

Mathematical Models of Systems

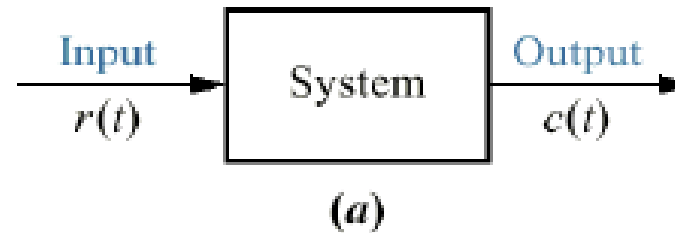
- Development of mathematical models from schematics of physical systems.

by applying the fundamental physical laws of science and engineering.

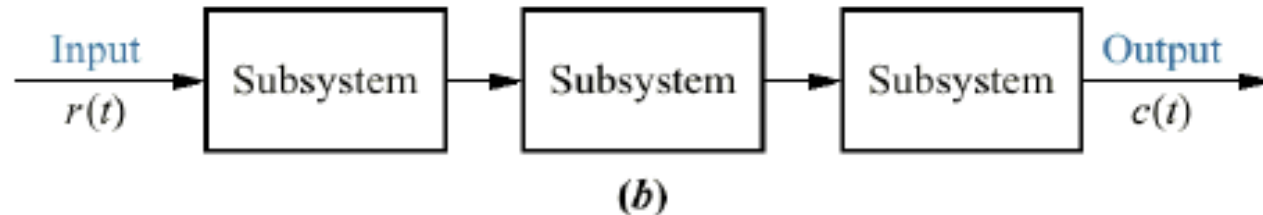
- Two Methods:**
 - Transfer functions* in the frequency domain,
 - State equations* in the time domain.

a. Block diagram representation of a system;

b. block diagram Representation of an interconnection of subsystems



From mathematical model equations we will obtain the relationship between the system's output and input.



Note: The input, $r(t)$, stands for *reference input*.
The output, $c(t)$, stands for *controlled variable*.

- Transfer function* (mathematical function), is inside each block.

Modeling in Frequency Domain

Laplace Transform Review

- A system represented by a differential equation is difficult to model as a block diagram.
- A differential equation can describe the relationship between the input and output of a system.
- By using *Laplace transform* we can represent the input, output, and system as separate entities.

$$A(s)Y(s) = B(s)U(s)$$

Laplace transform can be defined as:

$$\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st} dt$$

Where $s = \sigma + j\omega$, a complex variable

Inverse Laplace transform:

$$\mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j} \int_{\sigma-j\omega}^{\sigma+j\omega} F(s)e^{st} ds = f(t)u(t)$$

$$\text{Where, } \begin{aligned} u(t) &= 1 & t > 0 \\ &= 0 & t < 0 \end{aligned}$$

Multiplication of $f(t)$ by $u(t)$ yields a time function that is zero for $t < 0$.

Laplace Transform Table

Table 2.1
Laplace transform table

Item no.	$f(t)$	$F(s)$
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2 + \omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2 + \omega^2}$

Problem: Find the Laplace transform of

$$f(t) = Ae^{-at}u(t)$$

Solution:

$$\begin{aligned}
 F(s) &= \int_0^{\infty} f(t)e^{-st} dt = \int_0^{\infty} Ae^{-at}e^{-st} dt \\
 &= A \int_0^{\infty} e^{-(s+a)t} dt = -\frac{A}{s+a} e^{-(s+a)t} \Big|_{t=0}^{\infty} \\
 &= \frac{A}{s+a}
 \end{aligned}$$



Laplace Transform Theorems

Table 2.2

Item no.	Theorem	Name
1.	$\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st} dt$	Definition
2.	$\mathcal{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathcal{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathcal{L}[e^{-at}f(t)] = F(s + a)$	Frequency shift theorem
5.	$\mathcal{L}[f(t - T)] = e^{-sT}F(s)$	Time shift theorem
6.	$\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - f'(0-)$	Differentiation theorem
9.	$\mathcal{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - \sum_{k=1}^n s^{n-k}f^{(k-1)}(0-)$	Differentiation theorem
10.	$\mathcal{L}\left[\int_{0-}^t f(\tau)d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s \rightarrow 0} sF(s)$	Final value theorem ¹
12.	$f(0+) = \lim_{s \rightarrow \infty} sF(s)$	Initial value theorem ²

¹For this theorem to yield correct finite results, all roots of the denominator of $F(s)$ must have negative real parts, and no more than one can be at the origin.

²For this theorem to be valid, $f(t)$ must be continuous or have a step discontinuity at $t = 0$ (that is, no impulses or their derivatives at $t = 0$).

Inverse Laplace Transform

Problem: Find inverse Laplace Transform of

$$F_1(s) = \frac{1}{(s + 3)^2}$$

Solution:

From item 3 and 5 of Table 2.1,

$$f_1(t) = e^{-3t}tu(t)$$

Frequency shift theorem item 4 of Table 2.2,

$$\mathcal{L}[e^{-at}f(t)] = F(s + a).$$

Inverse Laplace: Partial-Fraction Expansion

A partial-fraction expansion: transform of a complicated function to a sum of simpler terms for which we know the Laplace transform of each term.

Case 1. (Roots of the Denominator of F(s) Are Real and Distinct)

Problem: $F_1(s) = \frac{s^3 + 4s^2 + 6s + 5}{s^2 + 3s + 2} = \frac{N(s)}{D(s)}$

The order of N(s) > order of D(s) we must perform the division until we obtain a remainder whose numerator is of order less than its denominator

Solution:

$$\begin{array}{r} s + 1 \\ s^2 + 3s + 2 \overline{) s^3 + 4s^2 + 6s + 5} \\ \underline{-(s^3 + 3s^2 + 2s)} \\ s^2 + 4s + 5 \\ \underline{-(s^2 + 3s + 2)} \\ s + 3 \end{array}$$

Partial-Fraction Expansion

$$F(s) = \frac{s + 3}{(s + 1)(s + 2)} = \frac{K_1}{(s + 1)} + \frac{K_2}{(s + 2)}$$

To find K_1 , multiply by $(s + 1)$. Thus

$$\frac{s + 3}{(s + 2)} = K_1 + \frac{(s + 1)K_2}{(s + 2)}$$

Letting $s = -1$, $K_1 = 2$

Similarly, $K_2 = -2$

$$F_1(s) = s + 1 + \frac{2}{s^2 + 3s + 2}$$

Taking the inverse
Laplace transform,
(Table 2.2 Item 7)

$$f_1(t) = \frac{d\delta(t)}{dt} + \delta(t) + \mathfrak{I}^{-1} \left[\frac{s + 3}{(s + 1)(s + 2)} \right]$$

$$f(t) = (2e^{-t} - 2e^{-2t})u(t)$$

Final solution: $f_1(t) = \frac{d\delta(t)}{dt} + \delta(t) + (2e^{-t} - 2e^{-2t})u(t)$

Laplace Transform Solution of a Differential Equation

Problem: Solve for $y(t)$, if all initial conditions are zero.

$$\frac{d^2 y}{dt^2} + 12 \frac{dy}{dt} + 32y = 32u(t)$$

Solution: The Laplace transform is,

(Table 2.2 Item 8)



$$s^2 Y(s) + 12sY(s) + 32Y(s) = \frac{32}{s}$$

$$Y(s) = \frac{32}{s(s^2 + 12s + 32)} = \frac{32}{s(s+4)(s+8)}$$

$$= \frac{K_1}{s} + \frac{K_2}{(s+4)} + \frac{K_3}{(s+8)}$$



$$K_1 = \left. \frac{32}{(s+4)(s+8)} \right|_{s \rightarrow 0} = 1$$

$$K_2 = \left. \frac{32}{s(s+8)} \right|_{s \rightarrow -4} = -2$$

$$K_3 = \left. \frac{32}{s(s+4)} \right|_{s \rightarrow -8} = 1$$

Taking inverse Laplace transform, we get

$$y(t) = (1 - 2e^{-4t} + e^{-8t})u(t)$$

inverse Laplace transform



$$\text{Hence, } Y(s) = \frac{1}{s} - \frac{2}{(s+4)} + \frac{1}{(s+8)}$$

Inverse Laplace: Partial-Fraction Expansion

Case 2. (Roots of the Denominator of $F(s)$ Are Real and Repeated)

Problem: Find inverse Laplace transform of

$$F(s) = \frac{2}{(s+1)(s+2)^2} \quad (1)$$

Solution:

We can write the partial-fraction expansion as a sum of terms

$$F(s) = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)^2} + \frac{K_3}{(s+2)} \quad (2)$$

reduced
multiplicity

$$K_1 = \left. \frac{2}{(s+2)^2} \right|_{s \rightarrow -1} = 2$$

To find K_2 , multiply (1) = (2) by $(s+2)^2$

$$\frac{2}{(s+1)} = (s+2)^2 \frac{K_1}{(s+1)} + K_2 + (s+2)K_3 \quad (3)$$

Letting $s \rightarrow -2$, we obtain $K_2 = -2$

To find K_3 , differentiate (3) w.r.t. s :

$$\frac{-2}{(s+1)^2} = \frac{(s+2)s}{(s+1)^2} K_1 + K_3$$

Letting $s \rightarrow -2$, we obtain $K_3 = -2$

Therefore, inverse Laplace transform is:

$$f(t) = 2e^{-t} - 2te^{-2t} - 2e^{-2t}$$

For repeated roots with multiplicity r , we have

$$F(s) = \frac{K_1}{(s+p_1)^r} + \frac{K_2}{(s+p_1)^{r-1}} + \cdots + \frac{K_r}{(s+p_1)}$$

$$K_i = \frac{1}{(i-1)!} \left. \frac{d^{i-1} F_1(s)}{ds^{i-1}} \right|_{s \rightarrow -p_1} \quad i = 1, 2, \dots, r;$$

$$F_1(s) = (s+p_1)^r F(s)$$

Inverse Laplace: Partial-Fraction Expansion

Case 3. (Roots of the Denominator of F(s) Are Complex or Imaginary)

Problem: Find inverse Laplace transform of

$$F(s) = \frac{3}{s(s^2 + 2s + 5)}$$

Solution:

This function can be expanded in the following form:

$$\frac{3}{s(s^2 + 2s + 5)} = \frac{K_1}{s} + \frac{K_2s + K_3}{s^2 + 2s + 5} \quad (1)$$

K_1 is found in the usual way: $\left. \frac{3}{s^2 + 2s + 5} \right|_{s \rightarrow 0} = \frac{3}{5} \equiv K_1$

To find K_2 and K_3 :

Multiply (1) by $s(s^2 + 2s + 5)$, and put $K_1 = 3/5$

$$\Rightarrow 3 = \frac{3}{5}(s^2 + 2s + 5) + K_2s^2 + K_3s$$

$$\Rightarrow 3 = \left(K_2 + \frac{3}{5}\right)s^2 + \left(K_3 + \frac{6}{5}\right)s + 3$$

Balancing coefficients(matching)

$$K_2 + \frac{3}{5} = 0$$

$$K_3 + \frac{6}{5} = 0$$

Hence,

$$\Rightarrow K_2 = -\frac{3}{5}, \text{ and } K_3 = -\frac{6}{5}$$

$$F(s) = \frac{3/5}{s} - \frac{3}{5} \frac{s+2}{s^2 + 2s + 5}$$

Using Item 7 in Table 2.1 and Items 2 and 4 in Table 2.2, we get

$$\begin{aligned} \Im[Ae^{-at} \cos \omega t] &= \frac{A(s+a)}{(s+a)^2 + \omega^2} \\ \Im[Be^{-at} \sin \omega t] &= \frac{B\omega}{(s+a)^2 + \omega^2} \end{aligned}$$

$$\text{Adding, } \Im[Ae^{-at} \cos \omega t + Be^{-at} \sin \omega t] = \frac{A(s+a) + B\omega}{(s+a)^2 + \omega^2}$$

$$\text{We have, } s^2 + 2s + 5 = s^2 + 2s + 1 + 4 = (s+1)^2 + 2^2$$

$$\Rightarrow a = 1 \text{ and } \omega = 2$$

$$\Rightarrow F(s) = \frac{3/5}{s} - \frac{3}{5} \frac{(s+1) + (1/2)(2)}{(s+1)^2 + 2^2}$$

$$f(t) = \frac{3}{5} - \frac{3}{5}e^{-t} \left(\cos 2t + \frac{1}{2} \sin 2t \right)$$

Transfer Function

- A *Transfer Function* is the ratio of the output of a system to the input of a system. It allows us to algebraically combine mathematical representations of subsystems to yield a total system representation.



- General n^{th} order, linear time-invariant differential equation:

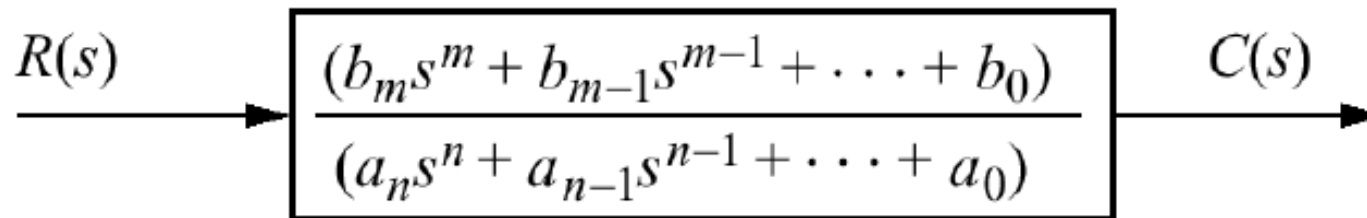
$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \dots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \dots + b_0 r(t) \quad \text{c: output, r: input}$$

Taking Laplace transform,

$$a_n s^n C(s) + a_{n-1} s^{n-1} C(s) + \dots + a_0 C(s) + [\text{Initial condition is zero}] = b_m s^m R(s) + b_{m-1} s^{m-1} R(s) + \dots + b_0 R(s) + [\text{Initial condition is zero}]$$

Transfer function:

$$\frac{C(s)}{R(s)} = G(s) = \frac{(b_m s^m + b_{m-1} s^{m-1} + \dots + b_0)}{(a_n s^n + a_{n-1} s^{n-1} + \dots + a_0)}$$



Block diagram of a transfer function

We can find the output

$$C(s) = G(s)R(s)$$

Transfer Function for a Differential Equation

Problem 1: Find the transfer function represented by $\frac{dc(t)}{dt} + 2c(t) = r(t)$

Solution:

Taking Laplace transform and assuming zero initial conditions, we have

$$sC(s) + 2C(s) = R(s)$$

Transfer function, $G(s)$, $\Rightarrow G(s) = \frac{C(s)}{R(s)} = \frac{1}{s+2}$

Problem 2: Find the transfer function represented by $\frac{d^3c}{dt^3} + 3\frac{d^2c}{dt^2} + 7\frac{dc}{dt} + 5c = \frac{d^2r}{dt^2} + 4\frac{dr}{dt} + 3r$.

Solution:

$$s^3C(s) + 3s^2C(s) + 7sC(s) + 5C(s) = s^2R(s) + 4sR(s) + 3R(s)$$
$$\Rightarrow G(s) = \frac{C(s)}{R(s)} = \frac{s^2 + 4s + 3}{s^3 + 3s^2 + 7s + 5}$$

Problem Solving

Problem: Find the ramp response for a system whose transfer function is

$$G(s) = \frac{s}{(s+4)(s+8)}$$

Solution:

The input (ramp)

$$C(s) = R(s)G(s) = \frac{1}{s^2} \times \frac{s}{(s+4)(s+8)} = \frac{1}{s(s+4)(s+8)}$$

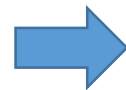
$$\Rightarrow C(s) = \frac{A}{s} + \frac{B}{(s+4)} + \frac{C}{(s+8)}$$

$$A = \left. \frac{1}{(s+4)(s+8)} \right|_{s \rightarrow 0} = \frac{1}{32}$$

$$B = \left. \frac{1}{s(s+8)} \right|_{s \rightarrow -4} = -\frac{1}{16}$$

$$C = \left. \frac{1}{s(s+4)} \right|_{s \rightarrow -8} = \frac{1}{32}$$

Hence,



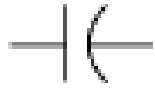


$$c(t) = \frac{1}{32} - \frac{1}{16}e^{-4t} + \frac{1}{32}e^{-8t}$$

Electric Network Transfer Functions

Apply the transfer function to the mathematical modeling of electronic circuits including passive networks and O-Amp circuits.

Table 2.3

Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

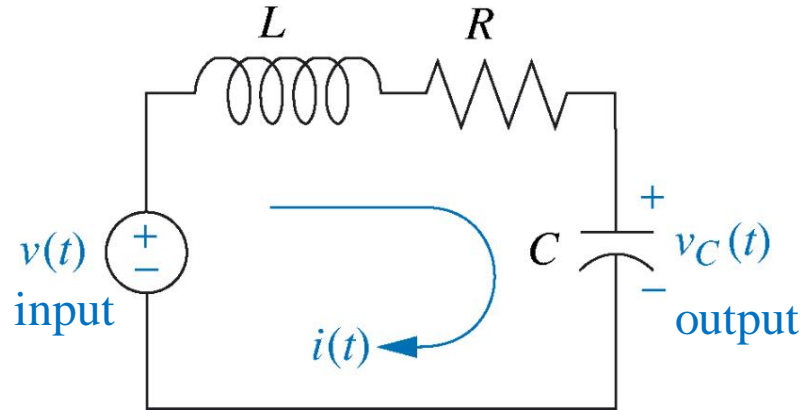
 Capacitor	$v(t) = \frac{1}{C} \int_0^t i(\tau) d\tau$	$i(t) = C \frac{dv(t)}{dt}$	$v(t) = \frac{1}{C} q(t)$	$\frac{1}{Cs}$	Cs
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R} v(t)$	$v(t) = R \frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
 Inductor	$v(t) = L \frac{di(t)}{dt}$	$i(t) = \frac{1}{L} \int_0^t v(\tau) d\tau$	$v(t) = L \frac{d^2 q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t) = V$ (volts), $i(t) = A$ (amps), $q(t) = Q$ (coulombs), $C = F$ (farads), $R = \Omega$ (ohms), $G = \mathcal{U}$ (mhos), $L = H$ (henries).

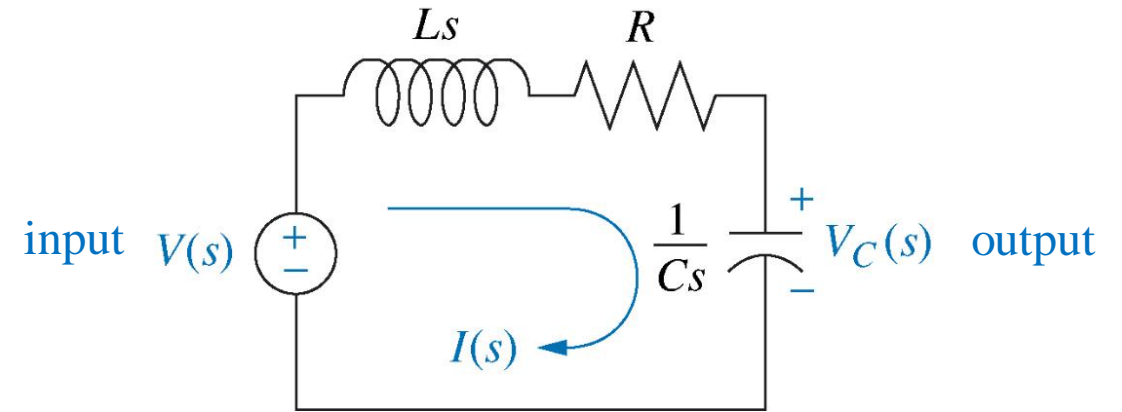
Transfer Function: Single Loop

1

Problem: Find the transfer function relating capacitor voltage, $V_C(s)$, to input voltage, $V(s)$.



Laplace-transform



Summing the voltages around the loop,

$$L \frac{di(t)}{dt} + Ri(t) + \frac{1}{C} \int_0^t i(\tau) d\tau = V(t)$$

take the Laplace transform

$$V_C(s) = \frac{1}{Cs} I(s) \quad V_L(s) = Ls I(s) \quad V_R(s) = R I(s)$$

Capacitor

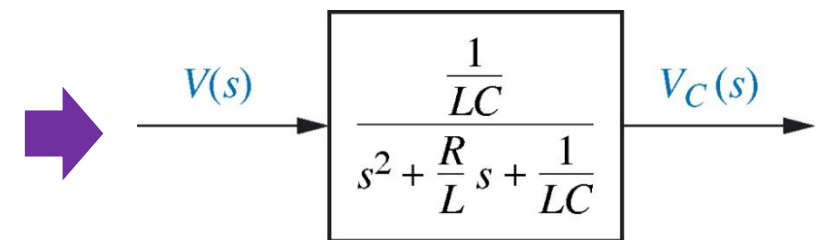
Inductor

Resistor

$$\left(Ls + R + \frac{1}{Cs} \right) I(s) = V(s) \Rightarrow \frac{I(s)}{V(s)} = \frac{1}{Ls + R + \frac{1}{Cs}}$$

$$\text{We know, } V_C(s) = I(s) \frac{1}{Cs} \Rightarrow \frac{V_C(s)}{V(s)} = \frac{I(s)}{V(s)} \times \frac{1}{Cs}$$

$$\begin{aligned} \Rightarrow V_C(s) &= \frac{V(s)}{Ls + R + \frac{1}{Cs}} \times \frac{1}{Cs} \\ \Rightarrow \frac{V_C(s)}{V(s)} &= \frac{1}{Ls + R + \frac{1}{Cs}} \times \frac{1}{Cs} = \frac{Cs}{s^2 LC + sRC + 1} \times \frac{1}{Cs} \\ &= \frac{1}{s^2 LC + sRC + 1} \\ &= \frac{1/LC}{s^2 + s \frac{R}{L} + \frac{1}{LC}} \end{aligned}$$



Transfer Function: Single Node

2

Transfer functions also can be obtained using Kirchhoff's current law and summing currents flowing from nodes. currents leaving the node are positive and currents entering the node are negative.

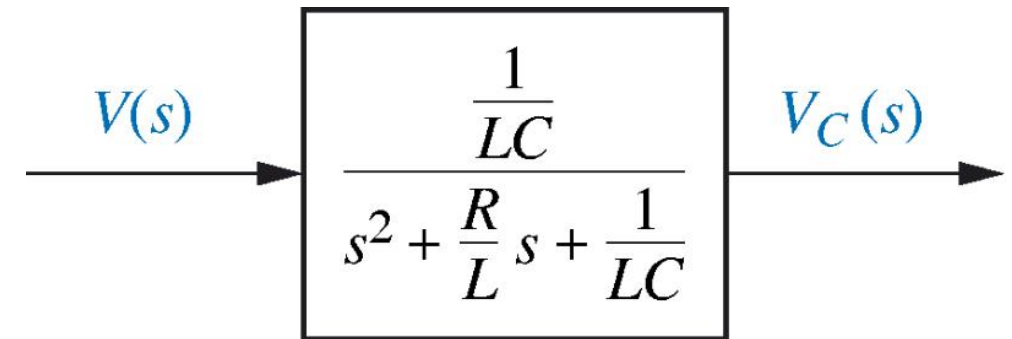
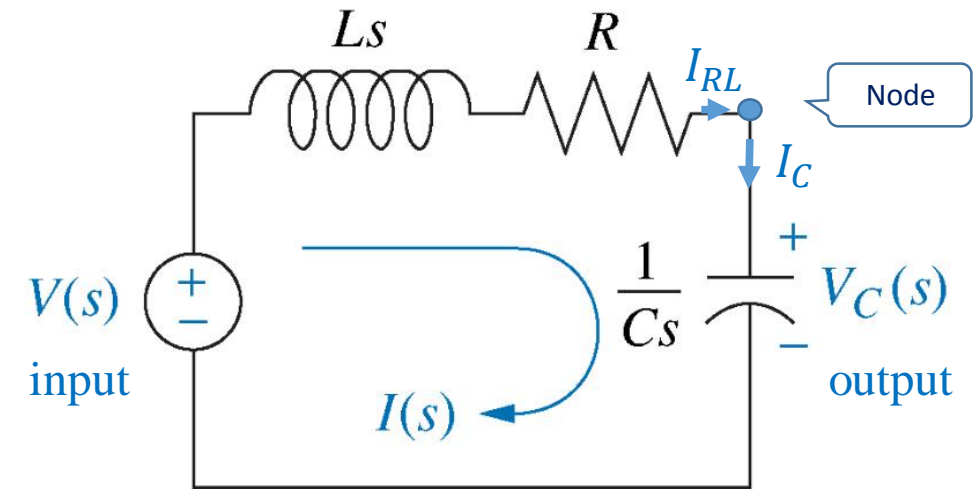
$$\sum I_{in} = \sum I_{out} \quad \Rightarrow \quad I_c(s) = I_{RL}(s) \quad \text{Same current}$$

$$\begin{aligned} \Rightarrow I_c(s) - I_{RL}(s) &= 0 \quad \Rightarrow \quad \frac{V_c}{Z_c} - \frac{V_{RL}}{Z_{RL}} = 0 \\ \Rightarrow \frac{V_c(s)}{\frac{1}{Cs}} + \frac{V_c(s) - V(s)}{R + Ls} &= 0 \quad \text{V}_{RL} = -(V_c - V) \end{aligned}$$

$$\Rightarrow V_c(s) \left(Cs + \frac{1}{R + Ls} \right) = \frac{V(s)}{R + Ls}$$

$$\Rightarrow \frac{V_c(s)}{V(s)} = \frac{1}{s^2 LC + sRC + 1}$$

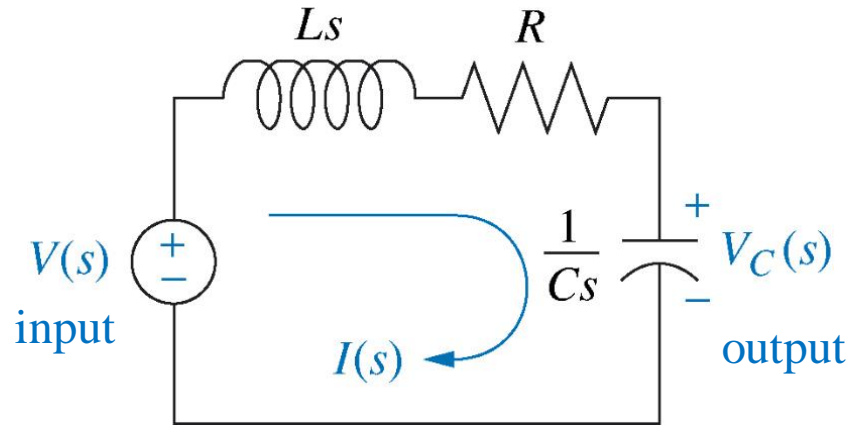
$$= \frac{1/LC}{s^2 + s \frac{R}{L} + \frac{1}{LC}}$$



Transfer Function: Single Loop via Voltage Division

3

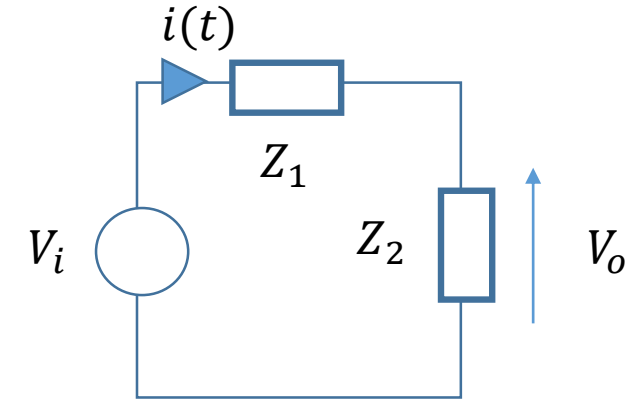
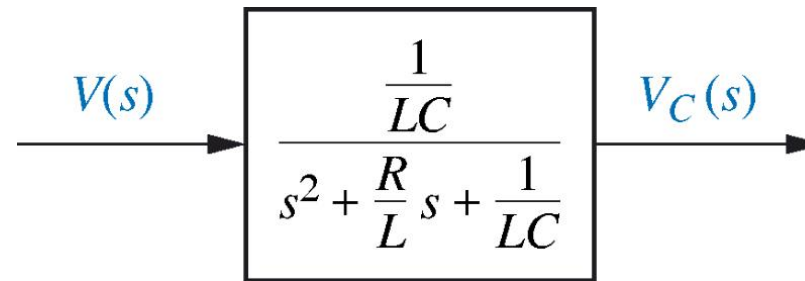
Voltage across capacitor is some proportion of the input voltage.



$$V_C(s) = \frac{1/Cs}{\left(Ls + R + \frac{1}{Cs}\right)} V(s)$$

$$\Rightarrow \frac{V_C(s)}{V(s)} = \frac{1/LC}{s^2 + s\frac{R}{L} + \frac{1}{LC}}$$

$$\rightarrow \frac{\text{Impedance of the capacitor}}{\text{Sum of the impedances}}$$



$$V_i = (Z_1 + Z_2) i(t) \quad (1)$$

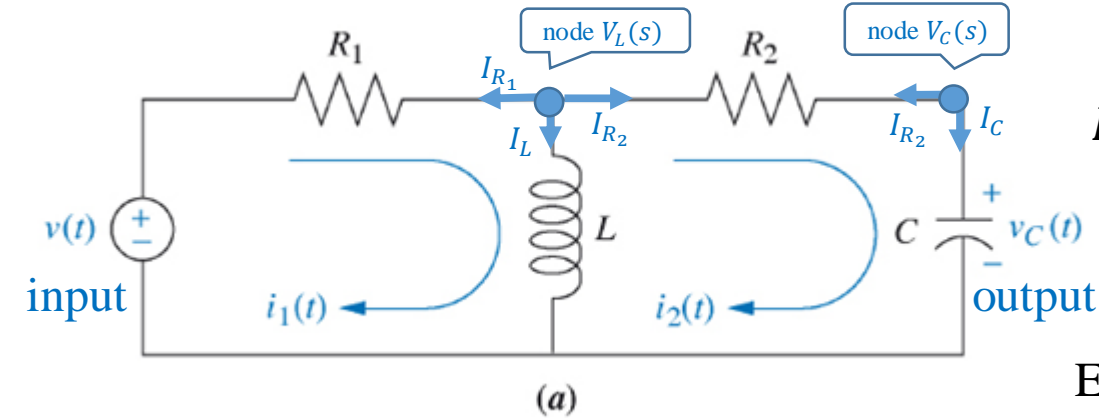
$$V_o = Z_2 i(t) \quad (2)$$

$$\frac{(2)}{(1)} \rightarrow \frac{V_o}{V_i} = \frac{Z_2}{(Z_1 + Z_2)}$$

Which one is the easiest? Method 1, method 2, or method 3?

Complex Circuits via Nodal Analysis₁

Complex electrical networks (those with multiple loops and nodes). We use nodal analysis to find the *transfer function* $\frac{V_C(s)}{V(s)}$



sum currents at the nodes

$$I_{R_1} + I_L + I_{R_2} = 0 \Rightarrow \frac{V_L(s) - V(s)}{R_1} + \frac{V_L(s)}{Ls} + \frac{V_L(s) - V_C(s)}{R_2} = 0$$

Current from node $V_L(s)$

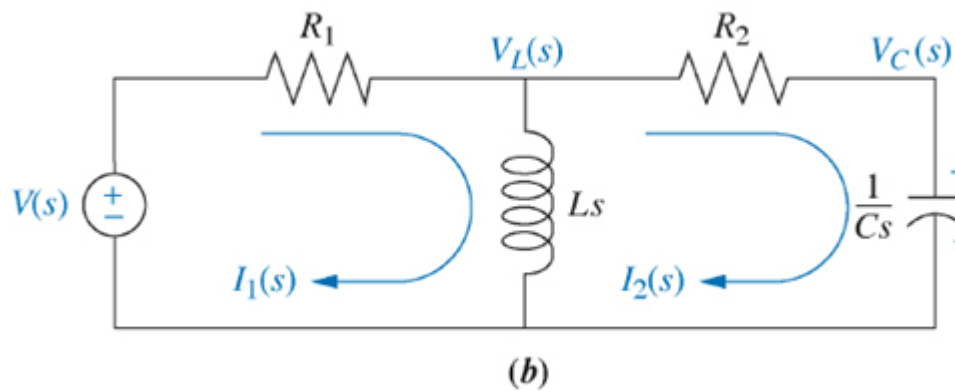
$$\text{And } I_C + I_{R_2} = 0 \Rightarrow CsV_C(s) + \frac{V_C(s) - V_L(s)}{R_2} = 0$$

Current from node $V_C(s)$

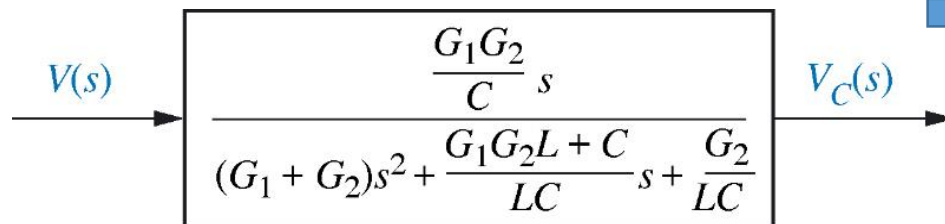
Expressing resistances as conductances, $G=1/R$

$$\left(G_1 + G_2 + \frac{1}{Ls}\right)V_L(s) - G_2V_C(s) = V(s)G_1 \quad (1)$$

$$-G_2V_L(s) + (G_2 + Cs)V_C(s) = 0 \Rightarrow V_L(s) = \frac{G_2 + Cs}{G_2} V_C(s) \quad (2)$$



(2) in (1) $\Rightarrow \frac{V_C(s)}{V(s)} = \frac{G_1 G_2 L s}{(G_1 + G_2) L C s^2 + (C + G_2(G_1 + G_2)L - G_2^2 L) s + G_2}$



Transfer function:

$$\frac{V_C(s)}{V(s)} = \frac{\frac{G_1 G_2}{C} s}{(G_1 + G_2) s^2 + \frac{G_1 G_2 L + C}{LC} s + \frac{G_2}{LC}}$$

Divide by LC

Complex Circuits - Mesh Equations via Inspection₂

For Mesh 1: $\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{around} \\ \text{Mesh 1} \end{bmatrix} \times I_1(s) - \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{common to} \\ \text{Mesh 1 and mesh2} \end{bmatrix} \times I_2(s) - \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{common to} \\ \text{Mesh 1 and mesh3} \end{bmatrix} \times I_3(s) = \begin{bmatrix} \text{Sum of applied} \\ \text{voltages around} \\ \text{Mesh 1} \end{bmatrix}$

For Mesh 2: $-\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{common to} \\ \text{Mesh 1 and mesh2} \end{bmatrix} \times I_1(s) + \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{around} \\ \text{Mesh 2} \end{bmatrix} \times I_2(s) - \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{common to} \\ \text{Mesh 2 and mesh3} \end{bmatrix} \times I_3(s) = \begin{bmatrix} \text{Sum of applied} \\ \text{voltages around} \\ \text{Mesh 2} \end{bmatrix}$

For Mesh 3: $-\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{common to} \\ \text{Mesh 1 and mesh3} \end{bmatrix} \times I_1(s) - \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{common to} \\ \text{Mesh 2 and mesh3} \end{bmatrix} \times I_2(s) + \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{around} \\ \text{Mesh 3} \end{bmatrix} \times I_3(s) = \begin{bmatrix} \text{Sum of applied} \\ \text{voltages around} \\ \text{Mesh 3} \end{bmatrix}$

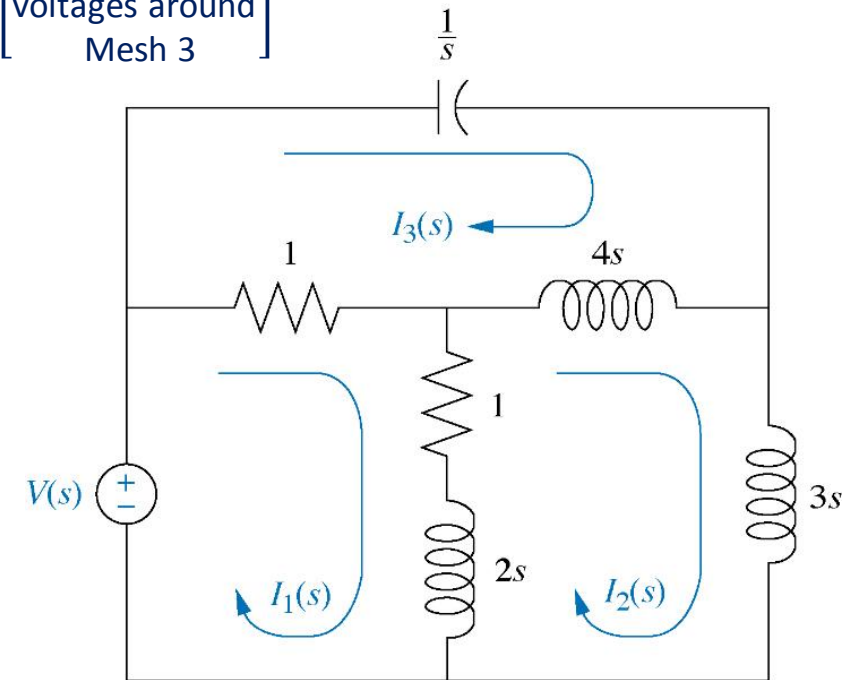
Similarly, Meshes 3, we obtain

For Mesh 1: $+(2s+2)I_1(s) - (2s+1)I_2(s) - I_3(s) = V(s)$

For Mesh 2: $-(2s+1)I_1(s) + (9s+1)I_2(s) - 4sI_3(s) = 0$

For Mesh 3: $-I_1(s) - 4sI_2(s) + \left(4s+1+\frac{1}{s}\right)I_3(s) = 0$

which can be solved simultaneously for any desired transfer function, for example, $I_3(s)/V(s)$



Three loop
electrical network

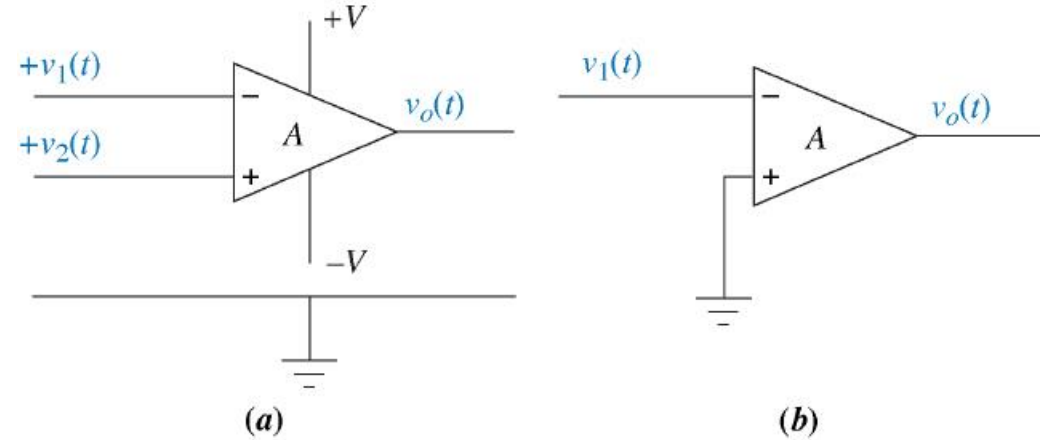
Operational Amplifier

An *operational amplifier* is an electronic amplifier used as a basic building block to implement transfer functions. It has the following characteristics:

1. Differential input, $v_2(t) - v_1(t)$
2. High input impedance, $Z_i = \infty$ (*ideal*)
3. Low output impedance, $Z_o = 0$ (*ideal*)
4. High constant gain amplification, $A = \infty$ (*ideal*)

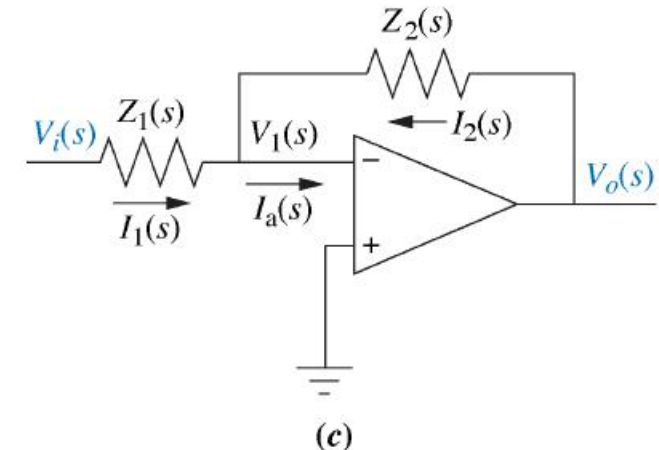
The output, $v_o(t)$, is given by: $v_o(t) = A(v_2(t) - v_1(t))$

- a. Operational amplifier;
- b. schematic for an inverting operational amplifier;
- c. Inverting operational amplifier configured for transfer function realization. Typically, the amplifier gain, A , is omitted.



$I_1(s) = -I_2(s)$, as $I_a(s) = 0$, because of high input impedance

$$\left. \begin{array}{l} V_i(s) = Z_1(s)I_1(s) \\ V_o(s) = Z_2(s)I_2(s) \\ I_1(s) = -I_2(s) \end{array} \right\} \Rightarrow \boxed{\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)}}$$



Problem Solving

Inverting Operational Amplifier

Problem: Find the transfer function, $\frac{V_o(s)}{V_i(s)}$, for the circuit below.

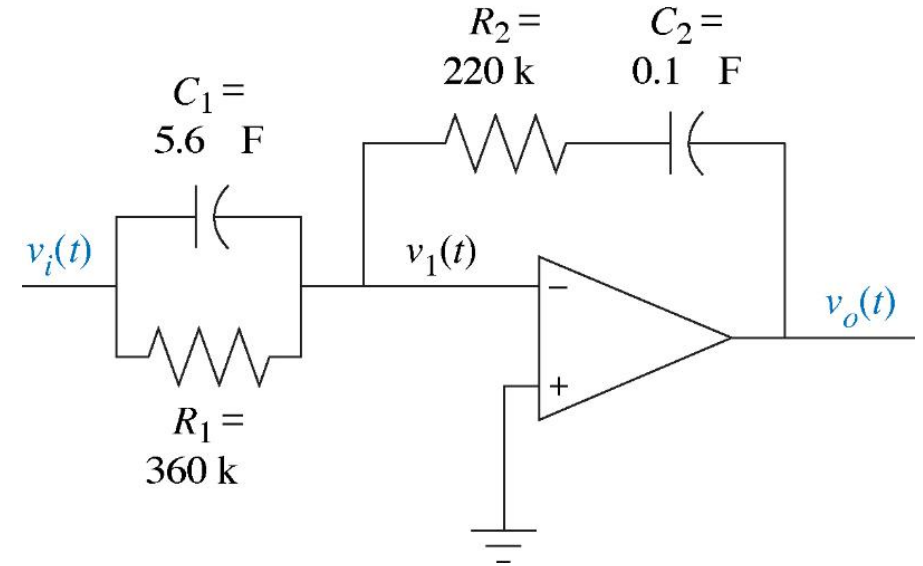
Solution:

For parallel components, $Z_1(s)$ is the reciprocal of the sum of the admittances.

$$Z_1(s) = \frac{1}{C_1 s + \frac{1}{R_1}} = \frac{1}{5.6 \times 10^{-6} s + \frac{1}{360 \times 10^3}}$$

For serial components, $Z_2(s)$ is the sum of the impedances.

$$Z_2(s) = R_2 + \frac{1}{C_2 s} = 220 \times 10^3 + \frac{10^7}{s}$$



$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)} = -1.232 \times \frac{s^2 + 45.95s + 22.55}{s}$$

Non-inverting Operational Amplifier

Using voltage division,

$$\left. \begin{aligned} V_o(s) &= [Z_1(s) + Z_2(s)] I(s) \\ V_i(s) &= Z_1(s) I(s) \end{aligned} \right\} \Rightarrow V_1(s) = \frac{Z_1(s)}{Z_1(s) + Z_2(s)} V_o(s)$$

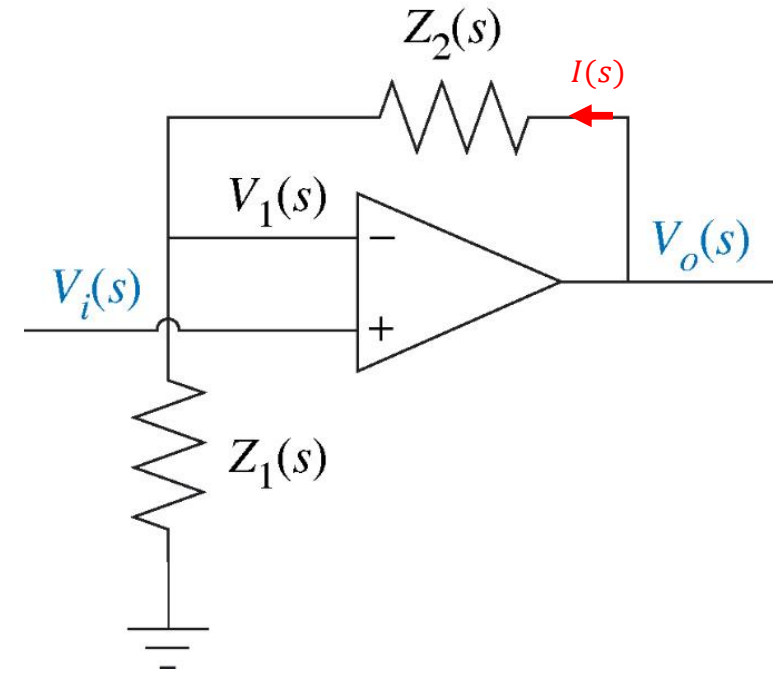
We have: $V_o(s) = A(V_i(s) - V_1(s)) \Rightarrow V_o(s) = A \left[V_i(s) - \frac{Z_1(s)}{Z_1(s) + Z_2(s)} V_o(s) \right]$

$$\Rightarrow V_o(s) \left[1 + A \frac{Z_1(s)}{Z_1(s) + Z_2(s)} \right] = A V_i(s)$$

$$\Rightarrow \frac{V_o(s)}{V_i(s)} = \frac{A}{1 + AZ_1(s)/(Z_1(s) + Z_2(s))}$$

For large A, we disregard '1' in the denominator.

$$\frac{V_o(s)}{V_i(s)} = \frac{Z_1(s) + Z_2(s)}{Z_1(s)}$$



Problem Solving

Non-Inverting Operational Amplifier

PROBLEM: Find the transfer function, $V_o(s)/V_i(s)$, for the Non-inverting operational amplifier circuit

SOLUTION:

We find each of the impedance functions,

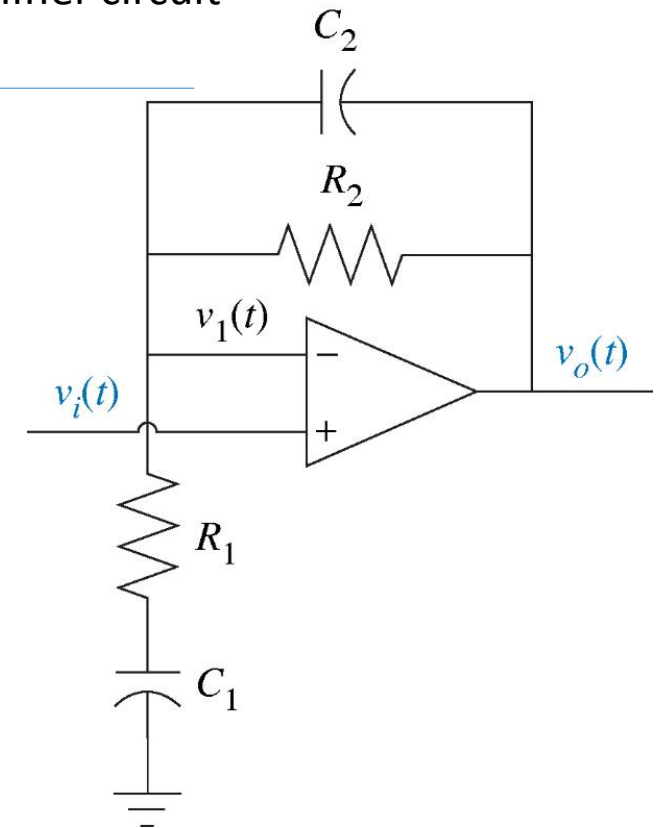
$$Z_1(s) = R_1 + \frac{1}{C_1 s} \quad \text{and} \quad Z_2(s) = \frac{1}{C_2 s + \frac{1}{R_2}} = \frac{R_2}{R_2 C_2 s + 1} = \frac{R_2 (1/C_2 s)}{R_2 + (1/C_2 s)}$$

Now use the following equation: $\frac{V_o(s)}{V_i(s)} = \frac{Z_1(s) + Z_2(s)}{Z_1(s)}$

$$Z_1 + Z_2 = \frac{R_1 C_1 s + 1}{C_1 s} + \frac{R_2}{R_2 C_2 s + 1} = \frac{(R_1 C_1 s + 1)(R_2 C_2 s + 1) + R_2 C_1 s}{C_1 s (R_2 C_2 s + 1)}$$

$$\Rightarrow \frac{Z_1 + Z_2}{Z_1} = \frac{(R_1 C_1 s + 1)(R_2 C_2 s + 1) + R_2 C_1 s}{C_1 s (R_2 C_2 s + 1)} \times \frac{C_1 s}{R_1 C_1 s + 1}$$

Substituting yields $\frac{V_o(s)}{V_i(s)} = \frac{C_2 C_1 R_2 R_1 s^2 + (C_2 R_2 + C_1 R_2 + C_1 R_1)s + 1}{C_2 C_1 R_2 R_1 s^2 + (C_2 R_2 + C_1 R_1)s + 1}$



Non-inverting operational amplifier circuit

Translational Mechanical System Transfer Functions

Mechanical systems (like electrical networks) have three passive linear components: **Spring** and the **mass** (energy-storage elements); and **viscous damper** (dissipates energy).

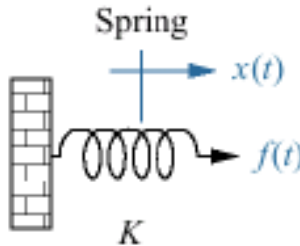
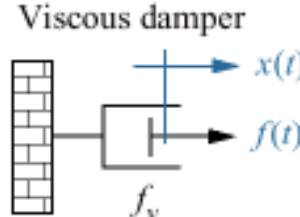
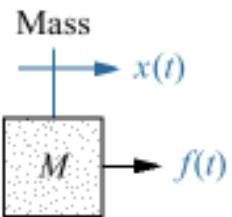
Table 2.4

Force-velocity, force-displacement, and impedance translational relationships for springs, viscous dampers, and mass

K : Spring constant

f_v : Coefficient of viscous friction

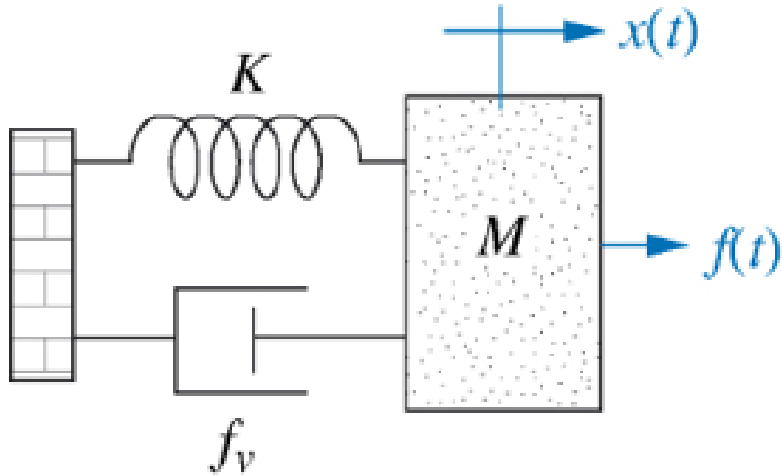
M : Coefficient of mass

Component	Force-velocity	Force-displacement	Impedance $Z_M(s) = F(s)/X(s)$
	$f(t) = K \int_0^t v(\tau) d\tau$	$f(t) = Kx(t)$	K
	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
	$f(t) = M \frac{dv(t)}{dt}$	$f(t) = M \frac{d^2 x(t)}{dt^2}$	Ms^2

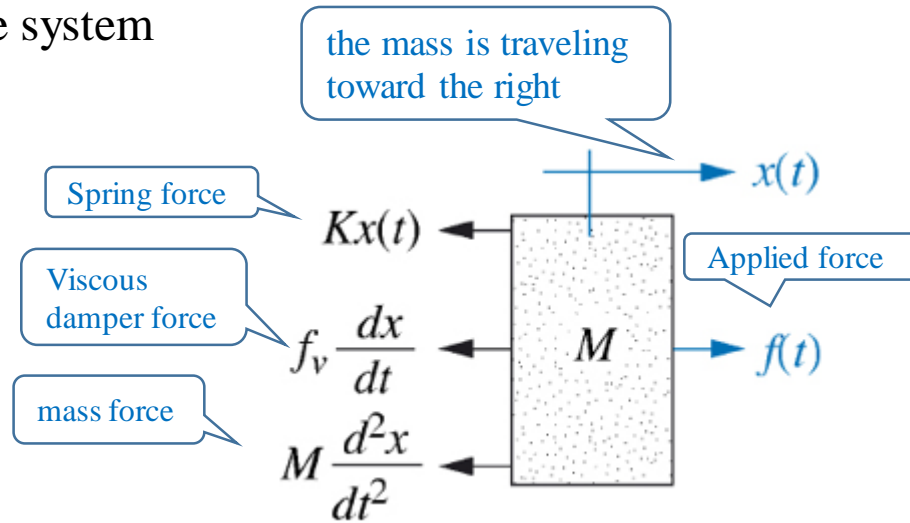
Note: The following set of symbols and units is used throughout this book: $f(t)$ = N (newtons), $x(t)$ = m (meters), $v(t)$ = m/s (meters/second), K = N/m (newtons/meter), f_v = N-s/m (newton-seconds/meter), M = kg (kilograms = newton-seconds²/meter).

Transfer Functions: One Degree of Freedom

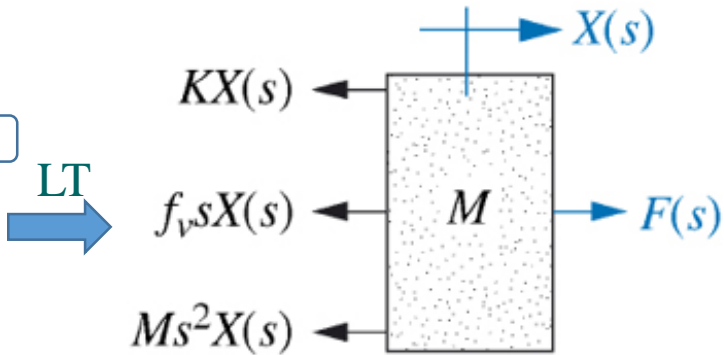
Find the transfer function, $X(s)/F(s)$, for the system



Mass, spring, and damper system



Free-body diagram of mass, spring, and damper system;



Transformed free body diagram

- All the forces impede (obstruct and block) the motion and act to oppose the applied force.

Differential equation of motion (Newton's law)

$$M \frac{d^2 x(t)}{dt^2} + f_v \frac{dx(t)}{dt} + Kx(t) = f(t)$$

Sum of impedances $\times X(s)$ = Sum of applied forces (zero initial conditions)

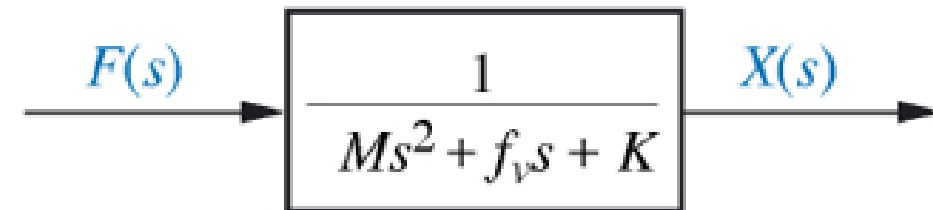
LT

$$(K + f_v s + Ms^2) \times X(s) = F(s)$$

Solving for the transfer function yields

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + f_v s + K}$$

→

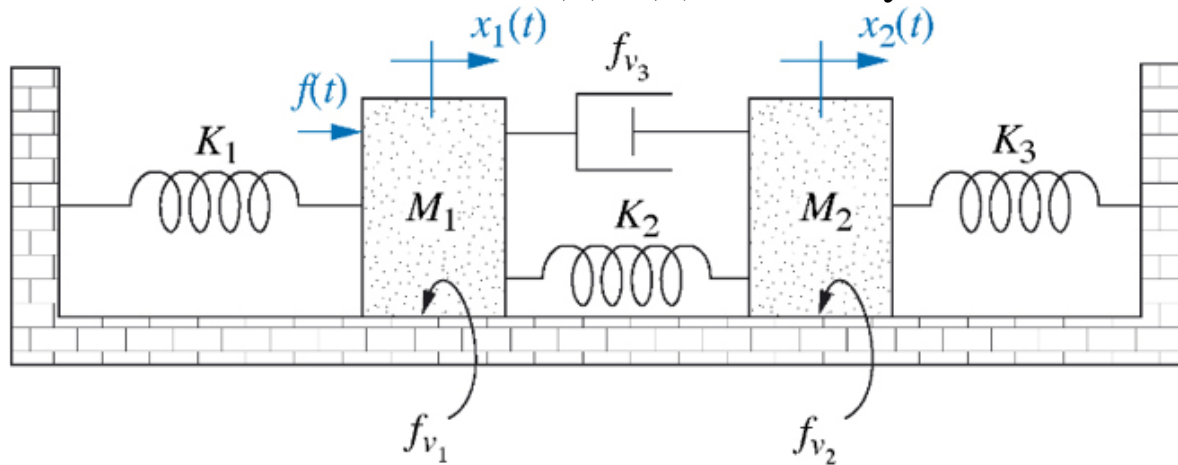


Transfer function

Transfer Functions: Two Degrees of Freedom

Number of differential equations required to describe the system is equal to the number of *linearly independent* motions (*degrees of freedom*).

- Find the transfer function, $X_2(s)/F(s)$, for the system



Two-degrees-of-freedom translational mechanical system

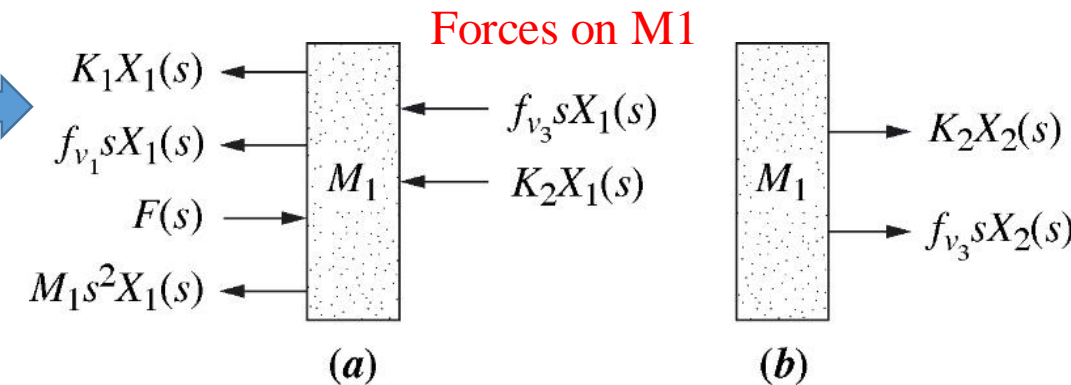
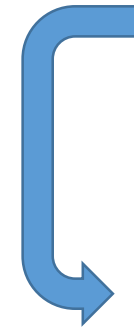
Two-degrees-of-freedom: since Each mass can be moved in the horizontal direction while the other is held still.

The Laplace transform of the equation of motion of M1

$$[M_1 s^2 + (f_{v1} + f_{v3})s + (K_1 + K_2)]X_1(s) - (f_{v3}s + K_2)X_2(s) = F(s) \quad (1)$$

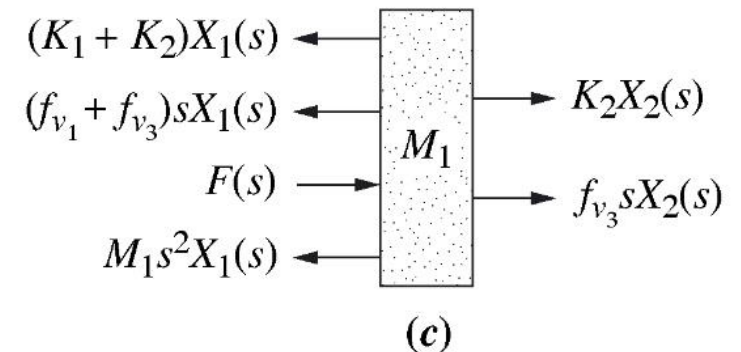
$$A X_1(s) - B X_2(s) = F \quad (1)$$

- Forces on M1 due only to motion of M1
- forces on M1 due only to motion of M2
- all forces on M1



hold M2 and move M1

hold M1 and move M2



total force on M1
(superposition or sum)

Transfer Functions: Two Degrees of Freedom Continued

The Laplace transform of the equation of motion of M_2

$$-(f_{v3}s + K_2)X_1(s) + [M_2s^2 + (f_{v2} + f_{v3})s + (K_2 + K_3)]X_2(s) = 0$$

$$-C X_1(s) + D X_2(s) = 0 \quad \Rightarrow \quad X_1(s) = \frac{D}{C} X_2(s) \quad (2)$$

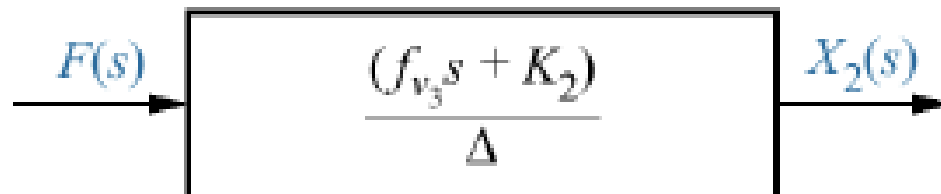
(2) in (1) $\Rightarrow A \frac{D}{C} X_2(s) - B X_2(s) = F \quad \Rightarrow \quad \frac{X_2(s)}{F(s)} = \frac{C}{AD - CB}$

Transfer function: $\frac{X_2(s)}{F(s)} = G(s) = \frac{(f_{v3}s + K_2)}{\Delta}$

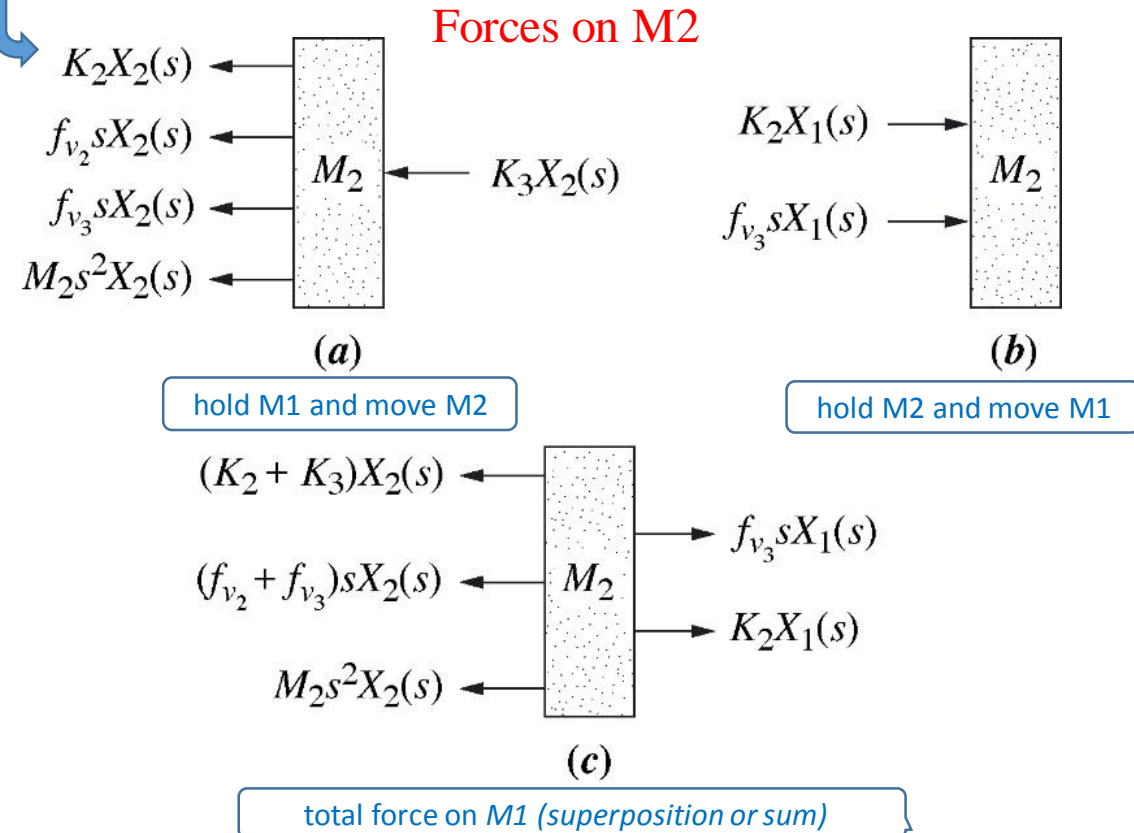
Determinant

where

$$\Delta = \begin{vmatrix} [M_1s^2 + (f_{v1} + f_{v3})s + (K_1 + K_2)] & -(f_{v3}s + K_2) \\ -(f_{v3}s + K_2) & [M_2s^2 + (f_{v2} + f_{v3})s + (K_2 + K_3)] \end{vmatrix}$$

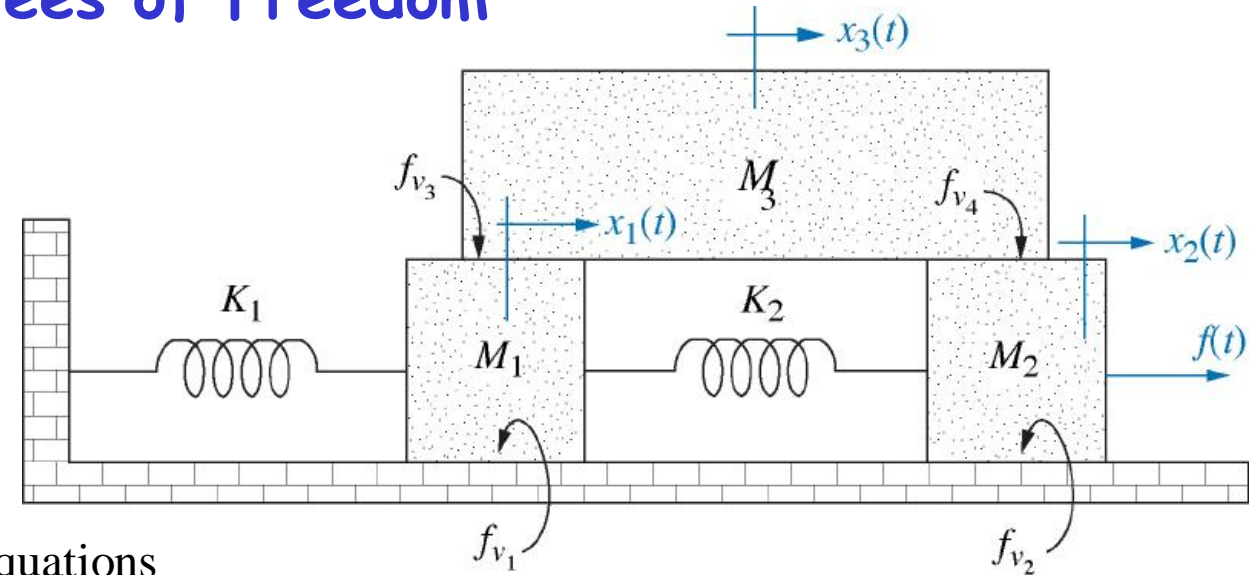


- a. Forces on M_2 due only to motion of M_2 ;
- b. forces on M_2 due only to motion of M_1 ;
- c. all forces on M_2



Transfer Functions: Three Degrees of Freedom

- Write, the equations of motion for the mechanical network
- The system has three degrees of freedom, since each of the three masses can be moved independently while the others are held still.



- The form of the equations will be similar to electrical mesh equations

For M1:

$$\left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } x_1 \end{array} \right] X_1(s) - \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ \text{x}_1 \text{ and } \text{x}_2 \end{array} \right] X_2(s) - \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ \text{x}_1 \text{ and } \text{x}_3 \end{array} \right] X_3(s) = \left[\begin{array}{c} \text{Sum of} \\ \text{applied forces} \\ \text{at } x_1 \end{array} \right]$$

Similarly, for M2 and M3, we obtain

$$\begin{aligned} \mathbf{M}_1: & [M_1 s^2 + (f_{v1} + f_{v3})s + (K_1 + K_2)]X_1(s) - K_2 X_2(s) - f_{v3} s X_3(s) = 0 \\ \mathbf{M}_2: & -K_2 X_1(s) + [M_2 s^2 + (f_{v2} + f_{v4})s + K_2]X_2(s) - f_{v4} s X_3(s) = F(s) \\ \mathbf{M}_3: & -f_{v3} s X_1(s) - f_{v4} s X_2(s) + [M_3 s^2 + (f_{v3} + f_{v4})s]X_3(s) = 0 \end{aligned}$$

Nonlinearity

Linear systems have two properties: (1) *additivity*, and (2) *homogeneity*.

1. *Additivity* (*superposition*):

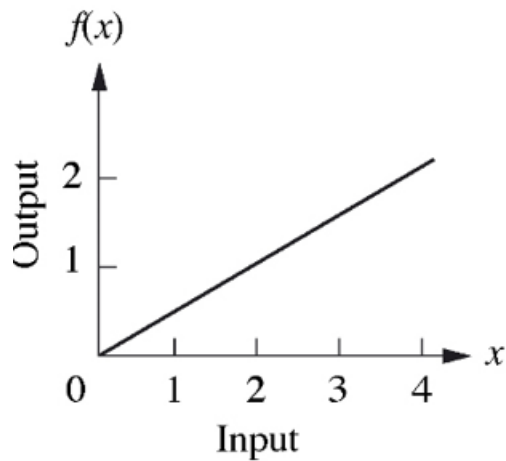
If $r_1(t) \rightarrow c_1(t)$ and $r_2(t) \rightarrow c_2(t)$, then $r_1(t) + r_2(t) \rightarrow c_1(t) + c_2(t)$

$r(t)$: Input

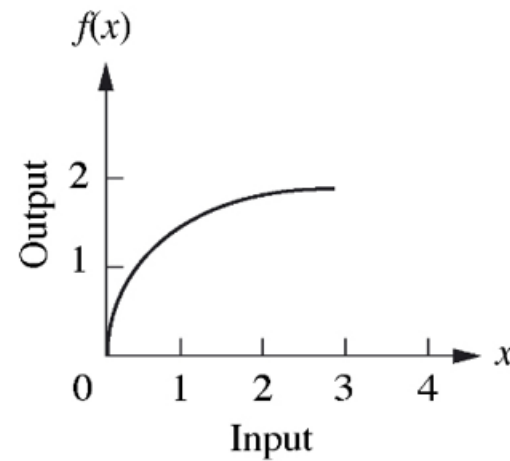
$c(t)$: Output

2. *Homogeneity*:

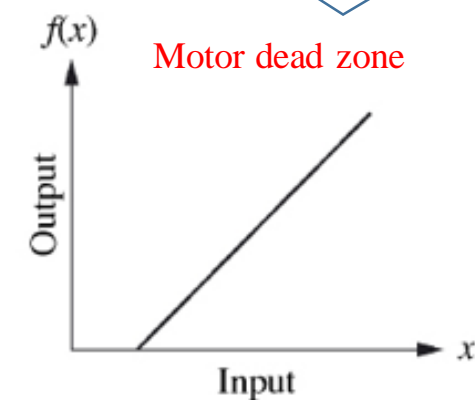
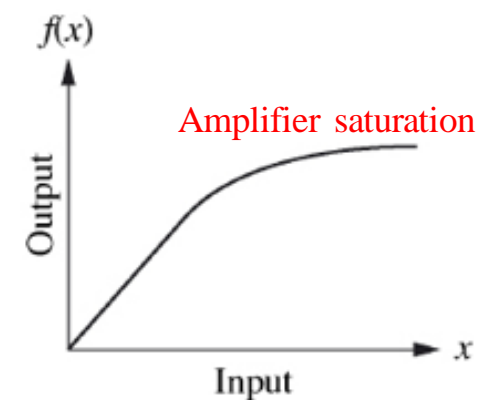
If $r_1(t) \rightarrow c_1(t)$, then $A r_1(t) \rightarrow A c_1(t)$



Linear system



Nonlinear system



motor does not respond at very low input voltages due to frictional forces exhibits a nonlinearity called *dead zone*

Some physical nonlinearities

Linearization

- If the system is nonlinear, we must linearize the system before we can find the transfer function.
- Making linear approximation to a nonlinear system.
- For a nonlinear system operating at point A: $[x_0, f(x_0)]$

Linear approximation :

related by the Slope m_A (line)
of the curve at the point A

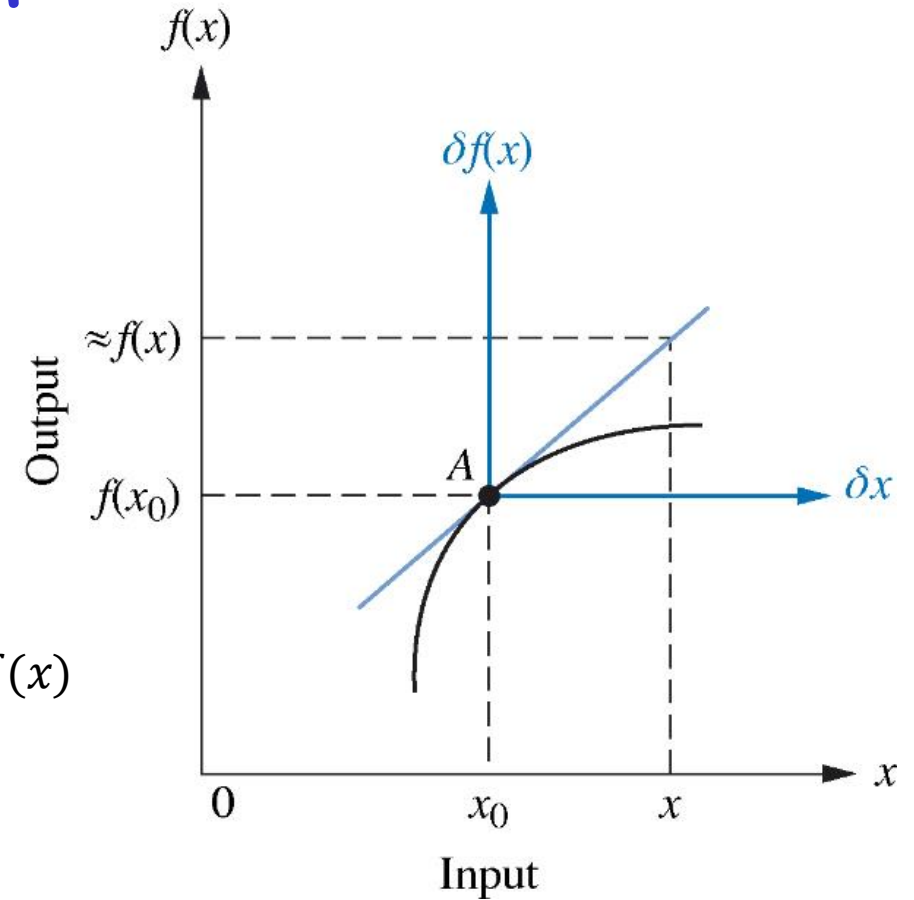
small changes in the input δx \longrightarrow small changes in the output $\delta f(x)$

Thus, $f(x) - f(x_0) \approx m_A(x - x_0)$

$\longrightarrow \delta f(x) \approx m_A \delta x$ Derivative of $f(x)$ at $x = x_0$

$\longrightarrow f(x) \approx f(x_0) + m_A(x - x_0) = f(x_0) + m_A \delta x$

$\longrightarrow \boxed{f(x) \approx f(x_0) + m_A \delta x}$



Linearization about a point A

Linearizing a Function

Problem: Linearize $f(x) = 5 \cos(x)$ about $x = \pi/2$.

Solution:

We first find that the derivative of $f(x)$ at $x = \pi/2$

$$\left. \frac{df}{dx} \right|_{x=\pi/2} = -5 \sin x \Big|_{x=\pi/2} = -5$$

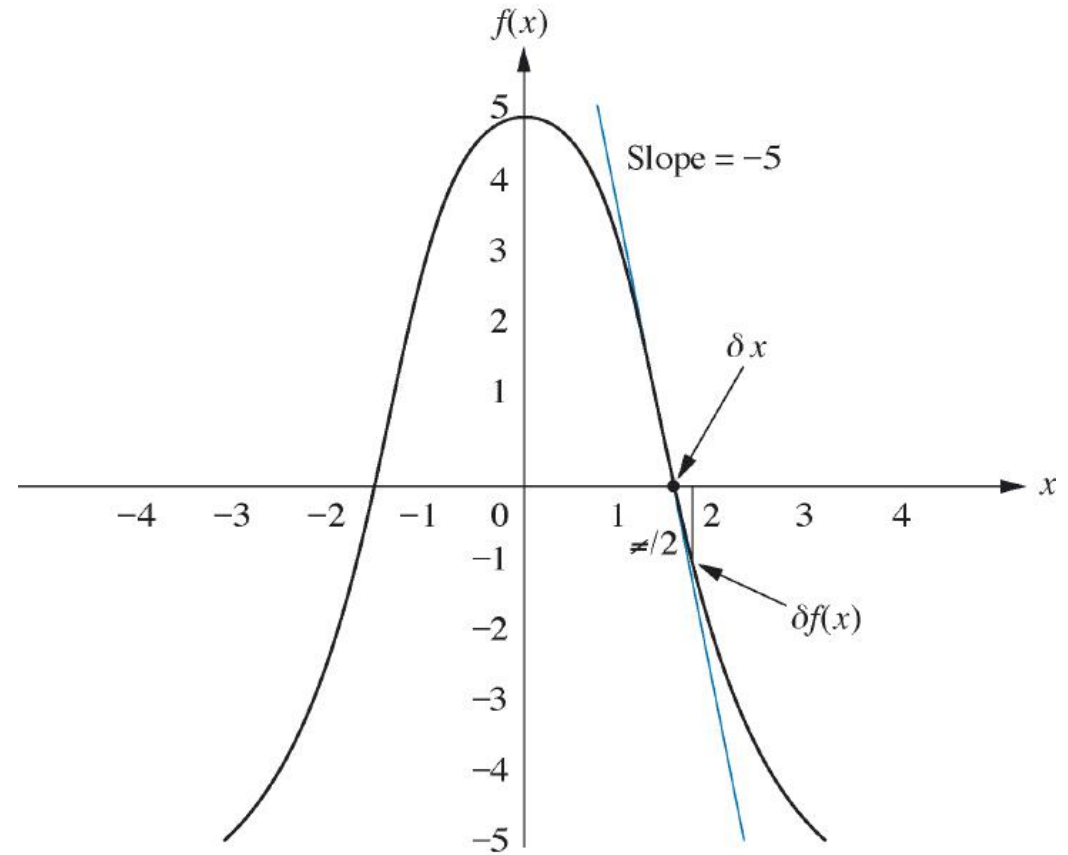
Slope at $x = \frac{\pi}{2}$

Also

$$f(x_0) = f(\pi/2) = 5 \cos(\pi/2) = 0$$

the system can be represented as

$$f(x) = -5 \delta x \text{ for small excursions of } x \text{ about } \pi/2$$



Modeling in The Time Domain

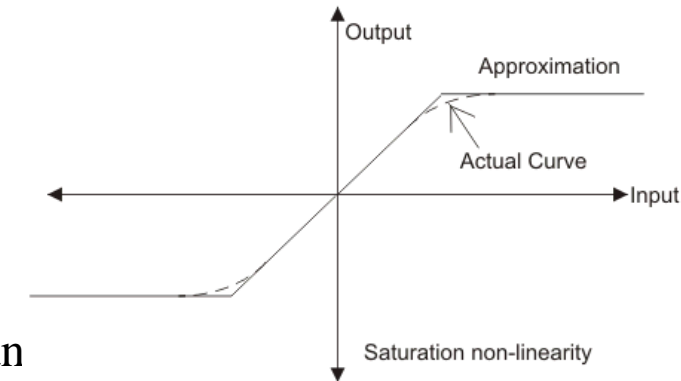
State-space Method

Two approaches are available for the analysis and design of feedback control systems.

1. Frequency domain approach (classical approach):

based on converting a system's differential equation to a transfer function.

- **Advantage:** rapidly providing stability and transient response information. Thus we can immediately see the effects of varying system parameters.
- **Disadvantage:** limited application. It can be applied only to linear, time-invariant systems or systems that can be approximated as such.

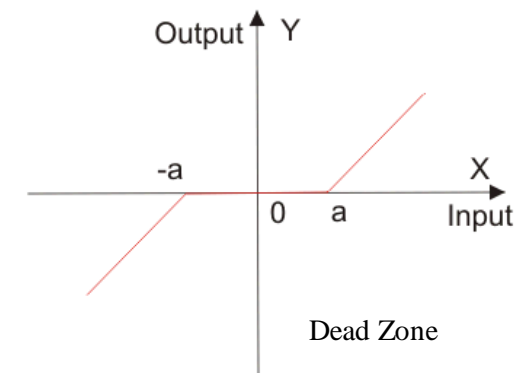


2. State-space approach (time domain / modern approach):

a sudden, forceful
backward movement

Can be used:

- a) To represent non-linear systems that have backlash, saturation, dead zone.
- b) It can handle systems with nonzero initial conditions.
- c) Multiple-inputs, multiple-outputs systems can easily be represented.
- d) Many commercial software packages are available.



Many calculation is needed
before actual realization.

RL Network: State-Space Representation

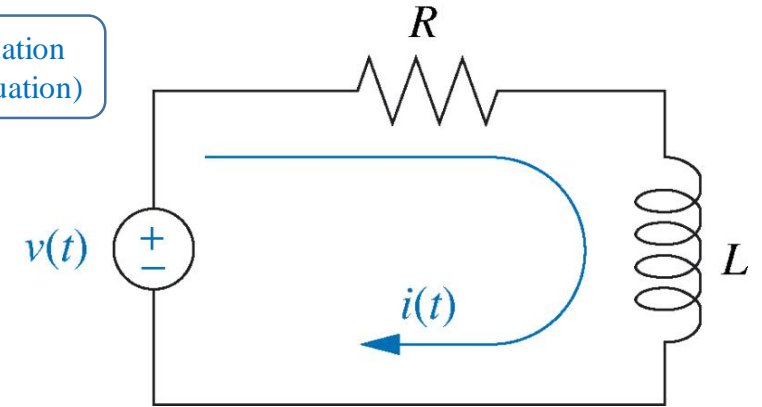
The *state-space* approach for representing physical systems (state equations and the output equations are a viable (feasible) representation of the system.).

1. Select a *state variable* (possible system variable) : say $i(t)$.

2. Write differential equation (in terms of the state variable $i(t)$).

$$L \frac{di}{dt} + Ri = v(t)$$

loop equation
(State Equation)



RL network

3. Take Laplace transform:

solve for $I(s)$: $L[sI(s) - i(0)] + RI(s) = V(s) \Rightarrow I(s) = \frac{V(s)}{(Ls + R)} + \frac{L i(0)}{Ls + R}$

If $v(t) = u(t)$, then $V(s) = 1/s$. $\Rightarrow I(s) = \frac{1}{s(Ls + R)} + \frac{L i(0)}{Ls + R}$

$\Rightarrow I(s) = \frac{A}{s} + \frac{B}{Ls + R} + \frac{L i(0)}{Ls + R} \Rightarrow I(s) = \frac{1}{R} \left(\frac{1}{s} - \frac{1}{s + \frac{R}{L}} \right) + \frac{i(0)}{s + \frac{R}{L}}$

Inverse Laplace transform: $i(t) = \frac{1}{R} (1 - e^{-(R/L)t}) + i(0)e^{-(R/L)t}$

4. Output equations:

Algebraically combine the state variables with the system's input and find all of the other system variables for $t \geq t_0$.

$$v_R(t) = Ri(t)$$

$$v_L(t) = v(t) - Ri(t)$$

$$\frac{di}{dt} = \frac{1}{L} [v(t) - Ri(t)]$$

If we know initial condition of i , $i(0)$, and input voltage, $v(t)$, then we can find the value of any network variables at time $t \geq t_0$.

Self Study

Do the State-space representation of RLC network.

The General State-Space Representation

Some Terminology

- *Linear combination:* (of n variables x_i)
$$S = K_n x_n + K_{n-1} x_{n-1} + \dots + K_1 x_1$$

none of the variables can be written as a linear combination of the others.
- *Linear independence:* S is zero if every K is zero and no x is zero: variables x are linearly independent.
- *System variable:* Any variable that responds to an input or initial conditions in a system.
- *State variables:* The smallest set of linearly independent system variables that completely determines (knowing the value at t_0) the value of system variables for $t \geq t_0$
- *State vector:* A vector whose elements are state variables.
- *State space:* The n -dimensional space whose axes are the state variables.
- *State equations:* A set of n simultaneous, first-order differential equations with n variables (state variables).
- *Output equations:* The equation that expresses the output variables of a system as linear combinations of the state variables and the inputs.

State-space Representation

- A system is represented in *state-space* by the following equations:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A} \mathbf{x} + \mathbf{B} \mathbf{u} & \leftarrow \text{State equation (state variables } \mathbf{x} \text{ are quantities that can be differentiated } \dot{\mathbf{x}}) \\ \mathbf{y} = \mathbf{C} \mathbf{x} + \mathbf{D} \mathbf{u} & \leftarrow \text{Output equation} \end{cases}$$

- This representation of a system provides *complete knowledge* of all variables of the system at any $t \geq t_0$
- The choice of state variables:
 - is not unique.
 - minimum number* (equals the order of the differential equation).
 - are *linearly independent*.

x = state vector (Contains the state variables)

$\dot{\mathbf{x}}$ = derivative of the state vector with respect to time

y = output vector

u = input or control vector

A = system matrix

B = input matrix

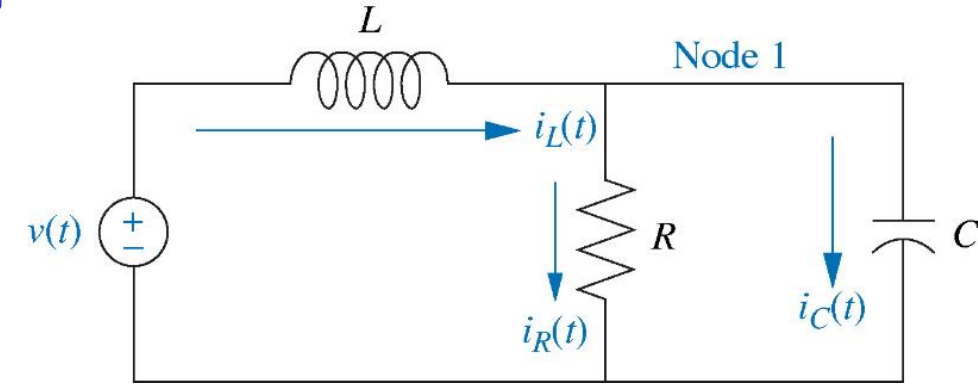
C = output matrix

D = feedforward matrix

Example-1: State-space Representation

Problem:

Find a state-state representation of the following electrical network if the *output is i_R* the current through the resistor, (*$v(t)$ is the input*).



Solution:

The following steps will yield a viable representation of the network in state space.

Step 1: Label all the branch currents in the network (These include i_L , i_R , and i_C).

Step 2: Select the state variables (*quantities that are differentiated* v_C and i_L , energy-storage elements, the inductor C and the capacitor L) and write derivative equations.

$$C \frac{dv_C}{dt} = i_C, \quad L \frac{di_L}{dt} = v_L$$

We can have $v_C = i_L$.

Step 3: Express *non-state variables* (right-hand side: i_C and v_L) as a linear combinations of the state variables (differentiated variables: v_C and i_L) and the input, $v(t)$.

$$i_C = -i_R + i_L = -\frac{1}{R}v_C + i_L$$

We have $v_R = v_C$.

Apply Kirchhoff's voltage and current laws, to obtain i_C and v_L in terms of the state variables, v_C and i_L .

$$v_L = -v_C + v(t)$$

Example-1: State-space Representation-contd.

Step 4: Obtain state equations: (by substituting the values and rearranging)

$$C \frac{dv_C}{dt} = -\frac{1}{R} v_C + i_L, \quad L \frac{di_L}{dt} = -v_C + v(t)$$

$$\Rightarrow \frac{dv_C}{dt} = -\frac{1}{RC} v_C + \frac{1}{C} i_L$$

$$\frac{di_L}{dt} = -\frac{1}{L} v_C + \frac{1}{L} v(t)$$

Matrix Form \Rightarrow

$$\begin{cases} \frac{dv_C}{dt} = -\frac{1}{RC} \cdot v_C + \frac{1}{C} \cdot i_L + 0 \cdot v(t) \\ \frac{di_L}{dt} = -\frac{1}{L} \cdot v_C + 0 \cdot i_L + \frac{1}{L} \cdot v(t) \end{cases}$$

State equation

Step 5: Find the output equation:

$$i_R = \frac{1}{R} v_C$$

Matrix Form \Rightarrow

$$i_R = \frac{1}{L} \cdot v_C + 0 \cdot i_L$$

output equation

Final result: Convert into vector-matrix form

$$\begin{cases} \dot{x} = A x + B u \\ y = C x + D u \end{cases}$$

$$\begin{bmatrix} \dot{v}_C \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} v(t)$$

$$i_R = \begin{bmatrix} \frac{1}{R} & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix}$$

Example-2: State-space Representation (with a dependent source)

PROBLEM: Find the state and output equations for the electrical network shown in Figure.

If the output vector is $y = [v_{R_2} \quad i_{R_2}]^T$

Step 1: Label all the branch currents in the network.

Step 2: Select the *state variables* (energy-storage elements: L and C) and write derivative equations (voltage-current relationships).

$$C \frac{dv_C}{dt} = i_C, \quad L \frac{di_L}{dt} = v_L \quad \mathbf{x}_1 = i_L; \quad \mathbf{x}_2 = v_C; \quad \text{the state variables (differentiated variables)}$$

Step 3: State equations (we find v_L and i_C in terms of the state variables)

mesh $LCR_2 \longrightarrow v_L = v_C + v_{R_2} = v_C + i_{R_2} R_2$

Node 2 \longrightarrow At node 2, $i_{R_2} = i_C + 4v_L$, so we get,

$$v_L = v_C + (i_C + 4v_L) R_2$$

$$\Rightarrow v_L = \frac{1}{1 - 4R_2} (v_C + i_C R_2)$$

$$\Rightarrow (1 - 4R_2)v_L - R_2 i_C = v_C \quad (1)$$

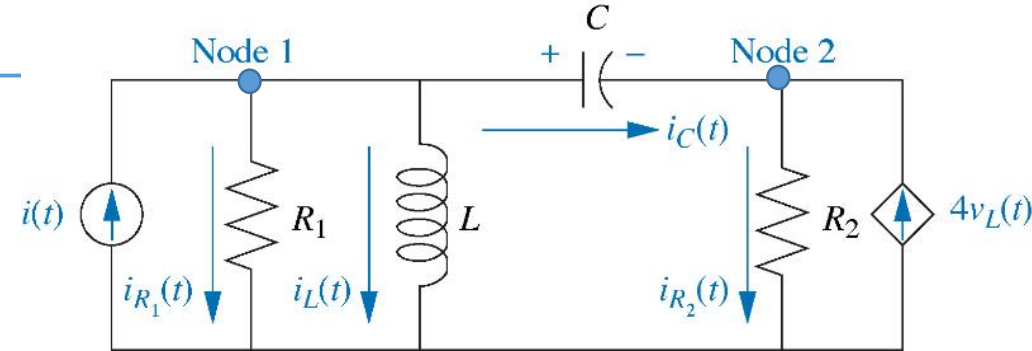
Node 1

$$i_C = i(t) - i_{R_1} - i_L$$

$$= i(t) - \frac{v_{R_1}}{R_1} - i_L$$

$$= i(t) - \frac{v_L}{R_1} - i_L \quad (v_{R_1} = v_L)$$

$$\Rightarrow -\frac{1}{R_1} v_L - i_C = i_L - i(t) \quad (2)$$



Example-2: State-space Representation

Solving (1) and (2) simultaneously for v_L and i_C yields

$$-\frac{1}{R_1}v_L - i_C = i_L - i(t) \quad \Rightarrow \quad i_C = -\frac{1}{R_1}v_L - i_L + i(t) \quad \xrightarrow{(1)} \quad (1 - 4R_2)v_L - R_2\left(-\frac{1}{R_1}v_L - i_L + i(t)\right) = v_C$$

$$\Rightarrow \left(1 - 4R_2 + \frac{R_2}{R_1}\right)v_L + R_2i_L - R_2i(t) = v_C \quad \Rightarrow \quad v_L = \frac{1}{\Delta}[R_2i_L - v_C - R_2i(t)] \quad \text{with } \Delta = -\left(1 - 4R_2 + \frac{R_2}{R_1}\right)$$

$$\text{and} \quad i_C = \frac{1}{\Delta}\left[(1 - 4R_2)i_L + \frac{1}{R_1}v_C - (1 - 4R_2)i(t)\right]$$

writing the result in vector-matrix form

$$\begin{bmatrix} \dot{i}_L \\ \dot{v}_C \end{bmatrix} = \begin{bmatrix} R_2/(L\Delta) & -1/(L\Delta) \\ (1 - 4R_2)/(C\Delta) & 1/(R_1C\Delta) \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix} + \begin{bmatrix} -R_2/(L\Delta) \\ -(1 - 4R_2)/(C\Delta) \end{bmatrix} i(t)$$

Step 4: Output equations $v_{R2} = -v_C + v_L$; $i_{R2} = i_C + 4v_L$;

vector-matrix form, the output equation is

$$\begin{bmatrix} v_{R2} \\ i_{R2} \end{bmatrix} = \begin{bmatrix} R_2/\Delta & -(1 + 1/\Delta) \\ 1/\Delta & (1 - 4R_1)/(\Delta R_1) \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix} + \begin{bmatrix} -R_2/\Delta \\ -1/\Delta \end{bmatrix} i(t)$$

Example-3: State-space Representation

(Translational Mechanical System)

For M1: $M_1 s^2 X_1(s) + DsX_1(s) + KX_1(s) - KX_2(s) = 0$

$$\Rightarrow M_1 \frac{d^2 x_1}{dt^2} + D \frac{dx_1}{dt} + Kx_1 - Kx_2 = 0$$

For M2:

$$-KX_1(s) + KX_2(s) + M_2 s^2 X_2(s) = F(s)$$

$$\Rightarrow -Kx_1 + Kx_2 + M_2 \frac{d^2 x_2}{dt^2} = f(t)$$

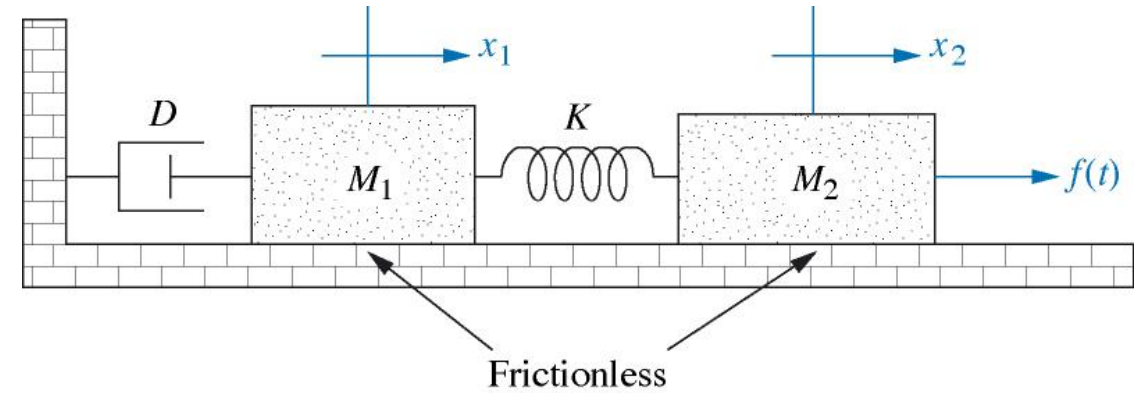
Let, $\frac{d^2 x_i}{dt^2} = \frac{dv_i}{dt}$

(acceleration = derivative of velocity)

Select x_1, x_2, v_1, v_2 as *state variables*.

$$\frac{dx_1}{dt} = v_1 \Rightarrow \frac{d^2 x_1}{dt^2} = \frac{dv_1}{dt} = \dot{v}_1$$

$$\frac{dx_2}{dt} = v_2 \Rightarrow \frac{d^2 x_2}{dt^2} = \frac{dv_2}{dt} = \dot{v}_2$$



State equations:

$$\begin{aligned} \frac{dx_1}{dt} &= v_1 \\ \frac{dv_1}{dt} &= -\frac{K}{M_1}x_1 - \frac{D}{M_1}v_1 + \frac{K}{M_1}x_2 \\ \frac{dx_2}{dt} &= v_2 \\ \frac{dv_2}{dt} &= +\frac{K}{M_2}x_1 - \frac{K}{M_2}x_2 + \frac{1}{M_2}f(t) \end{aligned}$$

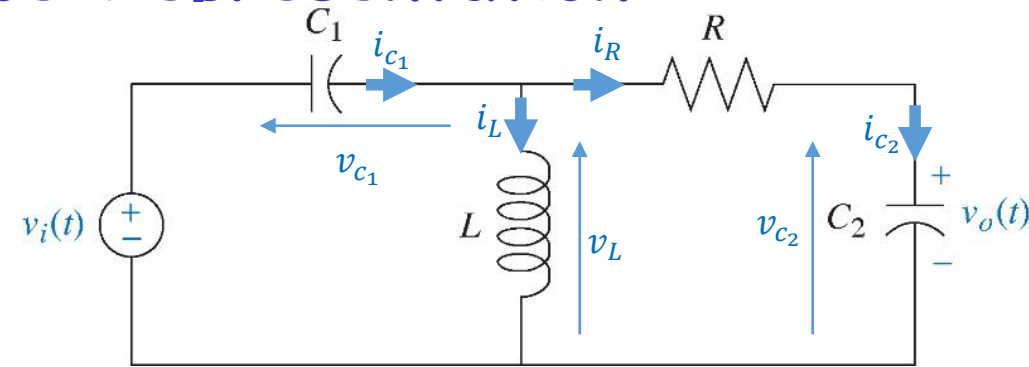
matrix form.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{v}_1 \\ \dot{x}_2 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -K/M_1 & -D/M_1 & K/M_1 & 0 \\ 0 & 0 & 0 & 1 \\ K/M_2 & 0 & -K/M_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/M_2 \end{bmatrix} f(t)$$

Example-4: State-space Representation

Problem:

Find the state-space representation of the electrical network shown in the figure. The output is $v_o(t)$.



Solution:

state variables

The derivative relations (one for each energy-storage element)

$$C_1 \frac{dv_{c1}}{dt} = i_{c1}$$

$$L \frac{di_L}{dt} = v_L$$

$$C_2 \frac{dv_{c2}}{dt} = i_{c2}$$

Using Kirchhoff's current and voltage laws:

$$i_{c1} = i_L + i_R = i_L + \frac{1}{R}(v_L - v_{c2})$$

$$v_L = v_i - v_{c1}$$

$$i_{c2} = i_R = \frac{1}{R}(v_L - v_{c2})$$

Find Matrix form

State vector

$$x = \begin{bmatrix} v_{c1} \\ i_L \\ v_{c2} \end{bmatrix}$$

state-space representation

$$\dot{x} = \begin{bmatrix} -1/RC_1 & 1/C_1 & -1/RC_1 \\ -1/L & 0 & 0 \\ -1/RC_2 & 0 & -1/RC_2 \end{bmatrix} x + \begin{bmatrix} 1/RC_1 \\ 1/L \\ 1/RC_2 \end{bmatrix} v_i(t)$$

$$i_{c1} = -\frac{1}{R}v_{c1} + i_L - \frac{1}{R}v_{c2} + \frac{1}{R}v_i$$

$$v_L = -v_{c1} + 0 i_L + 0 v_{c2} + v_i$$

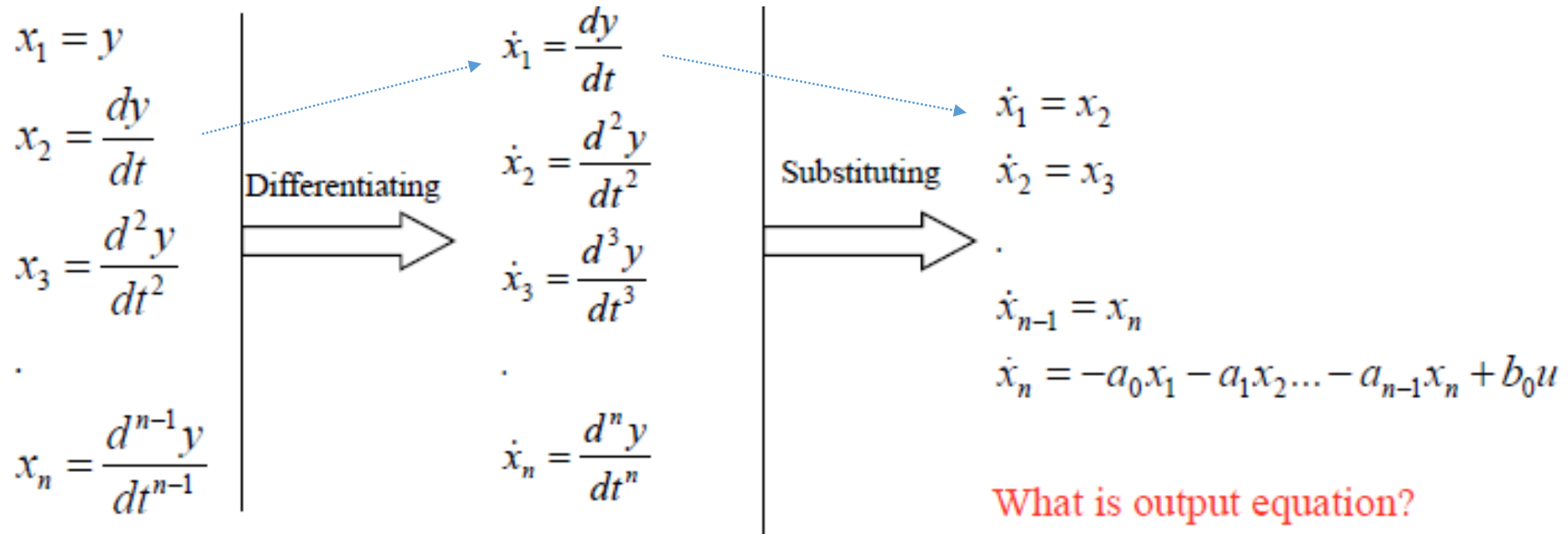
$$i_{c2} = -\frac{1}{R}v_{c1} + 0 i_L - \frac{1}{R}v_{c2} + \frac{1}{R}v_i$$

Converting a Transfer Function to State Space

Phase variables: A set of state variables where each state variable is defined to be the derivative of the previous state variable.

Consider a differential equation,
$$\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y = b_0 u$$

Choose the output, $y(t)$, and its derivatives as the state variables, x_i .



Converting a Transfer Function to State Space

matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & \dots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix}$$

output

$$y = [1 \quad 0 \quad 0 \quad \dots \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

PROBLEM: Find the state-space representation in phase-variable form for the transfer function

$$\frac{C(s)}{R(s)} = \frac{24}{(s^3 + 9s^2 + 26s + 24)}$$

Step 1 Find the associated differential equation
inverse Laplace transform,

$$(s^3 + 9s^2 + 26s + 24)C(s) = 24R(s)$$

$$\ddot{c} + 9\dot{c} + 26c + 24c = 24r$$

Step 2 Select the state variables.

$$x_1 = c$$

$$x_2 = \dot{c}$$

$$x_3 = \ddot{c}$$

the state equations.

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + 24r$$

$$y = c = x_1$$

matrix form,



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$