Chapter 4: Interval Estimation

Chapter 3 dealt with the point estimation of a parameter or made the inference of estimating the true value of the parameter to be a point. In this chapter, we might make the inference of estimating that true value of the parameter is contained in some interval that is called interval estimation.

Confidence Interval:

Let $X_1, X_2, \dots, X_n$ be a random sample from $f(x, \theta)$. Let $T_1 = t_1(X_1, \dots, X_n)$ and $T_2 = t_2(X_1, \dots, X_n)$ be two statistics satisfying $T_1 < T_2$ for which $P(T_1 < \tau(\theta) < T_2) = 1 - \alpha$, where α does not depend on θ, then the random interval (T_1, T_2) is called 100 $(1 - \alpha)$% **confidence interval** for $\tau(\theta)$, α is called the confidence coefficient and T_1 and T_2 are called the lower and upper confidence limits, respectively, for $\tau(\theta)$.

4.1 Confidence Intervals from Normal Distribution

In this section, we derive confidence intervals for the mean μ and the variance σ^2 when the random sample $X_1, X_2, \dots, X_n$ has normal distribution.

4.1.1 Confidence Interval for the Mean

There are two cases to consider depending on whether or not σ^2 is known.

First Case (σ^2 is known):

If the sample is selected from a normal population or, if n is large enough, (Theorem 2.3 and Theorem 2.4) the sampling distribution of the sample mean $\bar{X}$ when σ^2 is known is given by

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

Then, we establish a 100 $(1 - \alpha)$% confidence interval for μ when σ^2 is known as following:

$$P\left(-z_{1-\frac{\alpha}{2}} < \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < z_{1-\frac{\alpha}{2}}\right) = 1 - \alpha$$

$$P\left(\bar{X} - z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right) = 1-\alpha$$

where $z_{1-\frac{\alpha}{2}}$ is a value from z-table.

Second Case (σ^2 is unknown):

Now, we turn to the problem of finding a confidence interval for the mean μ of a normal distribution when we are not know the variance σ^2. In Theorem 2.11 we found that

$$\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t_{(n-1)}$$

where S is the sample standard deviation. Then, we can find $100\,(1-\alpha)\%$ confidence interval for μ when σ^2 is unknown as following:

$$P\left(-t_{(1-\frac{\alpha}{2},n-1)} < \frac{\bar{X}-\mu}{S/\sqrt{n}} < t_{(1-\frac{\alpha}{2},n-1)}\right) = 1-\alpha$$

$$P\left(\bar{X} - t_{(1-\frac{\alpha}{2},n-1)}\frac{S}{\sqrt{n}} < \mu < \bar{X} + t_{(1-\frac{\alpha}{2},n-1)}\frac{S}{\sqrt{n}}\right) = 1-\alpha$$

where $t_{(1-\frac{\alpha}{2},n-1)}$ is a value from t-table with $n-1$ degrees of freedom.

Example 4.1:

Let $X_1, X_2, \ldots, X_{10}$ be a random sample from $N(\mu, 16)$ and let the sample mean $\bar{X}$ be 3.67. Find 95% confidence interval for the population mean μ.

Solution:

Since population variance is known, $\sigma^2 = 16$, and $\bar{X} = 3.67$, $n = 10$; then 95% confidence interval for the population mean μ is

$$3.67 \pm z_{1-\frac{\alpha}{2}}\frac{4}{\sqrt{10}}$$

where, the value of z-table $z_{1-\frac{\alpha}{2}}$ is found as

$$1-\alpha = 0.95 \Rightarrow \alpha = 0.05 \Rightarrow \frac{\alpha}{2} = 0.025 \Rightarrow 1-\frac{\alpha}{2} = 0.975$$

$$\Rightarrow z_{1-\frac{\alpha}{2}} = z_{0.975} = 1.96$$

Then, $$3.67 \pm 1.96\frac{4}{\sqrt{10}} \Longrightarrow 3.67 \pm 2.4792$$

$$\Longrightarrow \mu \in (1.1908, 6.1492)$$

4.1.2 Confidence Interval for the Variance

Let the random variable X be $N(\mu, \sigma^2)$. We shall discuss the problem of finding a confidence interval for σ^2. Our discussion will consist of two parts: the first when μ is a know number, and second when μ is unknown.

First Case (μ is known):

Let $X_1, X_2, \dots, X_n$ denote a random sample of size n from distribution that is $N(\mu, \sigma^2)$, where μ is known. From Corollary 2.2, we got that

$$\frac{\sum_{i=1}^{n}(X_i-\mu)^2}{\sigma^2} \sim \chi_n^2$$

Let us select a probability, say $1-\alpha$, then 100 $(1-\alpha)$% confidence interval for σ^2 when μ is known is given by

$$P\left(\chi^2_{(1-\frac{\alpha}{2},n)} < \frac{\sum_{i=1}^{n}(X_i-\mu)^2}{\sigma^2} < \chi^2_{(\frac{\alpha}{2},n)}\right) = 1-\alpha$$

$$P\left(\frac{\sum_{i=1}^{n}(X_i-\mu)^2}{\chi^2_{(\frac{\alpha}{2},n)}} < \sigma^2 < \frac{\sum_{i=1}^{n}(X_i-\mu)^2}{\chi^2_{(1-\frac{\alpha}{2},n)}}\right) = 1-\alpha$$

where $\chi^2_{(\frac{\alpha}{2},n)}$ and $\chi^2_{(1-\frac{\alpha}{2},n)}$ are χ^2 values with n degrees of freedom.

Second Case (μ is unknown):

Now, we discuss the case when μ is not known. This case can be handled by making use of the facts from Theorem 2.8 that

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} \quad \text{or} \quad \frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{\sigma^2} \sim \chi^2_{n-1}$$

when the sample variance s^2 is computed. Then, for a fixed positive integer $n \geq 2$, we can find a 100 $(1-\alpha)$% confidence interval for σ^2 as

$$P\left(\chi^2_{(1-\frac{\alpha}{2},n-1)} < \frac{(n-1)S^2}{\sigma^2} < \chi^2_{(\frac{\alpha}{2},n-1)}\right) = 1-\alpha$$

$$P\left(\frac{(n-1)S^2}{\chi^2_{(\frac{\alpha}{2},n-1)}} < \sigma^2 < \frac{(n-1)S^2}{\chi^2_{(1-\frac{\alpha}{2},n-1)}}\right) = 1-\alpha$$

where $\chi^2_{(\frac{\alpha}{2},n-1)}$ and $\chi^2_{(1-\frac{\alpha}{2},n-1)}$ are χ^2 values with $n-1$ degrees of freedom.

Example 4.2:

Let $X_1, X_2, \dots, X_{25}$ be a random sample from normal distribution when the sample variance is equal to 2.3. Find 90% confidence interval for the population variance σ^2.

Solution:

We want to construct a confidence interval for σ^2 when the population is normal with unknown mean, thus we should use the following case

$$P\left(\frac{24(2.3)}{\chi^2_{(\frac{\alpha}{2},24)}} < \sigma^2 < \frac{24(2.3)}{\chi^2_{(1-\frac{\alpha}{2},24)}}\right) = 0.9$$

$$P\left(\frac{55.2}{\chi^2_{(\frac{\alpha}{2},24)}} < \sigma^2 < \frac{55.2}{\chi^2_{(1-\frac{\alpha}{2},24)}}\right) = 0.9$$

$$1-\alpha = 0.9 \Rightarrow \alpha = 0.1 \Rightarrow \frac{\alpha}{2} = 0.05 \Rightarrow 1-\frac{\alpha}{2} = 0.95$$

$$\Rightarrow \chi^2_{(0.05,24)} = 36.42 \quad \text{and} \quad \chi^2_{(0.95,24)} = 13.85$$

$$\Rightarrow \sigma^2 \in \left(\frac{55.2}{36.42}, \frac{55.2}{13.85}\right)$$

$$\Rightarrow \sigma^2 \in (1.5157, 3.9856)$$

4.2 Pivotal Quantity Method

Pivotal Quantity:

Let $X_1, X_2, \dots, X_n$ be a random sample from $f(x,\theta)$. Let $Q = q(X_1, \dots, X_n;\ \theta)$ be a function of $X_1, \dots, X_n$ and θ. If Q has a distribution that does not depend on θ, then Q is defined to be a **pivotal quantity**.

Example 4.3:

Let $X_1, X_2, \dots, X_n$ be a random sample from $N(\,\theta, 9)$. Then,

1. $\bar{X} - \ \theta \sim N\left(0, \frac{9}{n}\right)$ and $\frac{\bar{X}-\theta}{3/\sqrt{n}} \sim N(0,1)$ are pivotal quantities.
2. $\bar{X} - 2\theta \sim N\left(-\theta, \frac{1}{n}\right)$ is not pivotal quantity.

Pivotal Quantity Method:

If $Q = q(X_1, \dots, X_n;\ \theta)$ is apivotal quantity and has a probability distribution, then for any fixed $0 < \alpha < 1$ there will exist q_1 and q_2 such that $q_1 < q_2$ and

$$P(q_1 < Q < q_2) = 1 - \alpha$$

Therefore, we can find 100 $(1-\alpha)$% confidence interval for $\tau(\theta)$ as

$$P(t_1(x_1, \dots, x_n) < \tau(\theta) < t_2(x_1, \dots, x_n)) = 1 - \alpha$$

where t_1 and t_2 are functions of the random sample does not depend on θ.

Remark:

If $X_1, X_2, \dots, X_n$ is a random sample from $f(x, \theta)$, and the corresponding cumulative distribution function $F(x, \theta)$ is continuous in *x*. Then, a pivotal quantity can be given as

$$Q = -2\sum_{i=1}^{n} \log F(x_i, \theta) \sim \chi^2_{2n}$$

Then, the 100 $(1-\alpha)$% confidence interval for $\tau(\theta)$ is given as

$$P\left(\chi^2_{\left(1-\frac{\alpha}{2}, 2n\right)} < Q < \chi^2_{\left(\frac{\alpha}{2}, 2n\right)}\right) = 1 - \alpha$$

Example 4.4:

If $X_1, \dots, X_n$ be a random sample from the density function

$$f(x) = \theta\, x^{\theta - 1}, \quad 0 < x < 1$$

Find a pivotal quantity for θ and use it to construct $100(1-\alpha)$% confidence interval for θ.

Solution:

The cdf of x is given by

$$F(x) = \int_0^x \theta\, x^{\theta-1} dx = x^{\theta}, \quad 0 < x < 1$$

So, the pivotal quantity can be of the form

$$Q = -2\sum_{i=1}^{n} \log x_i^{\theta} = -2\theta \sum_{i=1}^{n} \log x_i$$

where $Q \sim \chi^2_{2n}$, then one can construct $100(1-\alpha)\%$ confidence interval for θ as

$$P\left(\chi^2_{(1-\frac{\alpha}{2},2n)} < Q < \chi^2_{(\frac{\alpha}{2},2n)}\right) = 1-\alpha$$

$$P\left(\chi^2_{(1-\frac{\alpha}{2},2n)} < -2\theta \sum_{i=1}^{n} \log x_i < \chi^2_{(\frac{\alpha}{2},2n)}\right) = 1-\alpha$$

$$P\left(\frac{-\chi^2_{(\frac{\alpha}{2},2n)}}{2\sum_{i=1}^{n} \log x_i} < \theta < \frac{-\chi^2_{(1-\frac{\alpha}{2},2n)}}{2\sum_{i=1}^{n} \log x_i}\right) = 1-\alpha$$

4.3 Large Sample Confidence Interval

From Section 3.3, the MLE $\hat{\theta}$ of θ, has an asymptotic normal distribution when n is large which is given by

$$\sqrt{n}(\hat{\theta}-\theta) \xrightarrow{d} N\left(0, \frac{1}{I(\theta)}\right) \quad \text{or} \quad \hat{\theta} \xrightarrow{d} N\left(\theta, \frac{1}{nI(\theta)}\right)$$

Thus, we can write

$$\frac{\hat{\theta}-\theta}{1/\sqrt{nI(\theta)}} \sim N(0,1)$$

Use the distribution of the MLE $\hat{\theta}$ to construct $100(1-\alpha)\%$ confidence interval for the parameter θ as following:

$$P\left(-z_{1-\frac{\alpha}{2}} < \frac{\hat{\theta}-\theta}{1/\sqrt{nI(\theta)}} < z_{1-\frac{\alpha}{2}}\right) = 1-\alpha$$

$$P\left(\hat{\theta} - z_{1-\frac{\alpha}{2}} \frac{1}{\sqrt{nI(\theta)}} < \theta < \hat{\theta} + z_{1-\frac{\alpha}{2}} \frac{1}{\sqrt{nI(\theta)}}\right) = 1-\alpha$$

In General:

Since the MLE $\hat{\tau}(\theta)$ of $\tau(\theta)$, has an asymptotic normal distribution when n is large as following:

$$\hat{\tau}(\theta) \xrightarrow{d} N\left(\tau(\theta), \frac{\left(\acute{\tau}(\theta)\right)^2}{nI(\theta)}\right)$$

$$\Rightarrow \frac{\hat{\tau}(\theta) - \tau(\theta)}{\acute{\tau}(\theta)/\sqrt{nI(\theta)}} \sim N(0,1)$$

Then, $100(1-\alpha)\%$ confidence interval for the parameter $\tau(\theta)$ is given by

$$P(-z_{1-\frac{\alpha}{2}} < \frac{\hat{\tau}(\theta) - \tau(\theta)}{\acute{\tau}(\theta)/\sqrt{nI(\theta)}} < z_{1-\frac{\alpha}{2}}) = 1-\alpha$$

$$P\left(\hat{\tau}(\theta) - z_{1-\frac{\alpha}{2}} \frac{\acute{\tau}(\theta)}{\sqrt{nI(\theta)}} < \tau(\theta) < \hat{\tau}(\theta) + z_{1-\frac{\alpha}{2}} \frac{\acute{\tau}(\theta)}{\sqrt{nI(\theta)}}\right) = 1-\alpha$$

Example 4.5:

Let $X \sim Exponential(\beta)$ with large sample size. Construct $100(1-\alpha)\%$ confidence interval for β.

Solution:

The pdf of the exponential distribution with parameter β is defined as

$$f(x,\beta) = \frac{1}{\beta} e^{-x/\beta}, \;\; x > 0$$

We found from Example 3.2 and Example 3.3 that the MLE of β is $\bar{X}$ and it is an unbiased estimator (i.e. $E(\bar{X}) = \beta$). Thus, $\bar{X}$ has an asymptotic normal distribution that is

$$\bar{X} \sim N\left(\beta, \frac{1}{nI(\theta)}\right)$$

Now we need to derive the Fisher information, $I(\theta)$:

$$\log f(x;\theta) = -\log \beta - \frac{x}{\beta}$$

$$\frac{\partial}{\partial \theta} \log f(x;\theta) = -\frac{1}{\beta} + \frac{x}{\beta^2}$$

$$I(\theta) = Var\left[\frac{\partial}{\partial\theta}\log f(x;\theta)\right] = Var\left[-\frac{1}{\beta} + \frac{X}{\beta^2}\right] = \frac{Var(X)}{\beta^2} = 1$$

The asymptotic normal distribution of the MLE is

$$\bar{X} \sim N\left(\beta, \frac{1}{n}\right) \quad \text{or} \quad \frac{\bar{X} - \beta}{1/\sqrt{n}} \sim N(0,1)$$

Thus, $100(1-\alpha)\%$ confidence interval for β is obtained as

$$P(-z_{1-\frac{\alpha}{2}} < \frac{\bar{X} - \beta}{1/\sqrt{n}} < z_{1-\frac{\alpha}{2}}) = 1 - \alpha$$

$$P\left(\bar{X} - z_{1-\frac{\alpha}{2}}\frac{1}{\sqrt{n}} < \beta < \bar{X} + z_{1-\frac{\alpha}{2}}\frac{1}{\sqrt{n}}\right) = 1 - \alpha$$