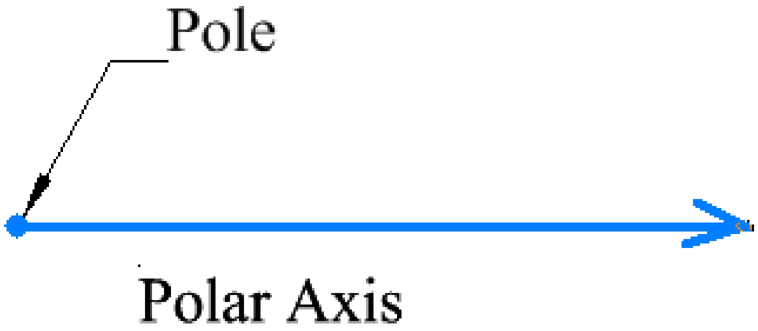
Polar Coordinates

A Polar Coordinate System

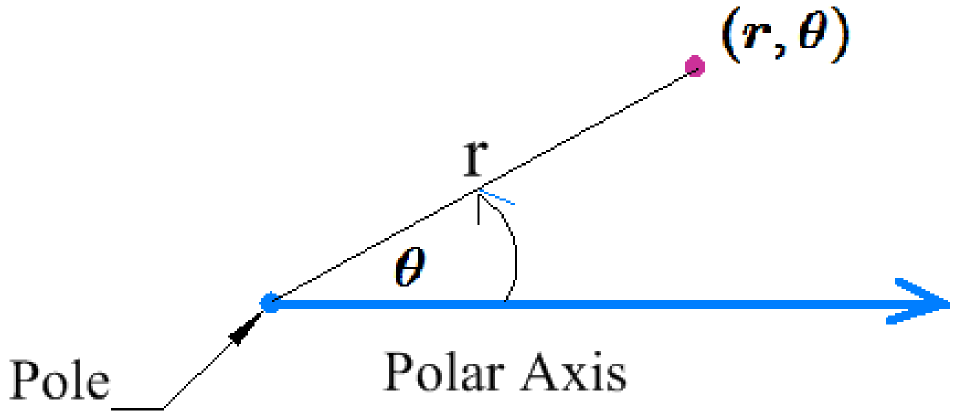
- To specify a point in rectangular coordinates you give an x value and a y value.
- To specify a point in a polar coordinate system you give a distance and an angle.

1^{st} You Need a Polar Axis

A polar axis is a ray emanating from a point called the pole. The polar axis is usually the positive x-axis, and the pole is the origin.



To specify a point, we need a distance and an angle.



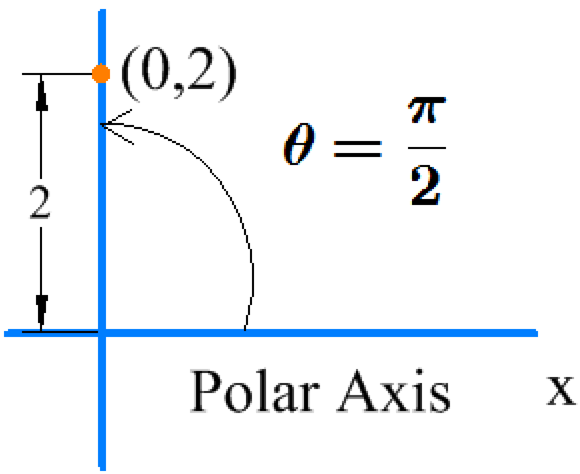
What are the polar coordinates of the rectangular point?

a) $(0, 2)$ b) $(-4, 0)$

Solution (part a)

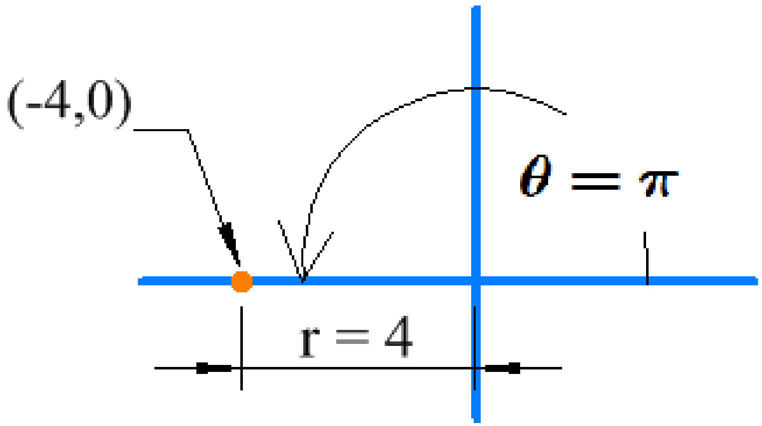
What are the polar coordinates of the rectangular point?

a) $(0, 2)$ b) $(-4, 0)$



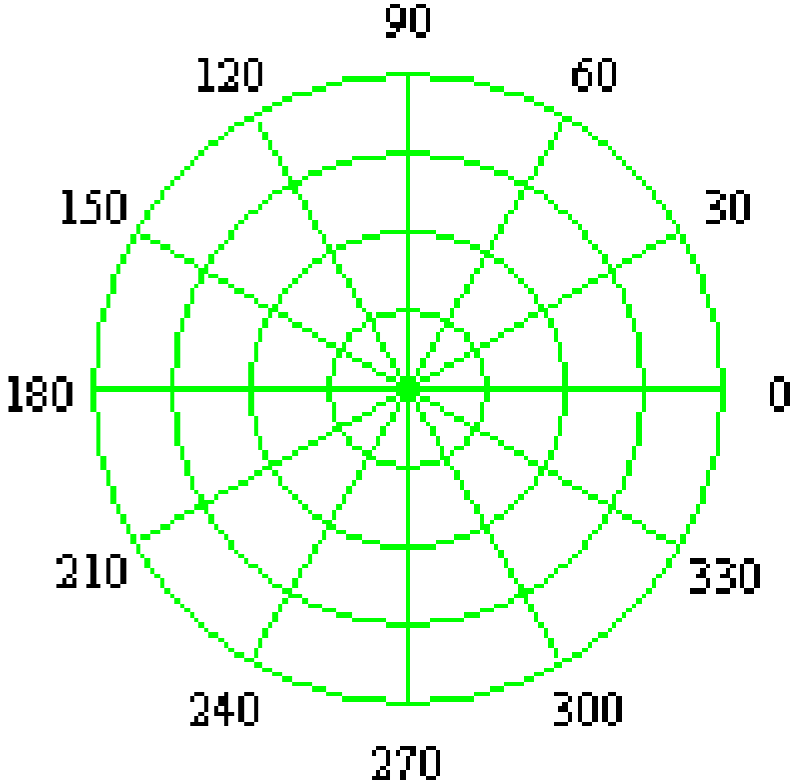
In polar coordinates, $(0, 2)$ can be expressed as $\left(2, \frac{\pi}{2}\right)$.

Solution (part b)

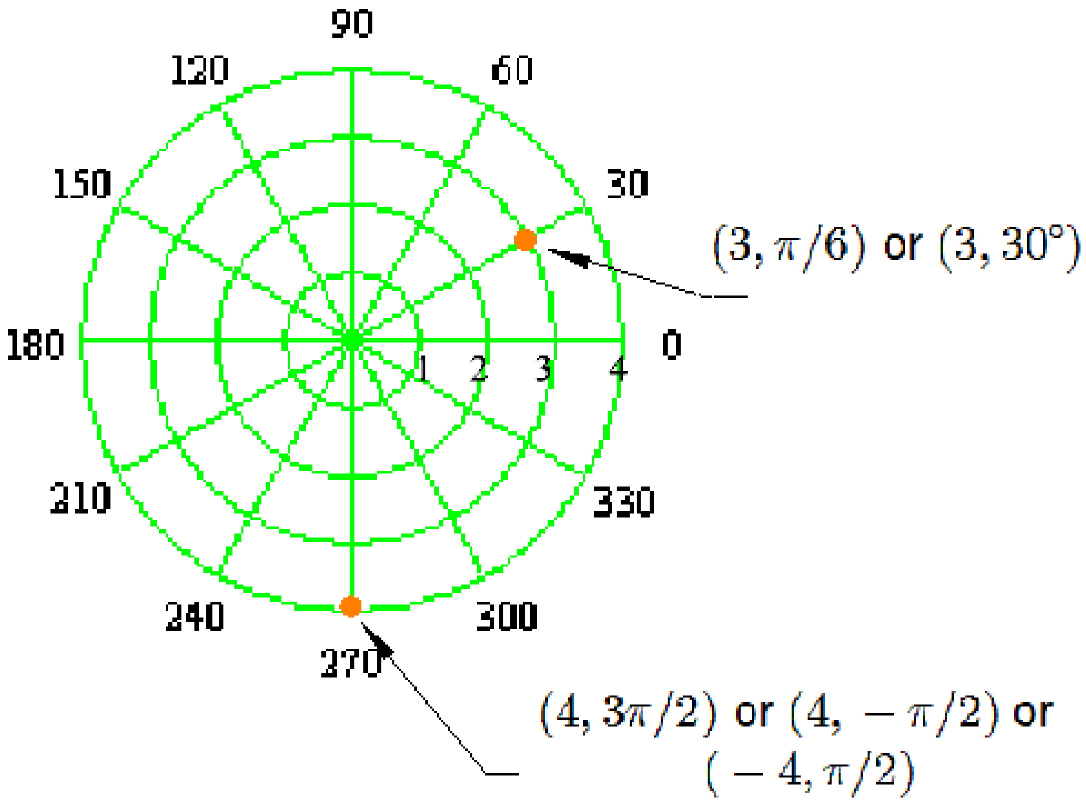


In polar coordinates, $(-4, 0)$ can be expressed as $\mathbf{(4, \pi)}$.

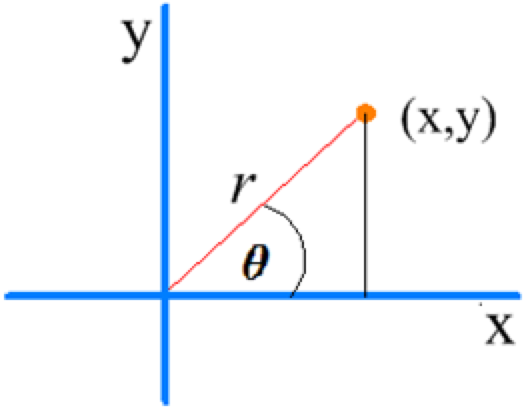
A Polar Coordinate System



Some Points Plotted



Relationships Between Polar and Rectangular Coordinates



$$r^2 = x^2 + y^2 \quad \left(r = \pm\sqrt{x^2 + y^2}\right)$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$\tan\theta = \frac{y}{x}$$

Note: θ is not always equal to $\tan^{-1}\frac{y}{x}$.

Polar to Rectangular

Find the retangular coordinates of the polar point $(3, -\pi/6)$.

$x = r\,cos\,\theta$ and $y = r\,sin\,\theta$

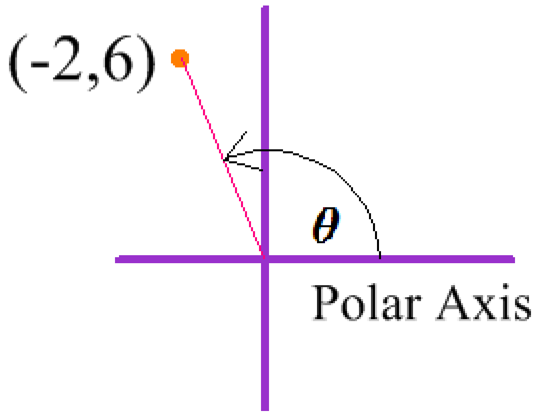
$$x = 3\,cos\left(-\frac{\pi}{6}\right) = 3 \cdot \frac{\sqrt{3}}{2} = 2.598$$

$$y = 3\,sin\left(-\frac{\pi}{6}\right) = 3 \cdot \left(-\frac{1}{2}\right) = -1.5$$

Ans: **The rectangular coordinates are (2.598, -1.5).**

Rectangular to Polar

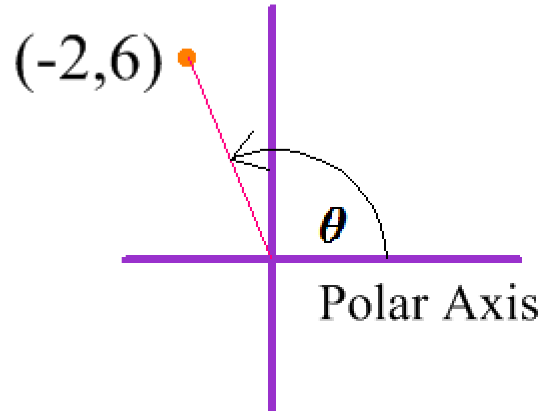
What are the polar coordinates of the rectangular point $(-2, 6)$?



$$r = \sqrt{4+36} = \sqrt{40} = 6.32456$$

Note that $\theta \neq tan^{-1}(-3)$.

Conclusion of Example



Find $tan^{-1}\left(\frac{6}{2}\right) = tan^{-1}(3)$ which is the reference angle for θ.

$tan^{-1}(3) = 1.24905$ or about 71°.

Remember θ is in the 2nd quadrant.

Therefore $\theta = \pi - 1.249 = \mathbf{1.893}$.

r for radians

The polar coordinates are $(\mathbf{6.325}, \mathbf{1.893}\,\boldsymbol{r})$.

Polar to Rectangular

Example: Find the polar coordinates of the rectangular point $(-6, -10)$.

The point is in the 3rd quadrant.

$$r = \sqrt{6^2 + 10^2} = \sqrt{136} \approx 11.662$$

The reference angle is $tan^{-1}(10/6) =$
$1.030\,r$ or $59.04°$.

Hence $\theta = \pi + tan^{-1}(5/3) =$
$\pi + 1.030 = 4.172\,r$
or $180° + 59.04° = 239.04°$.

Ans: $(11.662, 4.172r)$ or $(11.662, 239.04°)$.

Changing From a Rectangular Equation to a Polar Equation

Replace the Cartesian equation by an equivalent polar equation.

1. $x^2 + y^2 = 4$

2. $x = 7$

3. $x^2 + (y-2)^2 = 4$

Solutions

1. Ans: $r^2 = 4$ or $r = 2$

2. $x = 7$

$x = r\,cos\,\theta \;\Rightarrow\; r\,cos\,\theta = 7 \;\Rightarrow\; r = \dfrac{7}{cos\,\theta}$

$\Rightarrow$ Ans: $r = 7\,sec\,\theta$.

3. $x^2 + (y-2)^2 = 4$

$x^2 + y^2 - 4y + 4 = 4 \;\Rightarrow\; x^2 + y^2 - 4y = 0$

$\Rightarrow\; r^2 - 4r sin\,\theta = 0 \;\Rightarrow\; r(r - 4 sin\,\theta) = 0$

$\Rightarrow\; r = 0$ or $r = 4\,sin\,\theta$.

Ans: The eq. is simply $r = 4\,sin\,\theta$.

Changing from a Polar Equation to a Rectangular Equation

Replace the polar equation by an equivalent Cartesian equation.

1. $r = 4\,csc\,\theta$

2. $r = 6\,cos\,\theta$

3. $r^2 sin\,2\theta = 2$

Solutions

1. $r = 4\, csc\, \theta$

Remember that $sin\, \theta = \frac{y}{r} \quad \Rightarrow$

$$csc\ \theta = \frac{r}{y}\,.$$

Then $r = 4\, csc\, \theta \quad \Rightarrow \quad r = \frac{4r}{y}$.

Hold off on replacing r on the left side and the r cancels and we get

$1 = \frac{4}{y}$ or simply

$\boldsymbol{y = 4}$, a horizontal line.

2. $r = 6\, cos\, \theta \quad \Rightarrow \quad r = 6 \cdot \frac{x}{r}$.

Again, hold off on replacing r on the left side.

Therefore $r = \frac{6x}{r} \quad \Rightarrow \quad r^2 = 6x$

$$\Rightarrow \quad \boldsymbol{x^2 + y^2 = 6x}\,.$$

Solution to Problem 3

3. $r^2 sin\, 2\theta = 2$

Remember that $sin\, 2\theta = 2 sin\, \theta\, cos\, \theta$.

$$r^2 2\, sin\, \theta\, cos\, \theta = 2 \;\Rightarrow\; r^2 2 \frac{y}{r} \frac{x}{r} = 2 \;\Rightarrow$$

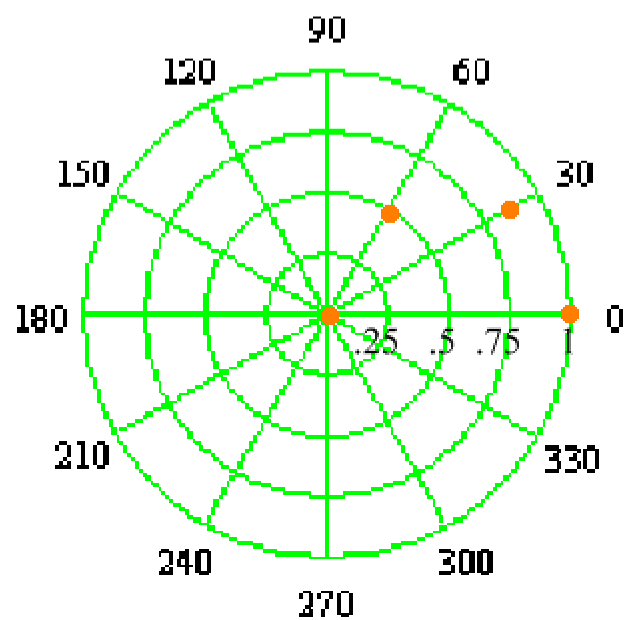
$$xy = 1\,.$$

Polar Graphs

Sketch the graph of the polar curve

$r = \cos\theta$.

θ	r
0	1
$\pi/6$	0.87
$\pi/3$	0.50
$\pi/2$	0

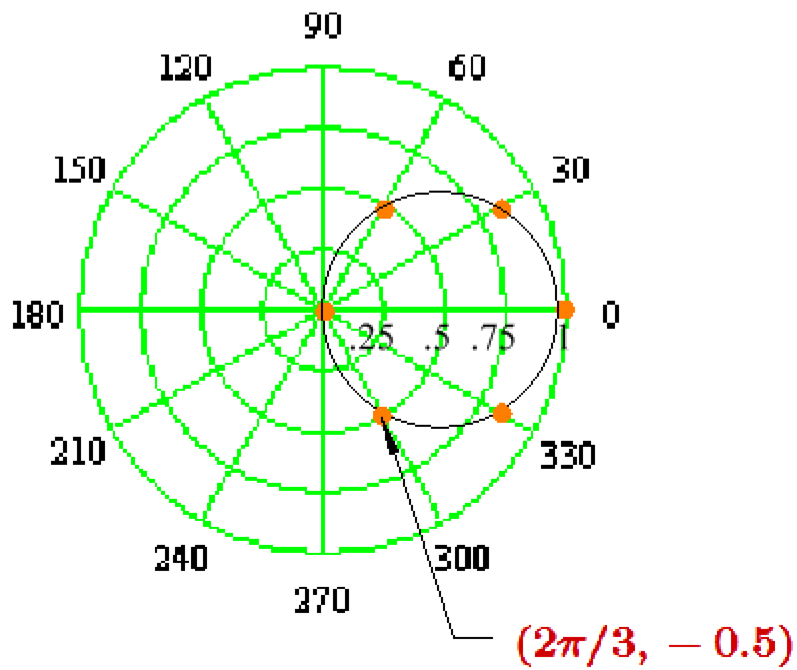


The Rest of the Graph

θ	r
0	1
$\pi/6$	0.87
$\pi/3$	0.50
$\pi/2$	0

θ	r
$2\pi/3$	-0.50
$5\pi/6$	-0.87
π	-1

These pts are on the lower ½ of circle

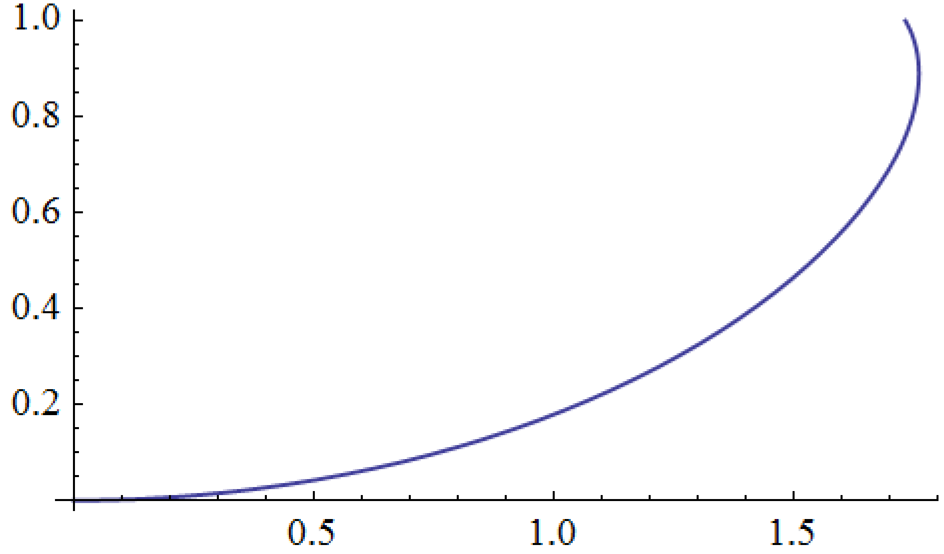


A Polar Graph

Sketch the graph of the polar equation

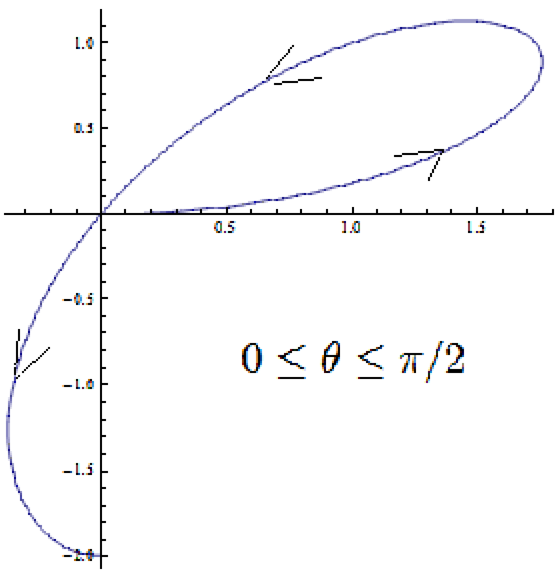
$$r = 2\,sin\,3\theta.$$

θ	3θ	$r = 2\,sin\,3\theta$
0	0	0
$\pi/6$	$\pi/2$	2

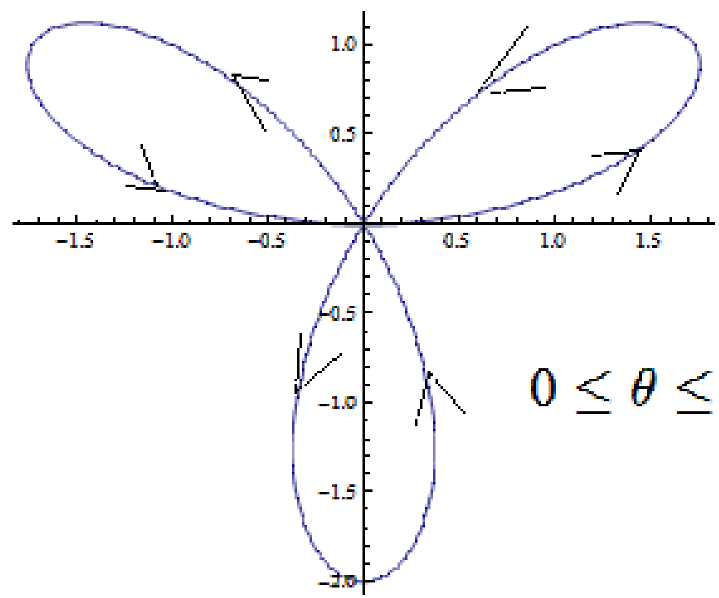


The Rest of the Graph

θ	3θ	$r = 2\,sin\,3\theta$
0	0	0
$\pi/6$	$\pi/2$	2
$2\pi/6$	π	0
$3\pi/6 = \pi/2$	$3\pi/2$	-2



$0 \leq \theta \leq \pi/2$



$0 \leq \theta \leq \pi$

Polar Calculus

If $r = f(\theta)$ then

$$x = r\,cos\,\theta = f(\theta)\,cos\,\theta \text{ and}$$
$$y = r\,sin\,\theta \;= f(\theta)\,sin\,\theta.$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

Example: Find the slope of the tangent line to the graph of $r = 2\,cos\,\theta$ at the point $(1, \pi/3)$.

Solution to Example

Example: Find the slope of the tangent line to the graph of $r = 2\,cos\,\theta$ at the point $(1, \pi/3)$.

$$x = r\,cos\,\theta = 2\,cos\,\theta\,cos\,\theta = 2cos^2\theta$$
$$y = r\,sin\,\theta = 2\,cos\,\theta\,sin\,\theta = sin\,2\theta$$

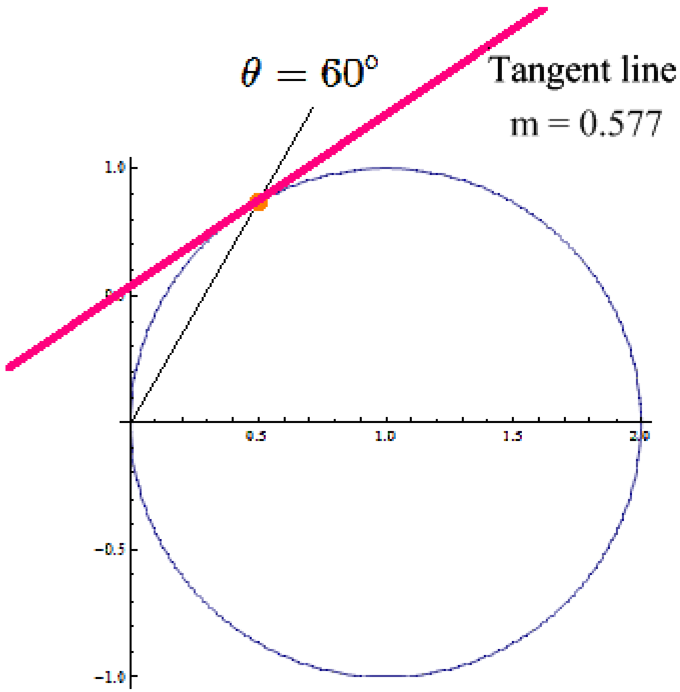
$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\,cos\,2\theta}{4cos\,\theta(-\,sin\,\theta)} =$$

$$-\frac{cos\,2\theta}{2sin\,\theta\,cos\,\theta} = -\frac{\boldsymbol{cos\,2\theta}}{\boldsymbol{sin\,2\theta}}.$$

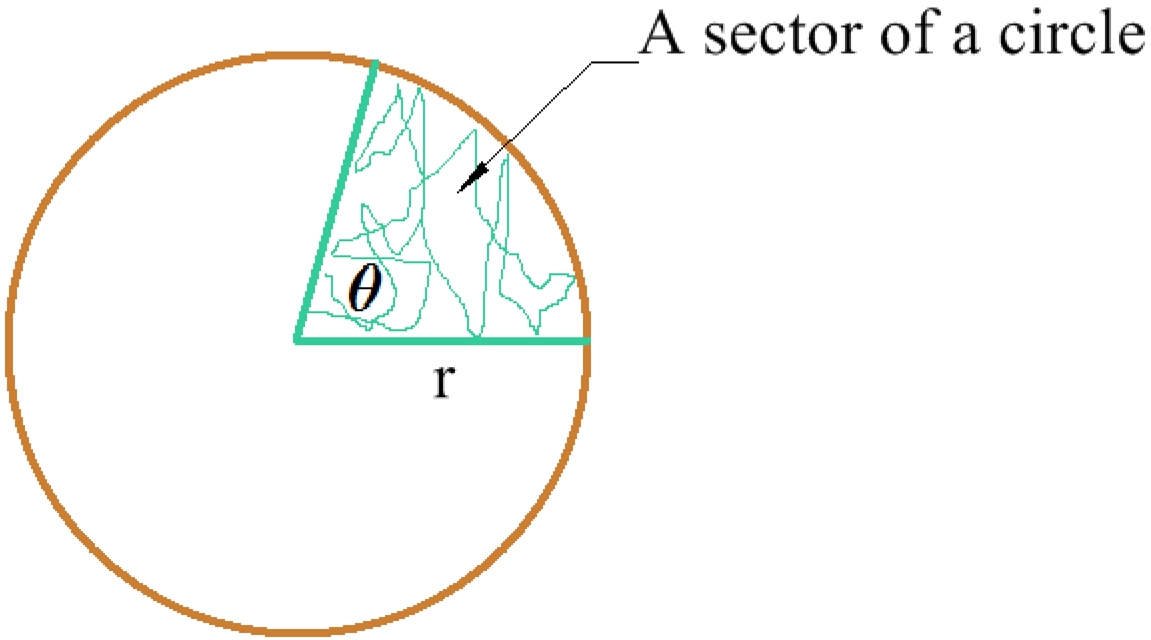
Conclusion

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/3} = -\frac{cos\,\dfrac{2\pi}{3}}{sin\,\dfrac{2\pi}{3}} = -\frac{-1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}$$

$$\approx \mathbf{0.577}\,.$$



The Area of a Sector

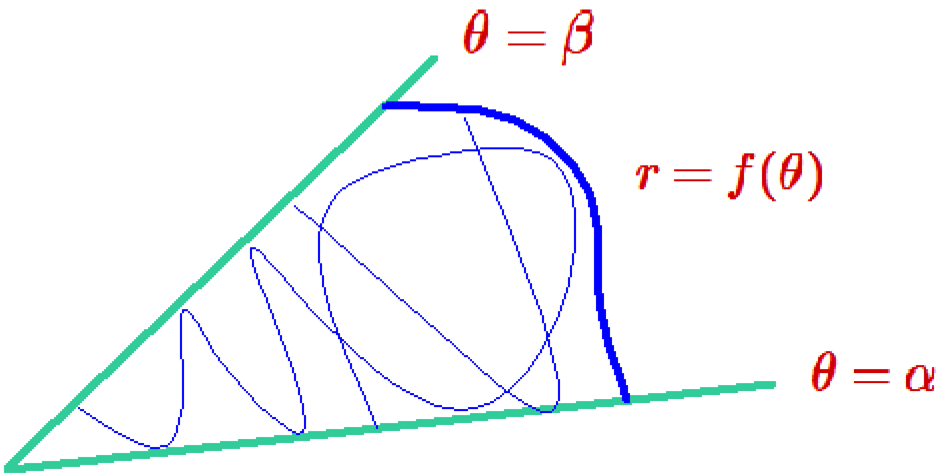


The sector has the area

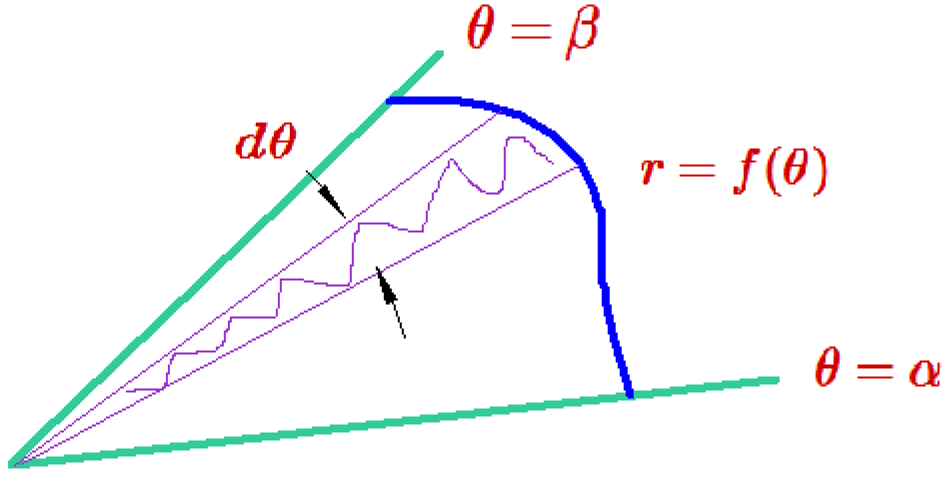
$$\frac{\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2} r^2 \theta \,.$$

Finding Areas Using Polar Coordinates

We want to be able to find an area like the one below bounded by a polar curve $r = f(\theta)$ and 2 rays emanating from the origin.



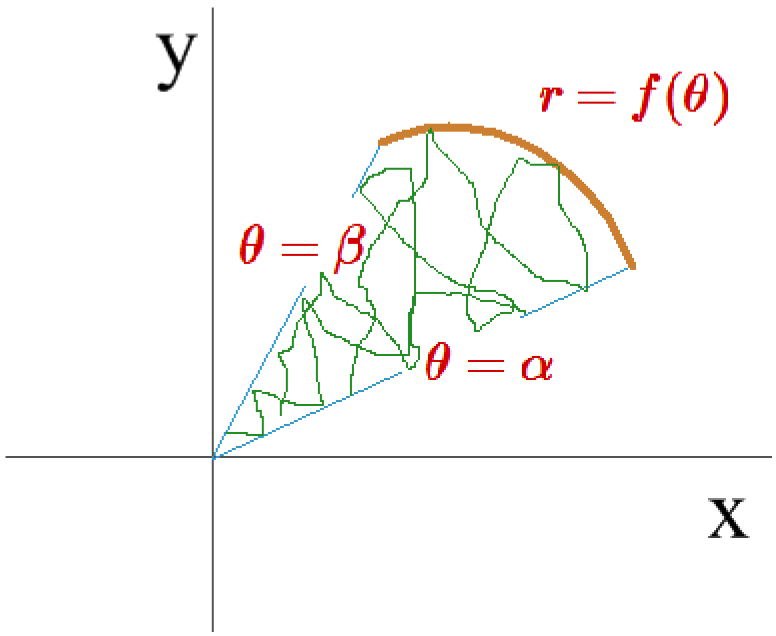
Finding the Area using Thin Sectors



$$dA = \frac{1}{2} r^2 d\theta$$

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$$

The Integral for Finding a Polar Area



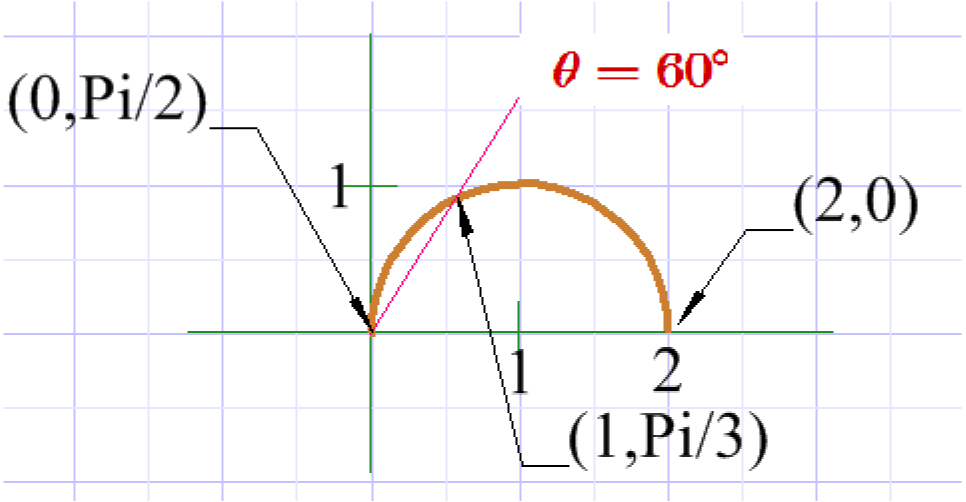
The shaded in area is equal to

$$\frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta \,.$$

Finding an Area of a Circle Using Integration

Find the area of the circle given by
$r = 2\cos\theta.$

θ	r
0	2
$\pi/3$	1
$\pi/2$	0



The Integral that Gives the Area

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$$

where $r = 2cos\,\theta$.

$1/2$ the area results when θ ranges from 0 to $\pi/2$. Therefore the area is

$$A = 2 \cdot \frac{1}{2}\int_{0}^{\pi/2} (2cos\,\theta)^2 d\theta =$$

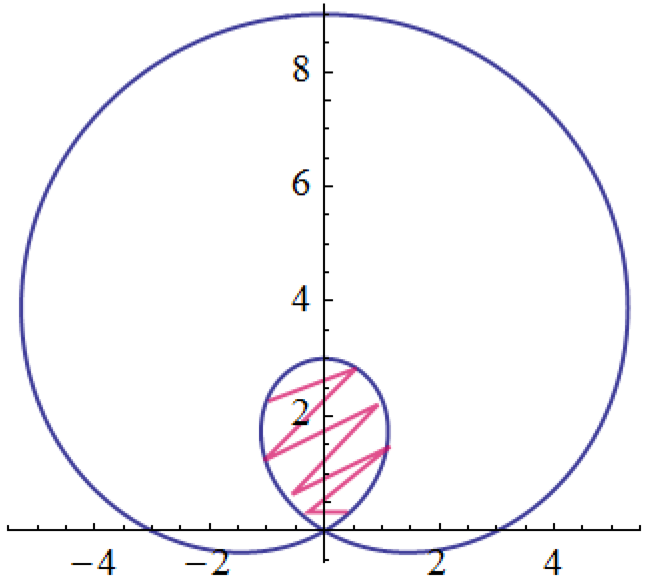
$$\int_{0}^{\pi/2} 4cos^2\theta d\theta = \pi.$$

Check: The area of a circle of radius 1 is $\pi r^2 = \pi \cdot 1 = \pi$.

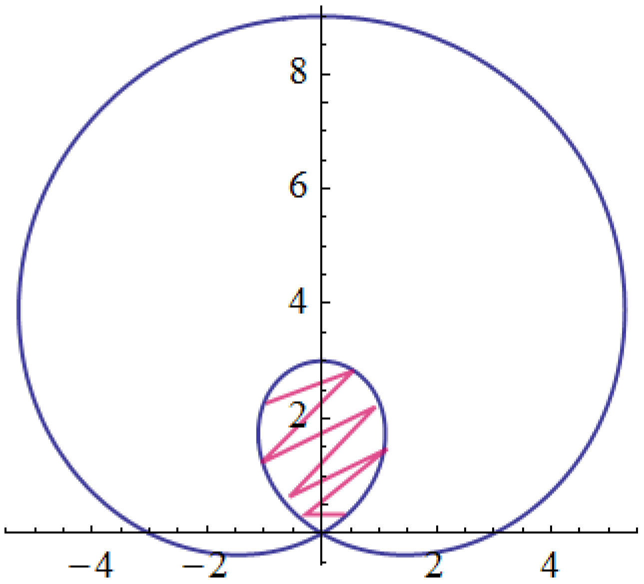
Another Example of Finding a Polar Area

Example: Find the area inside the small leaf of the graph of

$$r = -3 + 6 sin\,\theta.$$

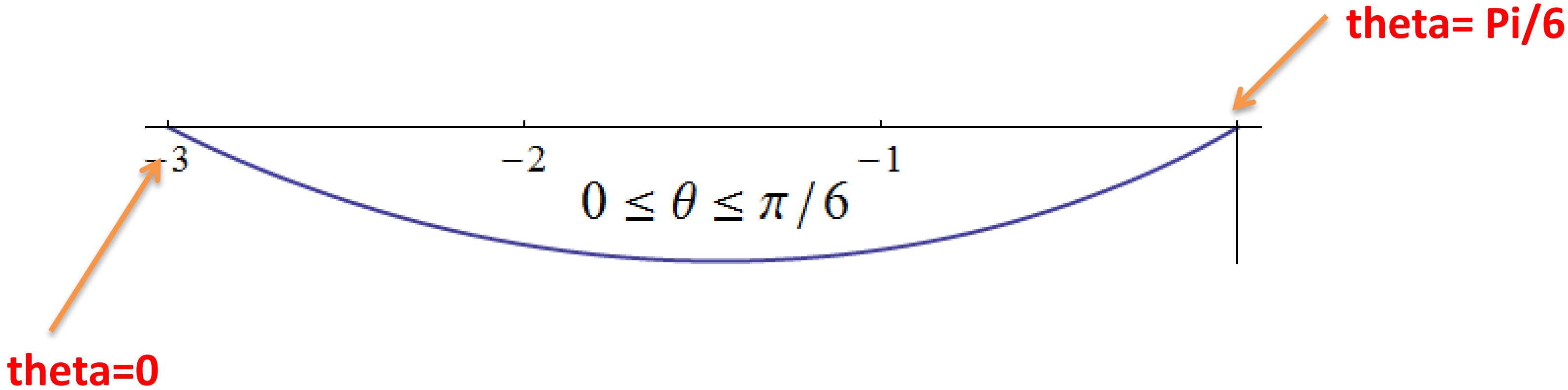


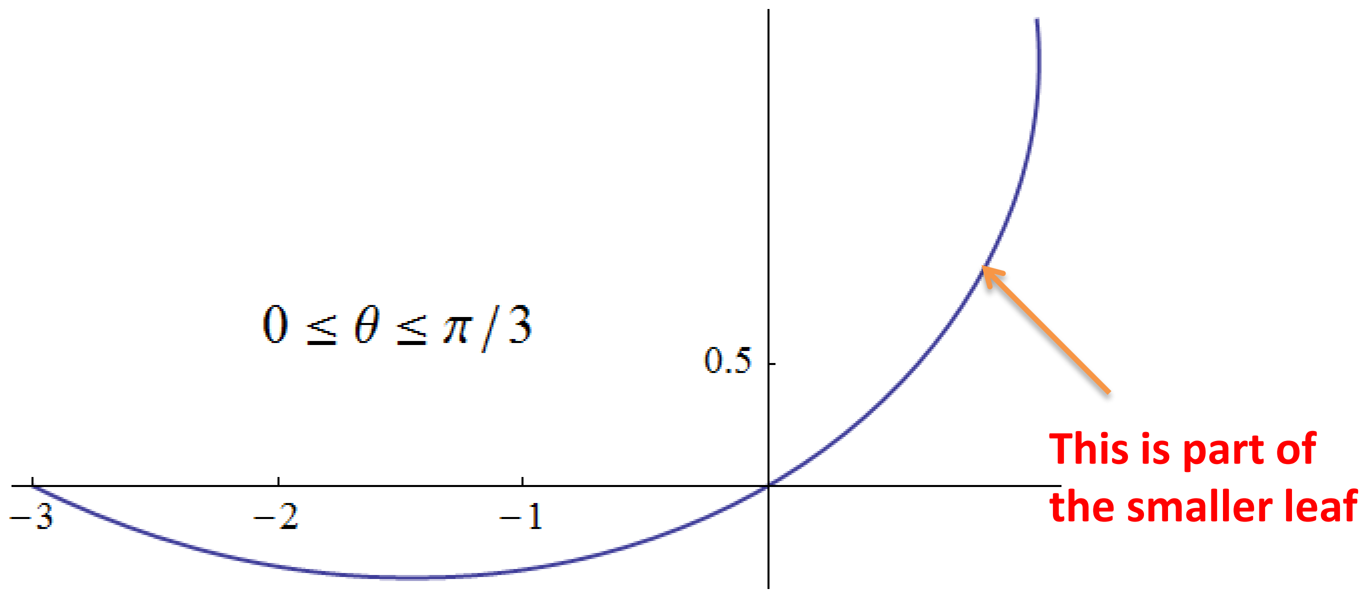
Continuation

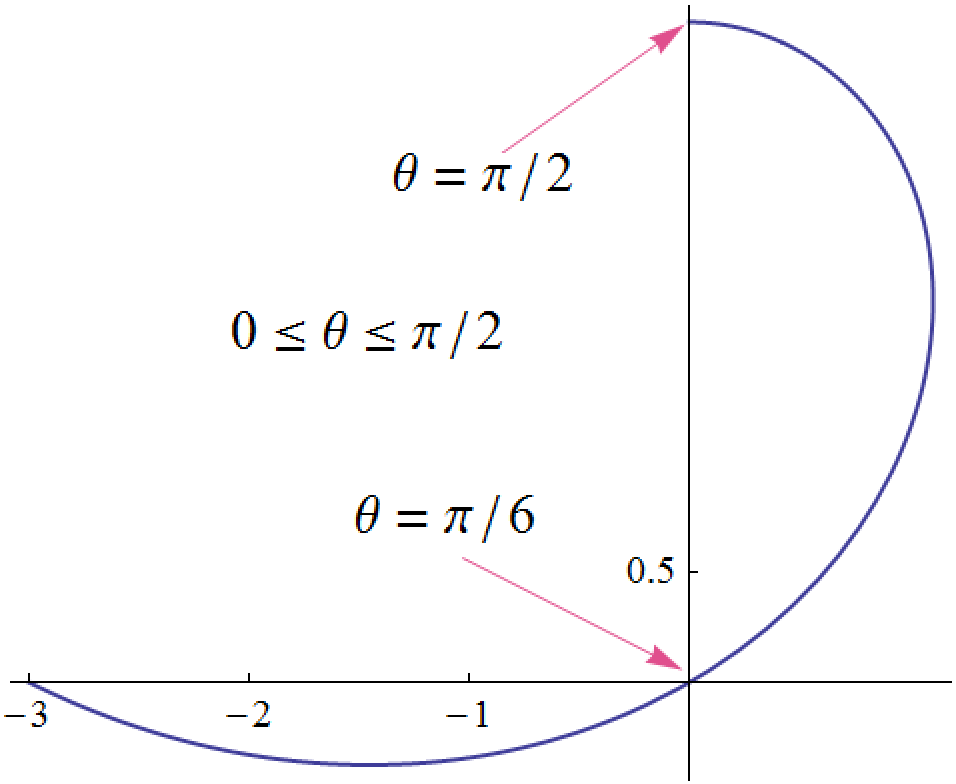


We must find the range of θ which results in the graph of the small leaf. Don't simply say θ goes from 0 to 2π.

Here's how the graph materializes.







1/2 the area results when θ ranges from $\pi/6$ to $\pi/2$. The area is equal to

$$2 \cdot \frac{1}{2}\int_{\pi/6}^{\pi/2} (-3 + 6 sin\,\theta)^2 d\theta = 4.892\,.$$