Recall:

The simple linear regression model and be formed in the matrix form as

$$Y = X\beta + \varepsilon,$$
$$E(\varepsilon) = 0, Var(\varepsilon) = \sigma^2 I,$$
$$E(Y) = X\beta, \quad Var(Y) = \sigma^2 I,$$

where

$$\underset{n\times 1}{\mathbf{Y}} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} \quad \underset{n\times 2}{\mathbf{X}} = \begin{bmatrix} 1 & X_1 \\ 1 & X_2 \\ \vdots & \vdots \\ 1 & X_n \end{bmatrix} \quad \underset{2\times 1}{\boldsymbol{\beta}} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} \quad \underset{n\times 1}{\boldsymbol{\varepsilon}} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

$$b = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} \widehat{\beta}_0 \\ \widehat{\beta}_1 \end{bmatrix} = \widehat{\beta} = (X'X)^{-1}X'Y$$

$$\widehat{Y} = Xb = X\widehat{\beta} = HY, \quad H = X(X'X)^{-1}X'$$

$$\begin{aligned} Var(e) &= Var((I-H)Y) = (I-H)\operatorname{var}(Y)(I-H) \\ &= (I-H)\sigma^2 I(I-H)' \\ &= \sigma^2 (I-H)I(I-H)' \\ &= \sigma^2 (I-H) = MSE(I-H), \end{aligned}$$

$$MSE = SSE/(n-2) = e'e/(n-2).$$

$$SSE = e'e$$

$$SSE = Y'Y - b'X'Y.$$

$$SSTO = Y'Y - \frac{1}{n}(Y'JY)$$

$$SSR = b'X'Y - \frac{1}{n}(Y'JY)$$

$$SSTO = \mathbf{Y}'\left[\mathbf{I} - \left(\frac{1}{n}\right)\mathbf{J}\right]\mathbf{Y}$$

$$SSE = \mathbf{Y}'(\mathbf{I} - \mathbf{H})\mathbf{Y}$$

$$SSR = \mathbf{Y}'\left[\mathbf{H} - \left(\frac{1}{n}\right)\mathbf{J}\right]\mathbf{Y}$$

Each of these sums of squares can now be seen to be of the form $\mathbf{Y}'\mathbf{AY}$, where the three $\mathbf{A}$ matrices are:

$$\mathbf{I} - \left(\frac{1}{n}\right)\mathbf{J}$$

$$\mathbf{I} - \mathbf{H}$$

$$\mathbf{H} - \left(\frac{1}{n}\right)\mathbf{J}$$

Inferences in Regression Analysis

Lemma

$$E(\hat{\beta}) = \beta$$

$$Var(\hat{\beta}) = MSE(X'X)^{-1}$$

Proof

$$\begin{aligned} E(\hat{\beta}) &= E(X'X)^{-1}X'Y = (X'X)^{-1}X'E(Y) \\ &= (X'X)^{-1}X'XB \\ &= I\beta \\ &= \beta. \end{aligned}$$

This show that the Least square estimate of β is an unbiased estimator.

$$
\begin{aligned}
Var(\widehat{\beta}) &= Var[(X'X)^{-1}X'Y] \\
&= (X'X)^{-1}X'Var(Y)[(X'X)^{-1}X']' \\
&= (X'X)^{-1}X'\sigma^2[(X'X)^{-1}X']' \\
&= \sigma^2(X'X)^{-1}X'X(X'X)^{-1} \\
&= \sigma^2(X'X)^{-1}I \\
&= \sigma^2(X'X)^{-1} \\
&= MSE(X'X)^{-1}.
\end{aligned}
$$

This can be written as

$$Var(\widehat{\beta}) = MSE(X'X)^{-1} =$$

$$
= \begin{bmatrix} \dfrac{\sigma^2}{n} + \dfrac{\sigma^2\bar{X}^2}{\sum(X_i - \bar{X})^2} & \dfrac{-\bar{X}\sigma^2}{\sum(X_i - \bar{X})^2} \\ \dfrac{-\bar{X}\sigma^2}{\sum(X_i - \bar{X})^2} & \dfrac{\sigma^2}{\sum(X_i - \bar{X})^2} \end{bmatrix}
$$

Or

$$Var(\widehat{\beta}) = MSE\,(X\,'X\,)^{-1}$$

$$= \begin{bmatrix} \dfrac{MSE}{n} + \dfrac{\bar{X}^2 MSE}{\sum(X_i - \bar{X})^2} & \dfrac{-\bar{X} MSE}{\sum(X_i - \bar{X})^2} \\ \dfrac{-\bar{X} MSE}{\sum(X_i - \bar{X})^2} & \dfrac{MSE}{\sum(X_i - \bar{X})^2} \end{bmatrix}$$

Example:

For Toluca Company example by matrix methods, we get

$$Var(\widehat{\beta}) = \sigma^2 (X\,'X\,)^{-1}$$

$$= MSE(\mathbf{X}'\mathbf{X})^{-1} = 2{,}384 \begin{bmatrix} .287475 & -.003535 \\ -.003535 & .00005051 \end{bmatrix}$$

$$= \begin{bmatrix} 685.34 & -8.428 \\ -8.428 & .12040 \end{bmatrix}$$

Mean Response

To estimate the mean response at X_h, let us define the vector:

$$\underset{2\times 1}{\mathbf{X}_h} = \begin{bmatrix} 1 \\ X_h \end{bmatrix} \qquad \text{or} \qquad \underset{1\times 2}{\mathbf{X}'_h} = [1 \quad X_h]$$

The fitted value in matrix notation then is:

$$\hat{Y}_h = \mathbf{X}'_h \mathbf{b}$$

since:

$$\mathbf{X}'_h \mathbf{b} = [1 \quad X_h] \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = [b_0 + b_1 X_h] = [\hat{Y}_h] = \hat{Y}_h$$

Note that $\mathbf{X}'_h \mathbf{b}$ is a 1×1 matrix; hence, we can write the final result as a scalar.

$$\begin{aligned} Var(\widehat{Y}_h) &= Var(X'_h b) = Var[X'_h (X'X)^{-1} X'Y] \\ &= X'_h (X'X)^{-1} X' Var(Y) [X'_h (X'X)^{-1} X']' \\ &= X'_h (X'X)^{-1} X' \sigma^2 [X'_h (X'X)^{-1} X']' \\ &= \sigma^2 X'_h (X'X)^{-1} X'X (X'X)^{-1} X_h \\ &= \sigma^2 X'_h (X'X)^{-1} X_h \\ &= MSE (X'_h (X'X)^{-1} X_h). \end{aligned}$$

Example

For Toluca Company example, the variance of the mean of the response when X=65 can be calculated using the matrix form as

$$Var(\widehat{Y}_h) = X'_h Var(b) X_h$$

$$= \begin{bmatrix} 1 & 65 \end{bmatrix} \begin{bmatrix} 685.34 & -8.428 \\ -8.428 & .12040 \end{bmatrix} \begin{bmatrix} 1 \\ 65 \end{bmatrix} = 98.37$$

Which is the same result as that obtained before.

Prediction of New Observation

Proceeding similarly, we get

$$Var(\widehat{Y}_{new}) = MSE\,(1 + X'_h (X'X)^{-1} X_h).$$