \rho_j \rho_k & \text{if } d = 3. \end{cases} \tag{3.6}$$

It follows from (3.5) that it is a scalar product on $\mathbb{P}_N(\Omega)$. Finally, let $\mathcal{I}_N$ denote the Lagrange interpolation operator at the nodes (ξ_i, ξ_j), $0 \le i, j \le N$, in dimension $d = 2$ and at the nodes (ξ_i, ξ_j, ξ_k), $0 \le i, j, k \le N$, in dimension $d = 3$, with values in $\mathbb{P}_N(\Omega)$.

From now on, we assume that the data $\boldsymbol{f}$ are continuous on $\overline{\Omega}$. The discrete problem is constructed from (2.9) by using the Galerkin method combined with numerical integration. It reads as follows:

Find $(\boldsymbol{\omega}_N, \boldsymbol{u}_N, p_N)$ *in* $\mathbb{C}_N \times \mathbb{D}_N \times \mathbb{M}_N$ *such that*

$$\begin{aligned} &\forall \boldsymbol{v}_N \in \mathbb{D}_N, \quad a_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N) + b_N(\boldsymbol{v}_N, p_N) = (\boldsymbol{f}, \boldsymbol{v}_N)_N, \\ &\forall q_N \in \mathbb{M}_N, \quad b_N(\boldsymbol{u}_N, q_N) = 0, \\ &\forall \boldsymbol{\vartheta}_N \in \mathbb{C}_N, \quad c_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{\vartheta}_N) = 0, \end{aligned} \tag{3.7}$$

where the bilinear forms $a_N(\cdot,\cdot;\cdot)$, $b_N(\cdot,\cdot)$, and $c_N(\cdot,\cdot;\cdot)$ are defined by

$$\begin{aligned}&a_N(\boldsymbol{\omega}_N,\boldsymbol{u}_N;\boldsymbol{v}_N)=\nu\,(\mathbf{curl}\,\boldsymbol{\omega}_N,\boldsymbol{v}_N)_N,\quad b_N(\boldsymbol{v}_N,q_N)=-(\operatorname{div}\boldsymbol{v}_N,q_N)_N,\\&c_N(\boldsymbol{\omega}_N,\boldsymbol{u}_N;\boldsymbol{\varphi}_N)=(\boldsymbol{\omega}_N,\boldsymbol{\varphi}_N)_N-(\boldsymbol{u}_N,\mathbf{curl}\,\boldsymbol{\varphi}_N)_N.\end{aligned}\tag{3.8}$$

It follows from (3.5) combined with Cauchy–Schwarz inequalities that the forms $a_N(\cdot,\cdot;\cdot)$, $b_N(\cdot,\cdot)$, and $c_N(\cdot,\cdot;\cdot)$ are continuous on $\big(\mathbb{C}_N\times\mathbb{D}_N\big)\times\mathbb{D}_N$, $\mathbb{D}_N\times\mathbb{M}_N$, and $\big(\mathbb{C}_N\times\mathbb{D}_N\big)\times\mathbb{C}_N$, respectively, with norms bounded independently of N. Moreover, as a consequence of the exactness property (3.4), the forms $b(\cdot,\cdot)$ and $b_N(\cdot,\cdot)$ coincide on $\mathbb{D}_N\times\mathbb{M}_N$.

The somewhat complex choice of the discrete spaces is justified by the following lemma (its finite element analogue is well known; see [24]).

LEMMA 3.1. *The range of $\mathbb{D}_N$ by the divergence operator is contained in $\mathbb{M}_N$. The range of $\mathbb{C}_N$ by the curl operator is contained in $\mathbb{D}_N$.*

Proof. For any $\boldsymbol{v}_N$ in $\mathbb{D}_N$, $\operatorname{div}\boldsymbol{v}_N$ belongs to $\mathbb{P}_{N-1}(\Omega)$ and the fact that it has a zero integral is derived from the property $\boldsymbol{v}_N\cdot\boldsymbol{n}=0$ on $\partial\Omega$ together with the Stokes formula. This yields the first part of the lemma. Similarly, for any $\boldsymbol{\vartheta}_N$ in $\mathbb{C}_N$, each component of $\mathbf{curl}\,\boldsymbol{\vartheta}_N$ is a polynomial of the right degree and the boundary conditions $\gamma_t(\boldsymbol{\vartheta}_N)=0$ on $\partial\Omega$ imply that $\mathbf{curl}\,\boldsymbol{\vartheta}_N\cdot\boldsymbol{n}$ vanishes on $\partial\Omega$, which concludes the proof.

In analogy with the continuous case, in order to investigate the properties of problem (3.7), we introduce the kernel

$$V_N=\big\{\boldsymbol{v}_N\in\mathbb{D}_N;\ \forall q_N\in\mathbb{M}_N,\ b_N(\boldsymbol{v}_N,q_N)=0\big\}.\tag{3.9}$$

The following result is easily derived from Lemma 3.1 by taking q_N equal to $\operatorname{div}\boldsymbol{v}_N$ in the previous line.

COROLLARY 3.2. *The kernel V_N is the space of divergence-free polynomials in $\mathbb{D}_N$; i.e., it coincides with $\mathbb{D}_N\cap V$.*

We now introduce the kernel

$$\mathcal{W}_N=\big\{(\boldsymbol{\vartheta}_N,\boldsymbol{v}_N)\in\mathbb{C}_N\times V_N;\ \forall\boldsymbol{\varphi}_N\in\mathbb{C}_N,\ c_N(\boldsymbol{\vartheta}_N,\boldsymbol{v}_N;\boldsymbol{\varphi}_N)=0\big\},\tag{3.10}$$

and we consider the following reduced discrete problem:

Find $(\boldsymbol{\omega}_N,\boldsymbol{u}_N)$ in $\mathcal{W}_N$ such that

$$\forall\boldsymbol{v}_N\in V_N,\quad a_N(\boldsymbol{\omega}_N,\boldsymbol{u}_N;\boldsymbol{v}_N)=(\boldsymbol{f},\boldsymbol{v}_N)_N.\tag{3.11}$$

We must now establish the analogues of (2.14) for the form $a_N(\cdot,\cdot;\cdot)$ on the discrete spaces. This requires several preliminary lemmas. We refer the reader to among others [15, Thm. 2.1] and [23] for analogous results in the finite element case.

LEMMA 3.3. *The kernel of the curl operator in $\mathbb{C}_N$ is reduced to $\{0\}$ in dimension $d=2$ and equal to the range of $H^1_0(\Omega)\cap\mathbb{P}_N(\Omega)$ by the gradient operator in dimension $d=3$.*

Proof. Since the lemma is obvious in dimension $d=2$, we prove it only in dimension $d=3$. Let $\boldsymbol{\vartheta}_N$ be a curl-free polynomial in $\mathbb{C}_N$. Using [22, Chap. I, Thm. 2.9] yields that it is the gradient of a function μ. It follows from the identity $\boldsymbol{\vartheta}_N=\mathbf{grad}\,\mu$ that μ belongs to $\mathbb{P}_N(\Omega)$. Moreover, the two tangential derivatives of μ vanish on all faces of Ω: Indeed, for instance on the faces contained in the planes $x=\pm1$, the second and third components of $\boldsymbol{\vartheta}_N$ are equal to zero thanks to the definition of $\mathbb{C}_N$, so that $\partial_y\mu$ and $\partial_z\mu$ vanish. So μ is constant on $\partial\Omega$ and, since it is

defined up to an additive constant, it can be taken equal to zero on $\partial\Omega$. The converse imbedding is readily checked.

The next lemma makes the second part of Lemma 3.1 more precise.

LEMMA 3.4. *The range of $\mathbb{C}_N$ by the curl operator is equal to V_N.*

Proof. Let $\boldsymbol{v}_N$ be any polynomial in V_N. We treat only the more complex case of dimension $d = 3$. Denoting the components of $\boldsymbol{v}_N$ by v_{Nx}, v_{Ny}, and v_{Nz}, we first define a function $\boldsymbol{\psi}_N = (\psi_{Nx}, \psi_{Ny}, \psi_{Nz})$ by

(3.12)
$$\psi_{Nx}(x,y,z) = \int_{-1}^{z} v_{Ny}(x,y,\zeta)\,d\zeta,$$
$$\psi_{Ny}(x,y,z) = -\int_{-1}^{z} v_{Nx}(x,y,\zeta)\,d\zeta, \quad \psi_{Nz} = 0.$$

The first two components of $\boldsymbol{\psi}_N$ belong to $\mathbb{P}_{N-1,N,N}(\Omega)$ and $\mathbb{P}_{N,N-1,N}(\Omega)$, respectively. This function is such that the first two components of its curl are equal to v_{Nx} and v_{Ny}. Moreover, since $\boldsymbol{v}_N$ belongs to V_N. we have

$$(\partial_x\psi_{Ny} - \partial_y\psi_{Nx})(x,y,z) = -\int_{-1}^{z} (\partial_x v_{Nx} + \partial_y v_{Ny})(x,y,\zeta)\,d\zeta$$
$$= \int_{-1}^{z} (\partial_z v_{Nz})(x,y,\zeta)\,d\zeta = v_{Nz}(x,y,z).$$

So $\mathbf{curl}\,\boldsymbol{\psi}_N$ is equal to $\boldsymbol{v}_N$. Moreover, it is readily checked that $\gamma_t(\boldsymbol{\psi}_N)$ vanishes on all faces of Ω but on the face Γ is contained in the plane $z = 1$. In a second step, we look for a function μ_N in $\mathbb{P}_N(\Omega)$ such that $\gamma_t(\mathbf{grad}\,\mu_N)$ is equal to zero on $\partial\Omega\setminus\Gamma$ and to $\gamma_t(\boldsymbol{\psi}_N)$ on Γ. Denoting by g_{Nx} and g_{Ny} the functions defined on Γ by

$$g_{Nx}(x,y) = \int_{-1}^{1} v_{Ny}(x,y,\zeta)\,d\zeta, \quad g_{Ny}(x,y) = -\int_{-1}^{1} v_{Nx}(x,y,\zeta)\,d\zeta,$$

we observe that the function $\boldsymbol{g}_N = (g_{Nx}, g_{Ny})$ belongs to $\mathbb{P}_{N-1,N}(\Gamma) \times \mathbb{P}_{N,N-1}(\Gamma)$, with obvious notation for these new spaces, has its tangential component equal to zero on the four edges of Γ, and satisfies, for the same reasons as previously,

$$(\partial_x g_{Ny} - \partial_y g_{Nx})(x,y) = \int_{-1}^{1} (\partial_z v_{Nz})(x,y,\zeta)\,d\zeta = 0.$$

Again applying [22, Chap. I, Thm. 2.9] yields that $\boldsymbol{g}_N$ is the tangential gradient of a function k_N^g, which is defined up to an additive constant. When choosing this constant such that $k_N^g(-1,-1)$ is zero, we easily derive that k_N^g belongs to $H_0^1(\Gamma) \cap \mathbb{P}_N(\Gamma)$. Then, using an appropriate lifting operator of polynomial traces as proposed in [9, Chap. II, Thm. 4.1], we derive the existence of a μ_N in $\mathbb{P}_N(\Omega)$ equal to 0 on $\partial\Omega\setminus\Gamma$ and to k_N^g on Γ. The function $\boldsymbol{\psi}_N - \mathbf{grad}\,\mu_N$ now belongs to $\mathbb{C}_N$ and has its curl equal to $\boldsymbol{v}_N$, whence the desired result.

It follows from Lemmas 3.3 and 3.4 that, for any function $\boldsymbol{v}_N$ in V_N, there exists a unique function $\boldsymbol{\psi}_N^*$ in $\mathbb{C}_N$ such that $\mathbf{curl}\,\boldsymbol{\psi}_N^*$ is equal to $\boldsymbol{v}_N$ and which, moreover, satisfies in dimension $d = 3$

$$\forall \mu_N \in H_0^1(\Omega) \cap \mathbb{P}_N(\Omega), \quad (\boldsymbol{\psi}_N^*, \mathbf{grad}\,\mu_N)_N = 0. \tag{3.13}$$

Let A_N be the operator defined from V_N into $\mathbb{C}_N$ by $A_N(\boldsymbol{v}_N) = \boldsymbol{\psi}_N^*$.

LEMMA 3.5. *There exists a constant c independent of N such that the following inequality holds:*

$$\forall \boldsymbol{v}_N \in V_N, \quad \|A_N(\boldsymbol{v}_N)\|_{H(\mathbf{curl},\Omega)} \le c\,\|\boldsymbol{v}_N\|_{L^2(\Omega)^d}. \tag{3.14}$$

Proof. Since $\mathbf{curl}\,A_N(\boldsymbol{v}_N)$ is equal to $\boldsymbol{v}_N$, it suffices to bound $\|A_N(\boldsymbol{v}_N)\|_{L^2(\Omega)^{\frac{d(d-1)}{2}}}$. There also, we consider only the case $d = 3$. The function $\boldsymbol{\psi}_N$ defined in (3.12) obviously satisfies

$$\|\boldsymbol{\psi}_N\|_{L^2(\Omega)^3} \le 2\,\|\boldsymbol{v}_N\|_{L^2(\Omega)^3}. \tag{3.15}$$

For the same reason, the function k_N^g introduced in the proof of Lemma 3.4 satisfies, thanks to the Poincaré–Friedrichs inequality,

$$\|k_N^g\|_{H^1(\Gamma)} \le c\,\|\boldsymbol{g}_N\|_{L^2(\Gamma)^2} \le c\sqrt{2}\|\boldsymbol{v}_N\|_{L^2(\Omega)^3}.$$

Thus applying [9, Chap. II, Thm. 4.1] leads to the estimate

$$\|\mathbf{grad}\,\mu_N\|_{L^2(\Omega)^3} \le c\,\|k_N^g\|_{H^1(\Gamma)},$$

whence

$$\|\mathbf{grad}\,\mu_N\|_{L^2(\Omega)^3} \le c'\,\|\boldsymbol{v}_N\|_{L^2(\Omega)^3}. \tag{3.16}$$

Finally, the Lax–Milgram lemma combined with (3.5) and the Poincaré–Friedrichs inequality yields that there exists a unique $\tilde{\mu}_N$ in $H_0^1(\Omega) \cap \mathbb{P}_N(\Omega)$ such that

$$\forall \rho_N \in H_0^1(\Omega) \cap \mathbb{P}_N(\Omega), \quad (\mathbf{grad}\,\tilde{\mu}_N, \mathbf{grad}\,\rho_N)_N = (\boldsymbol{\psi}_N - \mathbf{grad}\,\mu_N, \mathbf{grad}\,\rho_N)_N.$$

Moreover, this function satisfies

$$\|\mathbf{grad}\,\tilde{\mu}_N\|_{L^2(\Omega)^3} \le 3^{\frac{3}{2}}\,(\|\boldsymbol{\psi}_N\|_{L^2(\Omega)^3} + \|\mathbf{grad}\,\mu_N\|_{L^2(\Omega)^3}). \tag{3.17}$$

The choice of $\tilde{\mu}_N$ yields that the function $\boldsymbol{\psi}_N - \mathbf{grad}\,\mu_N - \mathbf{grad}\,\tilde{\mu}_N$ is equal to $A_N(\boldsymbol{v}_N)$, so that the desired estimate follows from (3.15) to (3.17).

We are now in a position to prove successively the two analogues of (2.14).

LEMMA 3.6. *The form $a_N(\cdot,\cdot;\cdot)$ satisfies the following positivity property:*

$$\forall \boldsymbol{v}_N \in V_N \setminus \{0\}, \quad \sup_{(\boldsymbol{\omega}_N, \boldsymbol{u}_N) \in \mathcal{W}_N} a_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N) > 0. \tag{3.18}$$

Proof. Let $\boldsymbol{v}_N$ be a polynomial in V_N such that $a_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N)$ vanishes for all pairs $(\boldsymbol{\omega}_N, \boldsymbol{u}_N)$ in $\mathcal{W}_N$. We set $\boldsymbol{\vartheta}_N = A_N(\boldsymbol{v}_N)$ and consider the equation:

Find $\boldsymbol{z}_N$ in V_N such that

$$\forall \boldsymbol{w}_N \in V_N, \quad (\boldsymbol{z}_N, \boldsymbol{w}_N)_N = (\boldsymbol{\vartheta}_N, A_N(\boldsymbol{w}_N))_N. \tag{3.19}$$

Since the norms $\|\cdot\|_{H(\mathrm{div},\Omega)}$ and $\|\cdot\|_{L^2(\Omega)^3}$ are equal on V_N, it follows from (3.5) that the bilinear form in the left-hand side is elliptic on V_N, so that this problem has a unique solution $\boldsymbol{z}_N$. Moreover, this function satisfies for any $\boldsymbol{\varphi}_N$ in $\mathbb{C}_N$

$$(\boldsymbol{z}_N, \mathbf{curl}\,\boldsymbol{\varphi}_N)_N = \big(\boldsymbol{\vartheta}_N, A_N(\mathbf{curl}\,\boldsymbol{\varphi}_N)\big)_N.$$

Note that $A_N(\mathbf{curl}\,\boldsymbol{\varphi}_N)$ is the sum of $\boldsymbol{\varphi}_N$ and of the gradient of a function μ_N in $H_0^1(\Omega) \cap \mathbb{P}_N(\Omega)$. Then it follows from the choice of $\boldsymbol{\vartheta}_N$ (see (3.13)) that

$$(\boldsymbol{z}_N, \mathbf{curl}\,\boldsymbol{\varphi}_N)_N = (\boldsymbol{\vartheta}_N, \boldsymbol{\varphi}_N)_N.$$

So the pair $(\boldsymbol{\vartheta}_N, \boldsymbol{z}_N)$ belongs to $\mathcal{W}_N$ and taking $(\boldsymbol{\omega}_N, \boldsymbol{u}_N)$ equal to $(\boldsymbol{\vartheta}_N, \boldsymbol{z}_N)$ yields, thanks to (3.5), that $\boldsymbol{v}_N = \mathbf{curl}\,\boldsymbol{\vartheta}_N$ is zero, which concludes the proof.

LEMMA 3.7. *There exists a positive constant α_* independent of N such that the form $a_N(\cdot,\cdot;\cdot)$ satisfies the following inf-sup condition:*

$$\forall (\boldsymbol{\omega}_N, \boldsymbol{u}_N) \in \mathcal{W}_N, \quad \sup_{\boldsymbol{v}_N \in V_N} \frac{a_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N)}{\|\boldsymbol{v}_N\|_{L^2(\Omega)^d}} \geq \alpha_* \big(\|\boldsymbol{\omega}_N\|_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u}_N\|_{L^2(\Omega)^d}\big). \tag{3.20}$$

Proof. For any $(\boldsymbol{\omega}_N, \boldsymbol{u}_N)$ in $\mathcal{W}_N$, we set $\boldsymbol{v}_N = \boldsymbol{u}_N + \mathbf{curl}\,\boldsymbol{\omega}_N$. Thanks to the definition of $\mathcal{W}_N$, this gives

$$a_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N) \geq \nu \, \|\boldsymbol{\omega}_N\|^2_{H(\mathbf{curl},\Omega)}.$$

On the other hand, again using the definition of $\mathcal{W}_N$ and (3.5), we write

$$\begin{aligned}\|\boldsymbol{u}_N\|^2_{L^2(\Omega)^d} \leq \big(\boldsymbol{u}_N, \mathbf{curl}\, A_N(\boldsymbol{u}_N)\big)_N &= \big(\boldsymbol{\omega}_N, A_N(\boldsymbol{u}_N)\big)_N \\ &\leq 3^d \, \|\boldsymbol{\omega}_N\|_{L^2(\Omega)^{\frac{d(d-1)}{2}}} \, \|A_N(\boldsymbol{u}_N)\|_{L^2(\Omega)^{\frac{d(d-1)}{2}}}.\end{aligned}$$

So by using Lemma 3.5 we obtain

$$\|\boldsymbol{u}_N\|_{L^2(\Omega)^d} \leq 3^d \, c \, \|\boldsymbol{\omega}_N\|_{L^2(\Omega)^{\frac{d(d-1)}{2}}}.$$

By combining these two inequalities and noting that

$$\|\boldsymbol{v}_N\|_{L^2(\Omega)^d} \leq \sqrt{2}\, \big(\|\boldsymbol{\omega}_N\|^2_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u}_N\|^2_{L^2(\Omega)^d}\big)^{\frac{1}{2}},$$

we derive the desired inf-sup condition.

The following result is a direct consequence of Lemmas 3.6 and 3.7; see [22, Chap. I, Lem. 4.1]. Note also from (3.5) that if $\mathcal{I}_N$ denotes the Lagrange interpolation operator introduced at the beginning of this section, the following property holds for any $\boldsymbol{v}_N$ in $\mathbb{D}_N$ (note that this requires the continuity of $\boldsymbol{f}$):

$$(\boldsymbol{f}, \boldsymbol{v}_N)_N = (\mathcal{I}_N \boldsymbol{f}, \boldsymbol{v}_N)_N \leq 3^d \, \|\mathcal{I}_N \boldsymbol{f}\|_{L^2(\Omega)^d} \|\boldsymbol{v}_N\|_{L^2(\Omega)^d}.$$

COROLLARY 3.8. *For any data $\boldsymbol{f}$ continuous on $\overline{\Omega}$, problem* (3.11) *has a unique solution $(\boldsymbol{\omega}_N, \boldsymbol{u}_N)$ in $\mathcal{W}_N$. Moreover, this solution satisfies for a constant c independent of N:*

$$\|\boldsymbol{\omega}_N\|_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u}_N\|_{L^2(\Omega)^d} \leq c \, \|\mathcal{I}_N \boldsymbol{f}\|_{L^2(\Omega)^d}. \tag{3.21}$$

To go further, we now state an inf-sup condition on the form $b_N(\cdot,\cdot)$. We refer the reader to [4, Lem. 3.1] for the main arguments of the proof in a slightly different case (and to [5] and [13, Thm. 24.6] for the basic ideas).

LEMMA 3.9. *There exists a positive constant β_* independent of N such that the form $b_N(\cdot,\cdot)$ satisfies the following inf-sup condition:*

$$\forall q_N \in \mathbb{M}_N, \quad \sup_{\boldsymbol{v}_N \in \mathbb{D}_N} \frac{b_N(\boldsymbol{v}_N, q_N)}{\|\boldsymbol{v}_N\|_{H(\mathrm{div},\Omega)}} \geq \beta_* \|q_N\|_{L^2(\Omega)}. \tag{3.22}$$

Note that Lemma 3.9 makes the first part of Lemma 3.1 more precise: Indeed, it implies that the range of $\mathbb{D}_N$ by the divergence operator is equal to $\mathbb{M}_N$. We skip the proof of the next theorem since it relies on exactly the same arguments as Theorem 2.5 with Corollary 2.4 replaced by Corollary 3.8 and (2.17) replaced by (3.22).

THEOREM 3.10. *For any data $\boldsymbol{f}$ continuous on $\overline{\Omega}$, problem* (3.7) *has a unique solution $(\boldsymbol{\omega}_N, \boldsymbol{u}_N, p_N)$ in $\mathbb{C}_N \times \mathbb{D}_N \times \mathbb{M}_N$. Moreover, this solution satisfies for a constant c independent of N:*

$$\|\boldsymbol{\omega}_N\|_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u}_N\|_{H(\mathrm{div},\Omega)} + \|p_N\|_{L^2(\Omega)} \leq c\, \|\mathcal{I}_N \boldsymbol{f}\|_{L^2(\Omega)^d}. \tag{3.23}$$

4. Error estimates. We now wish to derive the error estimates between the solution $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ of problem (2.9) and the solution $(\boldsymbol{\omega}_N, \boldsymbol{u}_N, p_N)$ of problem (3.7). The proof is rather technical and requires several lemmas. In all that follows, c stands for a generic constant which can vary from one line to the next one but is always independent of N.

LEMMA 4.1. *The following estimate holds for the error between the solution $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ of problem* (2.9) *and the solution $(\boldsymbol{\omega}_N, \boldsymbol{u}_N, p_N)$ of problem* (3.7)*:*

$$\begin{aligned} \|\boldsymbol{\omega} - \boldsymbol{\omega}_N\|_{H(\mathbf{curl},\Omega)} &+ \|\boldsymbol{u} - \boldsymbol{u}_N\|_{H(\mathrm{div},\Omega)} \\ &\leq c \inf_{(\boldsymbol{\vartheta}_N, \boldsymbol{w}_N) \in \mathcal{W}_N} \Big(\|\boldsymbol{\omega} - \boldsymbol{\vartheta}_N\|_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u} - \boldsymbol{w}_N\|_{L^2(\Omega)^d} \\ &\qquad + E_N^{\boldsymbol{f}} + E_N^a(\boldsymbol{\vartheta}_N, \boldsymbol{w}_N)\Big), \end{aligned} \tag{4.1}$$

where the quantities $E_N^{\boldsymbol{f}}$ and $E_N^a(\boldsymbol{\vartheta}_N, \boldsymbol{w}_N)$ are defined by

$$\begin{aligned} E_N^{\boldsymbol{f}} &= \sup_{\boldsymbol{v}_N \in \mathbb{D}_N} \frac{\langle \boldsymbol{f}, \boldsymbol{v}_N\rangle - (\boldsymbol{f}, \boldsymbol{v}_N)_N}{\|\boldsymbol{v}_N\|_{L^2(\Omega)^d}}, \\ E_N^a(\boldsymbol{\vartheta}_N, \boldsymbol{w}_N) &= \sup_{\boldsymbol{v}_N \in \mathbb{D}_N} \frac{(a - a_N)(\boldsymbol{\vartheta}_N, \boldsymbol{w}_N; \boldsymbol{v}_N)}{\|\boldsymbol{v}_N\|_{L^2(\Omega)^d}}. \end{aligned} \tag{4.2}$$

Proof. Let $(\boldsymbol{\vartheta}_N, \boldsymbol{w}_N)$ be an approximation of $(\boldsymbol{\omega}, \boldsymbol{u})$ in $\mathcal{W}_N$. By using (3.11), we have, for all $\boldsymbol{v}_N$ in V_N,

$$a_N(\boldsymbol{\omega}_N - \boldsymbol{\vartheta}_N, \boldsymbol{u}_N - \boldsymbol{w}_N; \boldsymbol{v}_N) = (\boldsymbol{f}, \boldsymbol{v}_N)_N - a_N(\boldsymbol{\vartheta}_N, \boldsymbol{w}_N; \boldsymbol{v}_N).$$

Then using problem (2.13) (we recall that V_N is contained in V) leads to

$$\begin{aligned} a_N(\boldsymbol{\omega}_N - \boldsymbol{\vartheta}_N, \boldsymbol{u}_N - \boldsymbol{w}_N; \boldsymbol{v}_N) = (\boldsymbol{f}, \boldsymbol{v}_N)_N - \langle \boldsymbol{f}, \boldsymbol{v}_N\rangle &+ a(\boldsymbol{\omega} - \boldsymbol{\vartheta}_N, \boldsymbol{u} - \boldsymbol{w}_N; \boldsymbol{v}_N) \\ &+ (a - a_N)(\boldsymbol{\vartheta}_N, \boldsymbol{w}_N; \boldsymbol{v}_N). \end{aligned}$$

By combining this identity with the inf-sup condition (3.20), we derive

$$\begin{aligned} \|\boldsymbol{\omega}_N - \boldsymbol{\vartheta}_N\|_{H(\mathbf{curl},\Omega)} &+ \|\boldsymbol{u}_N - \boldsymbol{w}_N\|_{L^2(\Omega)^d} \\ &\leq c\,\big(\|\mathbf{curl}\,(\boldsymbol{\omega} - \boldsymbol{\vartheta}_N)\|_{L^2(\Omega)^d} + E_N^{\boldsymbol{f}} + E_N^a(\boldsymbol{\vartheta}_N, \boldsymbol{w}_N)\big). \end{aligned}$$

We conclude thanks to a triangle inequality, by noting that both $\boldsymbol{u}$ and $\boldsymbol{u}_N$ are exactly divergence-free.

LEMMA 4.2. *The following estimate holds for the error between the solution* $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ *of problem* (2.9) *and the solution* $(\boldsymbol{\omega}_N, \boldsymbol{u}_N, p_N)$ *of problem* (3.7)*:*

(4.3)
$$\begin{aligned}\|p - p_N\|_{L^2(\Omega)} \le\ & c \inf_{q_N \in \mathbb{M}_N} \|p - q_N\|_{L^2(\Omega)} \\ & + c \inf_{(\boldsymbol{\vartheta}_N, \boldsymbol{w}_N) \in \mathcal{W}_N} \Big(\|\boldsymbol{\omega} - \boldsymbol{\vartheta}_N\|_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u} - \boldsymbol{w}_N\|_{L^2(\Omega)^d} \\ & \qquad + E_N^{\boldsymbol{f}} + E_N^a(\boldsymbol{\vartheta}_N, \boldsymbol{w}_N) \Big),\end{aligned}$$

where the quantities $E_N^{\boldsymbol{f}}$ *and* $E_N^a(\boldsymbol{\vartheta}_N, \boldsymbol{w}_N)$ *are defined in* (4.2).

Proof. It follows from problems (2.9) and (3.7) (note also that $b(\cdot,\cdot)$ and $b_N(\cdot,\cdot)$ coincide on $\mathbb{D}_N \times \mathbb{M}_N$) that, for any $\boldsymbol{v}_N$ in $\mathbb{D}_N$ and q_N in $\mathbb{M}_N$,

$$\begin{aligned}b_N(\boldsymbol{v}_N, p_N - q_N) = (\boldsymbol{f}, \boldsymbol{v}_N)_N - \langle \boldsymbol{f}, \boldsymbol{v}_N \rangle + a(\boldsymbol{\omega} - \boldsymbol{\omega}_N, \boldsymbol{u} - \boldsymbol{u}_N; \boldsymbol{v}_N) \\ + (a - a_N)(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N) + b(\boldsymbol{v}_N, p - q_N).\end{aligned}$$

Moreover, we use the identity

$$(a - a_N)(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N) = (a - a_N)(\boldsymbol{\vartheta}_N, \boldsymbol{w}_N; \boldsymbol{v}_N) + (a - a_N)(\boldsymbol{\omega}_N - \boldsymbol{\vartheta}_N, \boldsymbol{u}_N - \boldsymbol{w}_N; \boldsymbol{v}_N).$$

Combining the inf-sup condition (3.22) with Lemma 4.1 and a triangle inequality leads to (4.3).

In order to evaluate the distance from $(\boldsymbol{\omega}, \boldsymbol{u})$ to $\mathcal{W}_N$, we now prove an inf-sup condition on the form $c_N(\cdot,\cdot;\cdot)$.

LEMMA 4.3. *There exists a positive constant* γ_* *independent of* N *such that the form* $c_N(\cdot,\cdot;\cdot)$ *satisfies the following inf-sup condition:*

(4.4)
$$\forall \boldsymbol{\varphi}_N \in \mathbb{C}_N, \qquad \sup_{(\boldsymbol{\omega}_N, \boldsymbol{u}_N) \in \mathbb{C}_N \times V_N} \frac{c_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{\varphi}_N)}{\|\boldsymbol{\omega}_N\|_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u}_N\|_{L^2(\Omega)^d}} \ge \gamma_* \, \|\boldsymbol{\varphi}_N\|_{H(\mathbf{curl},\Omega)}.$$

Proof. For any $\boldsymbol{\varphi}_N$ in $\mathbb{C}_N$, we take $(\boldsymbol{\omega}_N, \boldsymbol{u}_N)$ equal to $(\boldsymbol{\varphi}_N, -\mathbf{curl}\,\boldsymbol{\varphi}_N)$ and note that it belongs to $\mathbb{C}_N \times V_N$; see Lemma 3.1. Next, we derive from (3.5) that

$$c_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{\varphi}_N) = (\boldsymbol{\varphi}_N, \boldsymbol{\varphi}_N)_N + (\mathbf{curl}\,\boldsymbol{\varphi}_N, \mathbf{curl}\,\boldsymbol{\varphi}_N)_N \ge \|\boldsymbol{\varphi}_N\|^2_{H(\mathbf{curl},\Omega)}.$$

On the other hand, we have

$$\|\boldsymbol{\omega}_N\|_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u}_N\|_{L^2(\Omega)^d} \le 2 \, \|\boldsymbol{\varphi}_N\|_{H(\mathbf{curl},\Omega)},$$

which leads to the desired inf-sup condition.

COROLLARY 4.4. *The following estimate holds:*

(4.5)
$$\begin{aligned}&\inf_{(\boldsymbol{\vartheta}_N, \boldsymbol{w}_N) \in \mathcal{W}_N} \Big(\|\boldsymbol{\omega} - \boldsymbol{\vartheta}_N\|_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u} - \boldsymbol{w}_N\|_{L^2(\Omega)^d} \Big) \\ &\qquad \le c \inf_{\boldsymbol{\zeta}_N \in \mathbb{C}_N} \inf_{\boldsymbol{z}_N \in V_N} \Big(\|\boldsymbol{\omega} - \boldsymbol{\zeta}_N\|_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u} - \boldsymbol{z}_N\|_{L^2(\Omega)^d} + E_N^c(\boldsymbol{\zeta}_N, \boldsymbol{z}_N) \Big),\end{aligned}$$

where the quantity $E_N^c(\boldsymbol{\zeta}_N, \boldsymbol{z}_N)$ *is defined by*

$$E_N^c(\boldsymbol{\zeta}_N, \boldsymbol{z}_N) = \sup_{\boldsymbol{\varphi}_N \in \mathbb{C}_N} \frac{(c - c_N)(\boldsymbol{\zeta}_N, \boldsymbol{z}_N; \boldsymbol{\varphi}_N)}{\|\boldsymbol{\varphi}_N\|_{H(\mathbf{curl},\Omega)}}. \tag{4.6}$$

Proof. For any $(\boldsymbol{\zeta}_N, \boldsymbol{z}_N)$ in $\mathbb{C}_N \times V_N$, we derive from the inf-sup condition (4.4) the existence of a pair $(\tilde{\boldsymbol{\zeta}}_N, \tilde{\boldsymbol{z}}_N)$ also in $\mathbb{C}_N \times V_N$ which satisfies, for all $\boldsymbol{\varphi}_N$ in $\mathbb{C}_N$,

$$c_N(\tilde{\boldsymbol{\zeta}}_N, \tilde{\boldsymbol{z}}_N; \boldsymbol{\varphi}_N) = c_N(\boldsymbol{\zeta}_N, \boldsymbol{z}_N; \boldsymbol{\varphi}_N),$$

and, moreover,

$$\|\tilde{\boldsymbol{\zeta}}_N\|_{H(\mathbf{curl},\Omega)} + \|\tilde{\boldsymbol{z}}_N\|_{L^2(\Omega)^d} \leq (\gamma_*)^{-1} \sup_{\boldsymbol{\varphi}_N \in \mathbb{C}_N} \frac{c_N(\boldsymbol{\zeta}_N, \boldsymbol{z}_N; \boldsymbol{\varphi}_N)}{\|\boldsymbol{\varphi}_N\|_{H(\mathbf{curl},\Omega)}}.$$

We also note that

$$c_N(\boldsymbol{\zeta}_N, \boldsymbol{z}_N; \boldsymbol{\varphi}_N) = -c(\boldsymbol{\omega} - \boldsymbol{\zeta}_N, \boldsymbol{u} - \boldsymbol{z}_N; \boldsymbol{\varphi}_N) - (c - c_N)(\boldsymbol{\zeta}_N, \boldsymbol{z}_N; \boldsymbol{\varphi}_N).$$

Since the pair $(\boldsymbol{\vartheta}_N, \boldsymbol{w}_N)$ with $\boldsymbol{\vartheta}_N = \boldsymbol{\zeta}_N - \tilde{\boldsymbol{\zeta}}_N$ and $\boldsymbol{w}_N = \boldsymbol{z}_N - \tilde{\boldsymbol{z}}_N$ belongs to $\mathcal{W}_N$, the desired estimate is easily derived from the two previous lines.

Remark 4.5. The same argument as in the previous proof, combined with the inf-sup condition (3.22), leads to the estimate

$$\inf_{\boldsymbol{z}_N \in V_N} \|\boldsymbol{u} - \boldsymbol{z}_N\|_{L^2(\Omega)^d} \leq c \inf_{\boldsymbol{v}_N \in \mathbb{D}_N} \|\boldsymbol{u} - \boldsymbol{v}_N\|_{H(\mathrm{div},\Omega)}. \tag{4.7}$$

However, we prefer to avoid dealing with the approximation error in the $H(\mathrm{div}, \Omega)$-norm and directly estimate the distance from $\boldsymbol{u}$ to V_N.

By combining Lemmas 4.1 and 4.2 and Corollary 4.4, we observe that the full error

$$\|\boldsymbol{\omega} - \boldsymbol{\omega}_N\|_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u} - \boldsymbol{u}_N\|_{H(\mathrm{div},\Omega)} + \|p - p_N\|_{L^2(\Omega)}$$

is bounded by the sum of the three terms of approximation error,

$$\inf_{\boldsymbol{\zeta}_N \in \mathbb{C}_N} \|\boldsymbol{\omega} - \boldsymbol{\zeta}_N\|_{H(\mathbf{curl},\Omega)}, \qquad \inf_{\boldsymbol{z}_N \in V_N} \|\boldsymbol{u} - \boldsymbol{z}_N\|_{L^2(\Omega)^d}, \qquad \inf_{q_N \in \mathbb{M}_N} \|p - q_N\|_{L^2(\Omega)},$$

plus the three quantities $E_N^{\boldsymbol{f}}$, $E_N^a(\boldsymbol{\vartheta}_N, \boldsymbol{w}_N)$, and $E_N^c(\boldsymbol{\zeta}_N, \boldsymbol{z}_N)$ which are issued from numerical integration.

In order to estimate these last ones, we introduce the orthogonal projection operator Π_{N-1} from $L^2(\Omega)$ onto $\mathbb{P}_{N-1}(\Omega)$. Indeed, we derive from (3.4) that, for any $\boldsymbol{v}_N$ in $\mathbb{D}_N$,

$$\begin{aligned}\langle \boldsymbol{f}, \boldsymbol{v}_N \rangle - (\boldsymbol{f}, \boldsymbol{v}_N)_N &= \langle \boldsymbol{f} - \Pi_{N-1}\boldsymbol{f}, \boldsymbol{v}_N \rangle - (\boldsymbol{f} - \Pi_{N-1}\boldsymbol{f}, \boldsymbol{v}_N)_N \\ &= \langle \boldsymbol{f} - \Pi_{N-1}\boldsymbol{f}, \boldsymbol{v}_N \rangle - (\mathcal{I}_N\boldsymbol{f} - \Pi_{N-1}\boldsymbol{f}, \boldsymbol{v}_N)_N,\end{aligned}$$

so that, owing to (3.5),

$$E_N^{\boldsymbol{f}} \leq (1 + 3^d)\, \|\boldsymbol{f} - \Pi_{N-1}\boldsymbol{f}\|_{L^2(\Omega)^d} + 3^d\, \|\boldsymbol{f} - \mathcal{I}_N\boldsymbol{f}\|_{L^2(\Omega)^d}. \tag{4.8}$$

Similarly, we have, for any $\boldsymbol{v}_N$ in $\mathbb{D}_N$,

$$\begin{aligned}(a - a_N)(\boldsymbol{\vartheta}_N, \boldsymbol{z}_N; \boldsymbol{v}_N) = \nu \int_\Omega &\big(\mathbf{curl}\,\boldsymbol{\vartheta}_N - \Pi_{N-1}(\mathbf{curl}\,\boldsymbol{\omega})\big)(\boldsymbol{x}) \cdot \boldsymbol{z}_N(\boldsymbol{x})\, d\boldsymbol{x} \\ &- \nu \big(\mathbf{curl}\,\boldsymbol{\vartheta}_N - \Pi_{N-1}(\mathbf{curl}\,\boldsymbol{\omega}), \boldsymbol{z}_N\big)_N,\end{aligned}$$

so that

(4.9)
$$E_N^a(\boldsymbol{\vartheta}_N, \boldsymbol{w}_N) \le (1+3^d)\big(\|\mathbf{curl}\,(\boldsymbol{\omega}-\boldsymbol{\vartheta}_N)\|_{L^2(\Omega)^d} + \|\mathbf{curl}\,\boldsymbol{\omega} - \Pi_{N-1}(\mathbf{curl}\,\boldsymbol{\omega})\|_{L^2(\Omega)^d}\big).$$

Note that a bound for the quantity $\|\mathbf{curl}\,(\boldsymbol{\omega}-\boldsymbol{\vartheta}_N)\|_{L^2(\Omega)^d}$ is provided by Corollary 4.4. Finally, the same arguments lead to

(4.10)
$$E_N^c(\boldsymbol{\zeta}_N, \boldsymbol{z}_N) \le (1+3^d)\big(\|\boldsymbol{\omega}-\boldsymbol{\zeta}_N\|_{L^2(\Omega)^{\frac{d(d-1)}{2}}} + \|\boldsymbol{\omega} - \Pi_{N-1}\boldsymbol{\omega}\|_{L^2(\Omega)^{\frac{d(d-1)}{2}}} + \|\boldsymbol{u}-\boldsymbol{z}_N\|_{L^2(\Omega)^d} + \|\boldsymbol{u}-\Pi_{N-1}\boldsymbol{u}\|_{L^2(\Omega)^d}\big).$$

We now recall from [13, Thms. 7.1 and 14.2] the approximation properties of the operators Π_{N-1} and $\mathcal{I}_N$: For any function g in $H^s(\Omega)$, $s \ge 0$,

$$\|g - \Pi_{N-1}g\|_{L^2(\Omega)} \le c\,N^{-s}\,\|g\|_{H^s(\Omega)}, \tag{4.11}$$

and, for any function g in $H^s(\Omega)$, $s > \frac{d}{2}$,

$$\|g - \mathcal{I}_N g\|_{L^2(\Omega)} \le c\,N^{-s}\,\|g\|_{H^s(\Omega)}. \tag{4.12}$$

Estimates (4.11) and (4.12), when applied to each component of $\boldsymbol{f}$ and combined with (4.8), lead to the desired bound for $E_N^{\boldsymbol{f}}$. When combined with (4.9) and (4.10), they allow us to reduce the evaluation of $E_N^a(\boldsymbol{\vartheta}_N, \boldsymbol{w}_N)$ and $E_N^c(\boldsymbol{\zeta}_N, \boldsymbol{z}_N)$ to a bound for the approximation error.

The approximation error for the pressure can also be estimated from (4.11). To go further we recall the following:

• In dimension $d = 2$, the orthogonal projection operator $\Pi_N^{1,0}$ from $H_0^1(\Omega)$ onto $\mathbb{C}_N$ satisfies, for all functions $\boldsymbol{\varphi}$ in $H^s(\Omega) \cap H_0^1(\Omega)$, $s \ge 1$,

$$\|\boldsymbol{\varphi} - \Pi_N^{1,0}\boldsymbol{\varphi}\|_{L^2(\Omega)} + N^{-1}\,|\boldsymbol{\varphi} - \Pi_N^{1,0}\boldsymbol{\varphi}|_{H^1(\Omega)} \le c\,N^{-s}\,\|\boldsymbol{\varphi}\|_{H^s(\Omega)}. \tag{4.13}$$

• In dimension $d = 3$, a spectral analogue $\mathcal{R}_N$ of the Nédélec operator [24, sect. 2] has been constructed in [7, sect. 4]. It maps smooth functions in $H_0(\mathbf{curl}, \Omega)$ onto $\mathbb{C}_N$ and satisfies, for all functions $\boldsymbol{\varphi}$ in $H^s(\Omega)^3 \cap H_0(\mathbf{curl}, \Omega)$, $s \ge 2$,

$$\|\boldsymbol{\varphi} - \mathcal{R}_N\boldsymbol{\varphi}\|_{L^2(\Omega)^3} \le c\,N^{-s}\,\|\boldsymbol{\varphi}\|_{H^s(\Omega)^3}, \tag{4.14}$$

and, for all functions $\boldsymbol{\varphi}$ in $H_0(\mathbf{curl}, \Omega)$ such that $\mathbf{curl}\,\boldsymbol{\varphi}$ belongs to $H^s(\Omega)^3$, $s \ge 1$,

$$\|\mathbf{curl}\,(\boldsymbol{\varphi} - \mathcal{R}_N\boldsymbol{\varphi})\|_{L^2(\Omega)^3} \le c\,N^{-s}\,\|\mathbf{curl}\,\boldsymbol{\varphi}\|_{H^s(\Omega)^3}. \tag{4.15}$$

Applying these estimates leads to a bound for the approximation error on $\boldsymbol{\omega}$. Moreover, since the velocity $\boldsymbol{u}$ is divergence-free and has a zero normal trace on $\partial\Omega$, it is equal to $\mathbf{curl}\,\boldsymbol{\psi}$ for a function $\boldsymbol{\psi}$ in $H_0(\mathbf{curl}, \Omega)$. Thus, thanks to Lemma 3.1, its best approximation in V_N can be bounded from (4.13) or (4.15).

To state the final estimate, we introduce the scale of spaces, for $s \ge 0$,

$$H^s(\mathbf{curl}, \Omega) = \big\{\boldsymbol{\varphi} \in H^s(\Omega)^{\frac{d(d-1)}{2}};\ \mathbf{curl}\,\boldsymbol{\varphi} \in H^s(\Omega)^d\big\}. \tag{4.16}$$

Note that this space coincides with $H^{s+1}(\Omega)$ in dimension $d = 2$.

THEOREM 4.6. *Assume that the data $\boldsymbol{f}$ belong to $H^\sigma(\Omega)^d$ for a real number $\sigma > \frac{d}{2}$ and that the solution $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ of problem* (2.9) *belongs to $H^s(\mathbf{curl}, \Omega) \times H^s(\Omega)^d \times H^s(\Omega)$ for a real number $s \geq d-1$. Then the following error estimate holds between this solution and the solution $(\boldsymbol{\omega}_N, \boldsymbol{u}_N, p_N)$ of problem* (3.7)*:*

$$\begin{aligned}
&\|\boldsymbol{\omega} - \boldsymbol{\omega}_N\|_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u} - \boldsymbol{u}_N\|_{H(\mathrm{div},\Omega)} + \|p - p_N\|_{L^2(\Omega)} \\
&\qquad \leq c\left(N^{-s}\left(\|\boldsymbol{\omega}\|_{H^s(\mathbf{curl},\Omega)} + \|\boldsymbol{u}\|_{H^s(\Omega)^d} + \|p\|_{H^s(\Omega)}\right) + N^{-\sigma}\, \|\boldsymbol{f}\|_{H^\sigma(\Omega)^d}\right).
\end{aligned} \tag{4.17}$$

Estimate (4.17) is fully optimal, which is especially interesting as far as the pressure is concerned since this optimality is not obtained for most spectral discretizations of the Stokes problem.

The regularity which is required ($s \geq d-1$) concerns only the vorticity $\boldsymbol{\omega}$ and seems reasonable at least in the case of a square. Moreover, it follows from [16] and [17] that both $\boldsymbol{\omega}$ and $\boldsymbol{u}$ can be written as a sum of a regular part and the gradient of a linear combination of the singular functions associated with the Laplace operator. These two terms can be approximated separately and, as usual in spectral methods [12], the approximation of the singular part is better than can be hoped from the general theory. This leads to the following result, where, in dimension $d = 2$, σ_Ω is equal to $4 - \varepsilon$ for any $\varepsilon > 0$.

COROLLARY 4.7. *Assume that the data $\boldsymbol{f}$ belong to $H^\sigma(\Omega)^d$ for a real number $\sigma > \frac{d}{2}$. Then the following error estimate holds between the solution $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ of problem* (2.9) *and the solution $(\boldsymbol{\omega}_N, \boldsymbol{u}_N, p_N)$ of problem* (3.7)*:*

$$\begin{aligned}
&\|\boldsymbol{\omega} - \boldsymbol{\omega}_N\|_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u} - \boldsymbol{u}_N\|_{H(\mathrm{div},\Omega)} + \|p - p_N\|_{L^2(\Omega)} \\
&\qquad\qquad \leq c\, N^{-\min\{\sigma,\sigma_\Omega\}}\, \|\boldsymbol{f}\|_{H^\sigma(\Omega)^d},
\end{aligned} \tag{4.18}$$

where σ_Ω is a real number ≥ 1 depending only on Ω.

5. Case of nonhomogeneous boundary conditions. We briefly explain how the results of the previous sections can be extended to the problem

$$\begin{cases}
\nu\, \mathbf{curl}\, \boldsymbol{\omega} + \mathbf{grad}\, p = \boldsymbol{f} & \text{in } \Omega, \\
\mathrm{div}\, \boldsymbol{u} = 0 & \text{in } \Omega, \\
\boldsymbol{\omega} = \mathbf{curl}\, \boldsymbol{u} & \text{in } \Omega, \\
\boldsymbol{u} \cdot \boldsymbol{n} = g & \text{on } \partial\Omega, \\
\gamma_t(\boldsymbol{\omega}) = \mathbf{0} & \text{on } \partial\Omega,
\end{cases} \tag{5.1}$$

where the function g belongs to $H^{-\frac{1}{2}}(\partial\Omega)$ and satisfies the compatibility condition (here $\langle \cdot, \cdot \rangle_{\partial\Omega}$ obviously denotes the duality pairing between $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$)

$$\langle g, 1 \rangle_{\partial\Omega} = 0. \tag{5.2}$$

We consider the following variational problem:

Find $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ in $H_0(\mathbf{curl}, \Omega) \times H(\mathrm{div}, \Omega) \times L^2_0(\Omega)$ such that

$$\boldsymbol{u} \cdot \boldsymbol{n} = g \quad \text{on } \partial\Omega \tag{5.3}$$

and that

$$\begin{aligned}
&\forall \boldsymbol{v} \in H_0(\mathrm{div}, \Omega), \quad && a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v}) + b(\boldsymbol{v}, p) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle, \\
&\forall q \in L^2_0(\Omega), && b(\boldsymbol{u}, q) = 0, \\
&\forall \boldsymbol{\vartheta} \in H_0(\mathbf{curl}, \Omega), && c(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{\vartheta}) = 0.
\end{aligned} \tag{5.4}$$

Thanks to the arguments given in section 2, it can be checked that problems (5.1) and (5.3)–(5.4) are equivalent, in the sense made precise in Proposition 2.2. To prove the well-posedness of problem (5.3)–(5.4), we need a lifting of the boundary condition (5.3).

LEMMA 5.1. *For any g in $H^{-\frac{1}{2}}(\partial\Omega)$ satisfying* (5.2)*, there exists a divergence-free and curl-free function $\boldsymbol{u}_b$ in $L^2(\Omega)^d$ such that $\boldsymbol{u}_b \cdot \boldsymbol{n}$ is equal to g on $\partial\Omega$. Moreover, this function satisfies*

$$\|\boldsymbol{u}_b\|_{H(\mathrm{div},\Omega)} \le c\,\|g\|_{H^{-\frac{1}{2}}(\partial\Omega)}. \tag{5.5}$$

Proof. The following variational problem,

Find μ in $H^1(\Omega) \cap L^2_0(\Omega)$ such that

$$\forall \rho \in H^1(\Omega) \cap L^2_0(\Omega), \qquad \int_\Omega \mathbf{grad}\,\mu \cdot \mathbf{grad}\,\rho\,d\boldsymbol{x} = \langle g, \rho\rangle_{\partial\Omega},$$

admits a unique solution μ which satisfies

$$|\mu|_{H^1(\Omega)} \le c\,\|g\|_{H^{-\frac{1}{2}}(\partial\Omega)}.$$

Then the function $\boldsymbol{u}_b = \mathbf{grad}\,\mu$ satisfies all the properties stated in the lemma.

THEOREM 5.2. *For any data $\boldsymbol{f}$ in the dual space of $H_0(\mathrm{div},\Omega)$ and g in $H^{-\frac{1}{2}}(\partial\Omega)$ satisfying* (5.2)*, problem* (5.3)–(5.4) *has a unique solution $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ in $H_0(\mathbf{curl},\Omega) \times H(\mathrm{div},\Omega) \times L^2_0(\Omega)$. Moreover, this solution satisfies*

$$\|\boldsymbol{\omega}\|_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u}\|_{H(\mathrm{div},\Omega)} + \|p\|_{L^2(\Omega)} \le c\,\big(\|\boldsymbol{f}\|_{H_0(\mathrm{div},\Omega)'} + \|g\|_{H^{-\frac{1}{2}}(\partial\Omega)}\big). \tag{5.6}$$

Proof. Using the function $\boldsymbol{u}_b$ introduced in Lemma 5.1 and setting $\boldsymbol{u}_0 = \boldsymbol{u} - \boldsymbol{u}_b$, we note that $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ is a solution of problem (5.3)–(5.4) if and only if $(\boldsymbol{\omega}, \boldsymbol{u}_0, p)$ is a solution of a problem similar to (2.9). So the existence and uniqueness of $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ follow from Theorem 2.5, and estimate (5.6) is derived by combining (2.18) and (5.5).

In order to write the discrete problem, we introduce the space

$$\overline{\mathbb{D}}_N = \begin{cases} \mathbb{P}_{N,N-1}(\Omega) \times \mathbb{P}_{N-1,N}(\Omega) & \text{if } d = 2, \\ \mathbb{P}_{N,N-1,N-1}(\Omega) \times \mathbb{P}_{N-1,N,N-1}(\Omega) \times \mathbb{P}_{N-1,N-1,N}(\Omega) & \text{if } d = 3. \end{cases} \tag{5.7}$$

Assuming that the function g belongs to $L^2(\partial\Omega)$, we define an approximation g_N of g as follows: On each edge ($d = 2$) or face ($d = 3$) Γ_r of Ω, $1 \le r \le 2d$, $g_{N|\Gamma_r}$ is equal to the image of $g_{|\Gamma_r}$ by the orthogonal projection operator from $L^2(\Gamma_r)$ onto $\mathbb{P}_{N-1}(\Gamma_r)$. Then we consider the following problem:

Find $(\boldsymbol{\omega}_N, \boldsymbol{u}_N, p_N)$ in $\mathbb{C}_N \times \overline{\mathbb{D}}_N \times \mathbb{M}_N$ such that

$$\boldsymbol{u}_N \cdot \boldsymbol{n} = g_N \quad \text{on } \partial\Omega \tag{5.8}$$

and that

$$\begin{aligned} &\forall \boldsymbol{v}_N \in \mathbb{D}_N, \quad a_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; v_N) + b_N(\boldsymbol{v}_N, p_N) = (\boldsymbol{f}, \boldsymbol{v}_N)_N, \\ &\forall q \in \mathbb{M}_N, \quad b_N(\boldsymbol{u}_N, q_N) = 0, \\ &\forall \boldsymbol{\vartheta}_N \in \mathbb{C}_N, \quad c_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{\vartheta}_N) = 0. \end{aligned} \tag{5.9}$$

Remark 5.3. The choice of g_N as the discrete boundary condition is justified at least in a first step by the following reasons:

(i) The normal trace operator on each Γ_r, $1 \le r \le 2d$, maps $\overline{\mathbb{D}}_N$ onto $\mathbb{P}_{N-1}(\Gamma_r)$, so that each $g_{N|\Gamma_r}$ belongs to the right space.

(ii) In dimension $d = 2$, on two adjacent edges (i.e., that share a vertex), the normal trace operator involves different components of any function in $H(\mathrm{div}, \Omega)$, so that g_N does not have to satisfy any compatibility conditions at the common vertex. The same remark holds in dimension $d = 3$ for two adjacent faces (i.e., that share an edge).

(iii) Property (5.2) is still satisfied with g replaced by g_N, which is essential since we intend to work with exactly divergence-free discrete velocities.

Moreover, the computation of g_N is not too expensive.

THEOREM 5.4. *For any data* $\boldsymbol{f}$ *continuous on* $\overline{\Omega}$ *and* g *in* $L^2(\partial\Omega)$ *satisfying* (5.2), *problem* (5.8)–(5.9) *has a unique solution* $(\boldsymbol{\omega}_N, \boldsymbol{u}_N, p_N)$ *in* $\mathbb{C}_N \times \overline{\mathbb{D}}_N \times \mathbb{M}_N$.

Proof. It is readily checked that problem (5.8)–(5.9) can be written as a square linear system. Moreover, it follows from Theorem 3.10 that the unique solution of this problem when the data $\boldsymbol{f}$ and g_N are zero is $(\boldsymbol{0}, \boldsymbol{0}, 0)$. This yields the existence and uniqueness property.

We briefly recall the arguments that can be used in order to derive the same error estimates as in section 4.

(1) It follows from [7, sect. 4] that, in dimension $d = 3$, an extension $\overline{\mathcal{R}}_N$ of the operator $\mathcal{R}_N$ introduced in section 4 can be constructed such that estimate (4.15) still holds but now for $s \ge \frac{3}{2}$ and that, for any smooth enough function $\boldsymbol{\varphi}$, the normal traces of $\mathbf{curl}\,(\overline{\mathcal{R}}_N\boldsymbol{\varphi})$ on each Γ_r coincides with the images of the normal traces of $\mathbf{curl}\,\boldsymbol{\varphi}$ by the projection operator from $L^2(\Gamma_r)$ onto $\mathbb{P}_{N-1}(\Gamma_r)$. A similar operator can obviously be constructed in dimension $d = 2$.

(2) Since the velocity $\boldsymbol{u}$ is divergence-free and thanks to (5.2), there exists a function $\boldsymbol{\psi}$ such that $\mathbf{curl}\,\boldsymbol{\psi} = \boldsymbol{u}$. Then the function $\boldsymbol{z}_N = \mathbf{curl}\,(\overline{\mathcal{R}}_N\boldsymbol{\psi})$ belongs to $\mathbb{D}_N$, is divergence-free, and has its normal trace equal to g_N on $\partial\Omega$. Moreover, the distance of $\boldsymbol{u}$ to $\boldsymbol{z}_N$ in $L^2(\Omega)^d$ can easily be evaluated from (4.15).

(3) Let $\overline{V}_N$ denote the space of divergence-free functions in $\overline{\mathbb{D}}_N$. Thanks to Lemma 4.3 (see also the proof of Corollary 4.4), for the previous function $\boldsymbol{z}_N$ and any $\boldsymbol{\zeta}_N$ in $\mathbb{C}_N$, there exists a $(\boldsymbol{\vartheta}_N, \boldsymbol{w}_N)$ in $\mathbb{C}_N \times \overline{V}_N$ such that the pair $(\boldsymbol{\vartheta}_N, \boldsymbol{w}_N)$ satisfies (4.5), the normal traces of $\boldsymbol{w}_N$ and $\boldsymbol{z}_N$ coincide on $\partial\Omega$, and, moreover,

$$\forall \boldsymbol{\varphi}_N \in \mathbb{C}_N,\ c_N(\boldsymbol{\vartheta}_N, \boldsymbol{w}_N; \boldsymbol{\varphi}_N) = 0.$$

(4) The pair $(\boldsymbol{\omega}_N - \boldsymbol{\vartheta}_N, \boldsymbol{u}_N - \boldsymbol{z}_N)$ now belongs to $\mathcal{W}_N$. So exactly the same arguments as in the proof of Lemma 4.1 lead to estimate (4.1).

THEOREM 5.5. *If the assumptions of Theorem* 4.6 *hold and the data* g *satisfies condition* (5.2) *and is such that each* $g_{|\Gamma_r}$, $1 \le r \le 2d$, *belongs to* $H^\tau(\Gamma_r)$ *for a nonnegative real number* τ, *the following error estimate holds between the solution* $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ *of problem* (5.3)–(5.4) *and the solution* $(\boldsymbol{\omega}_N, \boldsymbol{u}_N, p_N)$ *of problem* (5.8)–(5.9):

(5.10)
$$\begin{aligned}
&\|\boldsymbol{\omega} - \boldsymbol{\omega}_N\|_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u} - \boldsymbol{u}_N\|_{H(\mathrm{div},\Omega)} + \|p - p_N\|_{L^2(\Omega)} \\
&\quad \le c\,\Big(N^{-s}\,\big(\|\boldsymbol{\omega}\|_{H^s(\mathbf{curl},\Omega)} + \|\boldsymbol{u}\|_{H^s(\Omega)^d} + \|p\|_{H^s(\Omega)}\big) \\
&\qquad\qquad + N^{-\sigma}\,\|\boldsymbol{f}\|_{H^\sigma(\Omega)^d} + N^{-\tau-\frac{1}{2}} \sum_{r=1}^{2d} \|g\|_{H^\tau(\Gamma_r)}\Big).
\end{aligned}$$

COROLLARY 5.6. *Assume that the data* $(\boldsymbol{f}, g)$ *belong to* $H^\sigma(\Omega)^d \times H^{\sigma-\frac{1}{2}}(\partial\Omega)$ *for a real number* $\sigma > \frac{d}{2}$ *and that condition* (5.2) *is satisfied. Then the following error*

estimate holds between the solution $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ *of problem* (5.3)–(5.4) *and the solution* $(\boldsymbol{\omega}_N, \boldsymbol{u}_N, p_N)$ *of problem* (5.8)–(5.9) *for the same real number* σ_Ω *as in Corollary* 4.7*:*

$$\begin{aligned}(5.11)\qquad &\|\boldsymbol{\omega} - \boldsymbol{\omega}_N\|_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u} - \boldsymbol{u}_N\|_{H(\mathrm{div},\Omega)} + \|p - p_N\|_{L^2(\Omega)} \\ &\qquad\qquad \le c\, N^{-\min\{\sigma,\sigma_\Omega\}} \left(\|\boldsymbol{f}\|_{H^\sigma(\Omega)^d} + \|g\|_{H^{\sigma-\frac12}(\partial\Omega)}\right).\end{aligned}$$

6. Some numerical experiments. Before presenting the numerical experiments, we briefly describe how problem (3.7) is implemented. Let φ_j, $0 \le j \le N$, denote the Lagrange polynomials in $\mathbb{P}_N(-1,1)$ associated with the nodes ξ_j. We fix an integer j_* between 0 and N (usually equal to the integer part of $\frac{N}{2}$), define J^* as the set $\{0, \dots, N\} \setminus \{j^*\}$, and set

$$(6.1)\qquad \varphi_j^*(\zeta) = \varphi_j(\zeta)\, \frac{\xi_j - \xi_{j^*}}{\zeta - \xi_{j^*}}, \quad j \in J^*.$$

Then the unknowns $\boldsymbol{\omega}_N$ and $\boldsymbol{u}_N$ and a pseudopressure $\tilde{p}_N$ admit the expansions, in dimension $d = 2$ for simplicity,

$$\begin{aligned}\boldsymbol{\omega}_N(x,y) &= \sum_{i=1}^{N-1}\sum_{j=1}^{N-1} \omega_{ij}\, \varphi_i(x)\varphi_j(y),\\ u_{Nx}(x,y) &= \sum_{i=1}^{N-1}\sum_{j\in J^*} u_{ij}^x\, \varphi_i(x)\varphi_j^*(y), \quad u_{Ny}(x,y) = \sum_{i\in J^*}\sum_{j=1}^{N-1} u_{ij}^y\, \varphi_i^*(x)\varphi_j(y),\\ \tilde{p}_N(x,y) &= \sum_{i\in J^*, j\in J^*, (i,j)\neq(0,0)} p_{ij}\, \varphi_i^*(x)\varphi_j^*(y).\end{aligned}$$

The function $\tilde{p}_N$ vanishes in $(-1,-1)$ but no longer belongs to $L_0^2(\Omega)$; however, the real pressure p_N can easily be recovered in a postprocessing step, thanks to the formula

$$(6.2)\qquad p_N(x,y) = \tilde{p}_N(x,y) - \frac{1}{2^d}\,(\tilde{p}_N, 1)_N.$$

We denote by $\Omega^\diamond$, U, and P the vectors made of these coefficients. Their dimensions are equal to $\frac{d(d-1)}{2}\, N^{d-2}(N-1)^2$, $d\, N^{d-1}(N-1)$, and $N^d - 1$, respectively. Problem (3.7) can thus be written equivalently as the square linear system

$$(6.3)\qquad \begin{pmatrix} A & 0 & B \\ 0 & B^T & 0 \\ C_{\boldsymbol{\omega}} & C_{\boldsymbol{u}} & 0 \end{pmatrix} \begin{pmatrix} \Omega^\diamond \\ U \\ P \end{pmatrix} = \begin{pmatrix} F \\ 0 \\ 0 \end{pmatrix},$$

where B^T denotes the transposed matrix of B. The global matrix is not symmetric, even if the subblocks $\left(\begin{smallmatrix} 0 & B \\ B^T & 0 \end{smallmatrix}\right)$ and $C_{\boldsymbol{\omega}}$ are symmetric. Note that, up to the multiplicative constant $-\nu^{-1}$, the matrix $C_{\boldsymbol{u}}$ coincides with A^T. The system is very similar in the case of nonhomogeneous boundary conditions, except that a further vector $-\tilde{B}^T G$ appears in the second line of the right-hand side.

In what follows, system (6.3) is solved via the GMRES method, so that it has not to be assembled. As a preconditioner, we use the matrix issued from an incomplete LU factorization of the global matrix in (6.3). Moreover, as standard in spectral methods, it follows from the tensorization properties of the polynomial spaces that each product of this matrix by a vector is realized with $c\, N^{d+1}$ operations, which highly reduces the cost of the inversion.

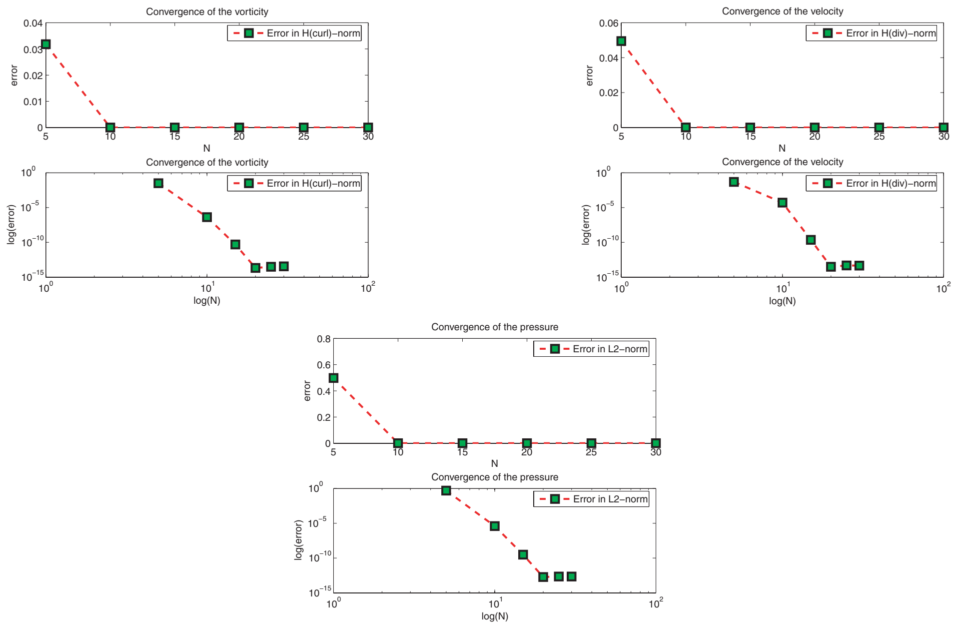

FIG. 1. *Error curves for the solution defined from* (6.4).

Two-dimensional experiments. We first work in the square $\Omega =]-1, 1[^2$, with $\nu = 1$. We consider a given solution constructed thanks to the formulas $\boldsymbol{u} = \mathbf{curl}\,\psi$ and $\boldsymbol{\omega} = \mathbf{curl}\,\boldsymbol{u}$ in the two situations

(i) of functions ψ and p of class $\mathcal{C}^\infty$, defined by

$$\psi(x, y) = \sin(\pi x)\sin(\pi y), \quad p(x, y) = xy, \tag{6.4}$$

and

(ii) of functions ψ and p of limited regularity, defined by

$$\psi(x, y) = (1 - x^2)^3(1 - y^2)^{\frac{7}{2}}, \quad p(x, y) = x(1 - x^2)^{\frac{3}{2}}(1 + y^2)^{-\frac{1}{2}}. \tag{6.5}$$

Figure 1 for the solution issued from (6.4) and Figure 2 for the solution issued from (6.5) present the convergence curves of the relative errors on $\boldsymbol{\omega}$, $\boldsymbol{u}$, and p in the corresponding norms, both in standard and logarithmic scales, for N varying from 5 to 30.

In Figure 1, the convergence is exponential and the three errors are smaller than 10^{-10} from $N = 15$. The convergence is, of course, slower in Figure 2. It can be noted that the vorticity and the pressure have the same regularity near the edges of Ω contained in the lines $x = \pm 1$ (they behave like $(1-x^2)^{\frac{3}{2}}$) and that the error slopes are the same.

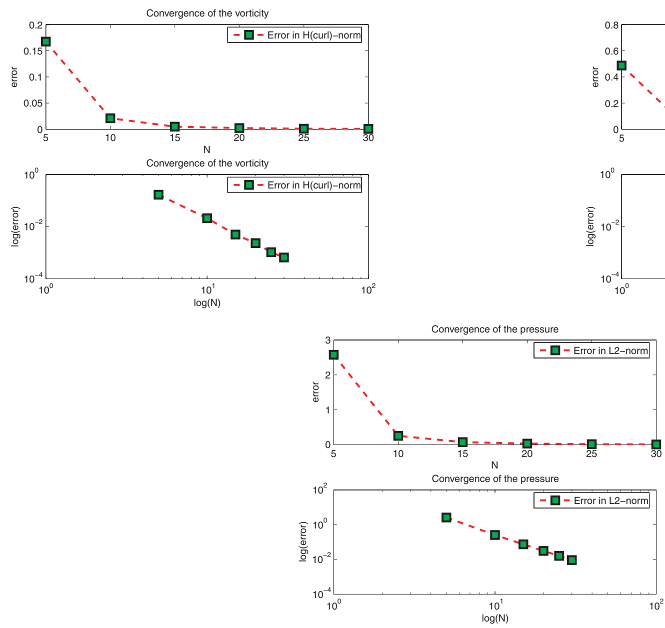

FIG. 2. *Error curves for the solution defined from* (6.5).

Figure 3 presents, from left to right and top to bottom, the values of the vorticity of the two components of the velocity and of the pressure corresponding to the data $\boldsymbol{f} = (f_x, f_y)$, with

$$f_x = 0, \quad f_y = xy^2, \tag{6.6}$$

in the case $g = 0$ of homogeneous boundary conditions, obtained with $N = 40$.

Figure 4 presents, from left to right and top to bottom, the values of the vorticity of the two components of the velocity and of the pressure corresponding to the data $\boldsymbol{f} = (f_x, f_y)$, given in (6.6) and with g given by

$$g(-1, y) = -(1 - y^2)^{\frac{3}{2}}, \qquad g(1, y) = (1 - y^2)^{\frac{3}{2}}, \qquad g(x, \pm 1) = 0, \tag{6.7}$$

obtained with $N = 40$. It can be noted that the vorticity $\boldsymbol{\omega}_N$ and pressure p_N are nearly the same in Figures 3 and 4.

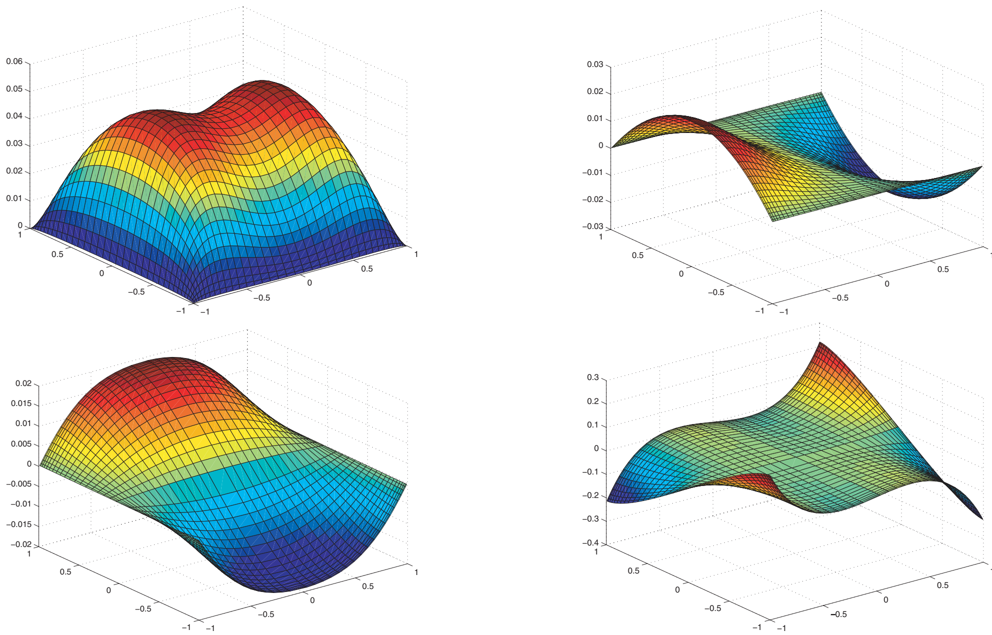

FIG. 3. *The solution* $(\boldsymbol{\omega}, u_x, u_y, p)$ *for the data* $\boldsymbol{f}$ *defined in* (6.6) *and* $g = 0$.

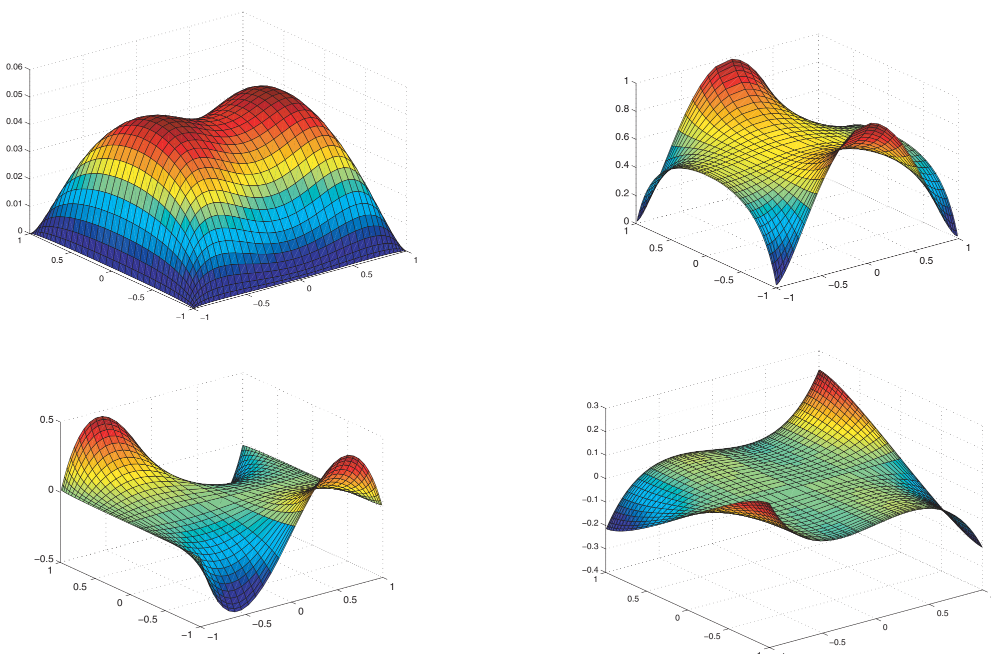

FIG. 4. *The solution* $(\boldsymbol{\omega}, u_x, u_y, p)$ *for the data* $(\boldsymbol{f}, g)$ *defined in* (6.6)–(6.7).

Three-dimensional experiments. We now work in the cube $\Omega =]-1, 1[^3$, with $\nu = 1$ and always in the case $g = 0$ of homogeneous boundary conditions. We consider a given solution constructed thanks to the formulas $\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{\psi}$ and $\boldsymbol{\omega} = \mathbf{curl}\,\boldsymbol{u}$, with

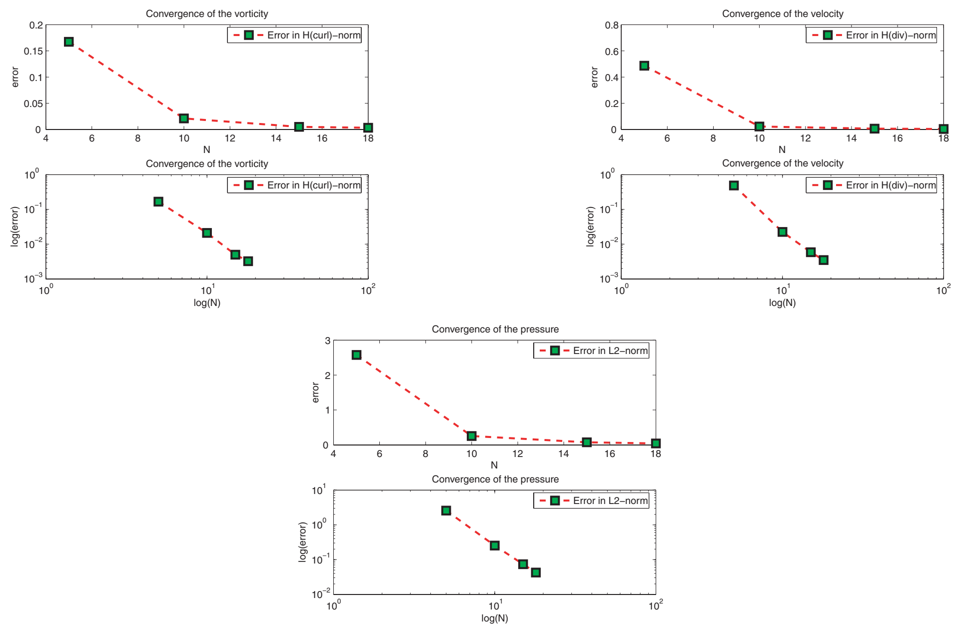

FIG. 5. *Error curves for the solution defined from* (6.8).

$\boldsymbol{\psi} = (\psi_x, \psi_y, \psi_z)$ and p defined by

$$\begin{aligned} \psi_x(x,y,z) &= (1-y^2)^3(1-z^2)^{\frac{7}{2}}, \quad \psi_y(x,y,z) = (1-x^2)^{\frac{7}{2}}(1-z^2)^3, \\ \psi_z(x,y,z) &= (1-x^2)^3(1-y^2)^{\frac{7}{2}}, \quad p(x,y,z) = \frac{x(1-x^2)^{\frac{3}{2}}}{(1+y^2)^{\frac{1}{2}}(1+z^2)^{\frac{1}{2}}}. \end{aligned} \tag{6.8}$$

Figure 5 presents the convergence curves of the relative errors on $\boldsymbol{\omega}$, $\boldsymbol{u}$, and p, both in standard and logarithmic scales, for N varying from 5 to 18. It can be noted that the regularity of the solution is the same as for the two-dimensional solution defined from (6.5) and that the slopes of the error are very similar to those in Figure 2. We do not present the convergence curves for a solution of class $\mathcal{C}^\infty$ since they are exactly the same as in Figure 1.

Figure 6 presents, from left to right and top to bottom, the curves of isovalues in the plane $x = 0$ of the three components of the vorticity and the velocity and of the pressure corresponding to the data $\boldsymbol{f} = (f_x, f_y, f_z)$, with

$$f_x = x, \qquad f_y = 0, \qquad f_z = yz^2, \tag{6.9}$$

obtained with $N = 18$.

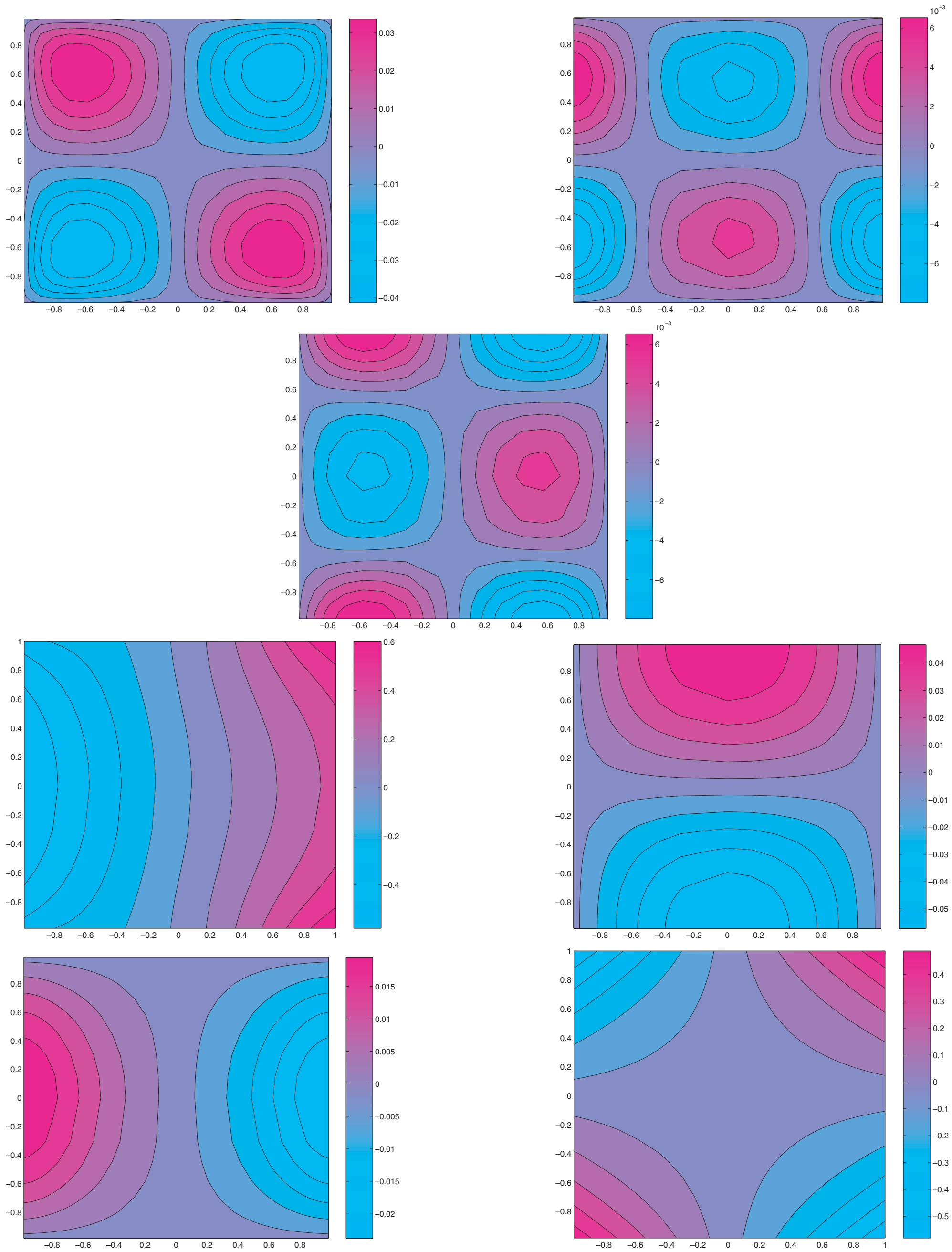

FIG. 6. *The solution* $(\omega_x, \omega_y, \omega_z, u_x, u_y, u_z, p)$ *for the data* $\boldsymbol{f}$ *defined in* (6.9).

As a conclusion, both two- and three-dimensional experiments are in perfect agreement with the error estimates derived in sections 4 and 5 and bring to light the efficiency of the spectral discretization for this type of problem.

REFERENCES

[1] M. Amara, D. Capatina-Papaghiuc, E. Chacon-Vera, and D. Trujillo, *Vorticity–velocity–pressure formulation for Navier–Stokes equations*, Comput. Vis. Sci., 6 (2004), pp. 47–52.

[2] M. Amara, D. Capatina, and D. Trujillo, *Stabilized finite element method for the Navier–Stokes equations, with non standard boundary conditions*, J. Sci. Comput., submitted.

[3] C. Amrouche, C. Bernardi, M. Dauge, and V. Girault, *Vector potentials in three-dimensional nonsmooth domains*, Math. Methods Appl. Sci., 21 (1998), pp. 823–864.

[4] M. Azaïez, F. Ben Belgacem, M. Grundmann, and H. Khallouf, *Staggered grids hybrid-dual spectral element method for second order elliptic problems. Application to high-order time splitting for Navier-Stokes equations*, Comput. Meth. Appl. Mech. Engrg., 166 (1998), pp. 183–199.

[5] M. Azaïez, C. Bernardi, and M. Grundmann, *Méthodes spectrales pour les équations du milieu poreux*, East-West J. Numer. Math., 2 (1994), pp. 91–105.

[6] C. Bègue, C. Conca, F. Murat, and O. Pironneau, *Les équations de Stokes et de Navier–Stokes avec des conditions aux limites sur la pression*, in Nonlinear Partial Differential Equations and Their Applications, Collège de France Seminar, Vol. IX, H. Brezis and J.-L. Lions, eds., Longman Scientific and Technical, Harlow, UK, 1988, pp. 179–264.

[7] F. Ben Belgacem and C. Bernardi, *Spectral element discretization of the Maxwell equations*, Math. Comp., 68 (1999), pp. 1497–1520.

[8] J.-M. Bernard, *Spectral discretizations of the Stokes equations with non standard boundary conditions*, J. Sci. Comput., 20 (2004), pp. 355–377.

[9] C. Bernardi, M. Dauge, and Y. Maday, *Polynomials in the Sobolev World*, Internal Report, Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, Paris, France, 2003.

[10] C. Bernardi and V. Girault, *Espaces duaux des domaines des opérateurs divergence et rotationnel avec trace nulle*, Internal Report, Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, Paris, France, 2003.

[11] C. Bernardi, F. Hecht, and O. Pironneau, *Coupling Darcy and Stokes equations for porous media with cracks*, M2AN Math. Model. Numer. Anal., 39 (2005), pp. 7–35.

[12] C. Bernardi and Y. Maday, *Polynomial approximation of some singular functions*, Appl. Anal., 42 (1991), pp. 1–32.

[13] C. Bernardi and Y. Maday, *Spectral Methods*, in Handbook of Numerical Analysis V, P.G. Ciarlet and J.-L. Lions, eds., North–Holland, Amsterdam, 1997, pp. 209–485.

[14] C. Bernardi and Y. Maday, *Uniform inf-sup conditions for the spectral discretization of the Stokes problem*, Math. Models Methods Appl. Sci., 9 (1999), pp. 395–414.

[15] A. Buffa, M. Costabel, and M. Dauge, *Algebraic convergence for anisotropic edge elements in polyhedral domains*, Numer. Math., 101 (2005), pp. 29–65.

[16] M. Costabel and M. Dauge, *Espaces fonctionnels Maxwell: Les gentils, les méchants et les singularités*, http://perso.univ-rennes1.fr/monique.dauge (1998).

[17] M. Costabel and M. Dauge, *Computation of resonance frequencies for Maxwell equations in non-smooth domains*, in Topics in Computational Wave Propagation, M. Ainsworth, P. Davies, D. Duncan, P. Martin, and B. Rynne, eds., Springer, Berlin, 2003, pp. 125–161.

[18] F. Dubois, *Vorticity–velocity–pressure formulation for the Stokes problem*, Math. Methods Appl. Sci., 25 (2002), pp. 1091–1119.

[19] F. Dubois, M. Salaün, and S. Salmon, *Vorticity–velocity–pressure and stream function-vorticity formulations for the Stokes problem*, J. Math. Pures Appl. (9), 82 (2003), pp. 1395–1451.

[20] A. Ern, J.-L. Guermond, and L. Quartapelle, *Vorticity–velocity formulation of the Stokes problem in 3D*, Math. Methods Appl. Sci., 22 (1999), pp. 531–546.

[21] V. Girault, *Incompressible finite element methods for Navier–Stokes equations with nonstandard boundary conditions in* $\mathbb{R}^3$, Math. Comp., 51 (1988), pp. 55–74.

[22] V. Girault and P.-A. Raviart, *Finite Element Methods for Navier–Stokes Equations, Theory and Algorithms*, Springer, Berlin, 1986.

[23] R. Hiptmair, *Finite elements in computational electromagnetism*, Acta Numer., 11 (2002), pp. 237–339.

[24] J.-C. Nédélec, *Mixed finite elements in* $\mathbb{R}^3$, Numer. Math., 35 (1980), pp. 315–341.

[25] F.-X. Roux, *Utilisation d'éléments finis conformes* $H(\mathrm{rot})$ *pour la résolution de l'équation de Navier-Stokes tridimensionnelle*, Séminaire du Laboratoire d'Analyse Numérique, Paris, France, 1984.

[26] S. Salmon, *Développement numérique de la formulation tourbillon–vitesse–pression pour le problème de Stokes*, Thesis, Université Pierre et Marie Curie, Paris, France, 1999.