The Chaos and Control of Food Chain Model Using Nonlinear Feedback1

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Abstract

This paper studies the chaos and control of a continuous time food chain model which contains one prey, one predator and super-predator. We show that this system can be asymptotically stabilized using a nonlinear feedback control inputs. The necessary feedback control law for asymptotic stability of this system is obtained. The system appears to exhibit a chaotic behavior for a range of parametric values. The range of the system parameters for which the subsystems converge to limit cycles is determined. Numerical examples and analysis of the results are presented.

Keywords: Food chain, Limit cycles, Chaotic behavior, chaos control, Asymptotic stability, Liapunov function

1 Introduction

Food chains and webs in the environment are very important systems in many different fields such ecological science, applied mathematics, economic and engineering science. Food chains and webs can be modeled by systems of differential equations which approximate species or functional feeding group behavior with different functional responses.

The subjects of chaos and chaos control are growing rapidly in many different fields such biological systems, structural engineering, ecological models, aerospace science, and economics [5, 6, 10, 11]. Food chain modeling provides challenges in the fields of both theoretical ecology and applied mathematics. The simple food chain model describes by a nonlinear mathematical model

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that includes varying numbers of stability of equilibrium states and limit cycles. Determining the equilibrium states and bifurcations of equilibria in a nonlinear system is an important problem in many different areas such ecology model, applied mathematical model and others.

The dynamic relationship between predators and their preys has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance. Recently, many authors have explored the dynamics of a class of the so-called semi ratio-dependent predator-prey systems with functional responses see for example [6, 7].

In the recent years, interest in adaptive control systems has increased rapidly along with interest and progress in control topics. The adaptive control has a variety of specific meanings, but it often implies that the system is capable of accommodating unpredictable environmental changes, whether these changes arise within the system or external to it. Adaptation is a fundamental characteristic of living organisms such as prey-predator systems and many other biological models since these systems attempt to maintain physiological equilibrium in the midst changing environmental conditions [18].

Prey-predator phenomena have many important applications in many different field, such as biology, economic, ecology and others sciences. The study of prey-predator phenomena is now a dominant problem in many ecological sciences see for example Refs [14, 17, 18].

The mathematical ecology has emerged and developed rapidly. A variety of mathematical methods can be used in ecological science. One of the main problems of ecosystems is to study the stability and instability of these systems [1, 2, 3, 5, 9]. A model was introduced by Volterra for community in which organisms of one population provided food for those of the other is very important. Similar phenomena can be observed in communities with one population parasitizing on the organisms of another species. The communities of such type are usually termed prey-predator or host-parasite models [10, 11, 12, 13, 14].

A stochastic analysis of the Lotka-Volterra model for the prey-predator when the birth rate of the prey and death rate of the predator are perturbed by independent white noises and others related models are presented in [20].

The evolution of this model is also investigated for a large length of the time interval. One of the first successes of mathematical ecology was the demonstration of population periodic oscillations in a stationary medium. The biological literature abounds in works where systems are either observed in nature or simulated on models populations in laboratory conditions [18, 19]

This paper is organized as follows. We start in section *2* defining the three species population that consists one prey, one predator and super-predator. The nonlinear system of differential equations that govern this system is introduced. In section *3* we discuss the linear stability analysis and the chaotic

behavior of this system. In this section the equilibrium states are classified according to their stability. Further, the convergence of the subsystems of this system to limit cycles is studied. The ranges of the system parameters for which the system converges to limit cycles are determined. In section 4 the adaptive control and Liapunov asymptotic stability of the food chain model are discussed¿ In this section the necessary control inputs for this asymptotic stability is obtained as nonlinear feedback of the population densities. Extensive Numerical examples and simulation are introduced in section 5.

2   Food Chain Mathematical Model

In this section, we will describe the three-species prey-predator system which consists of two competing predator and one prey. Such system can be describe by the following set of nonlinear differential equations:

$$\left.\begin{aligned}
\dot{x}_1 &= x_1\Big[r(1 - k^{-1}x_1) - \frac{a_2 x_2}{b_2 + x_1}\Big], \\
\dot{x}_2 &= x_2\Big[\frac{e_2 a_2 x_1}{b_2 + x_1} - \frac{a_2 x_3}{b_3 + x_2} - d_2\Big], \\
\dot{x}_3 &= x_3\Big[\frac{e_3 a_3 x_2}{b_3 + x_2} - d_3\Big],
\end{aligned}\right\} \tag{1}$$

where x_1, x_2 and x_3 are prey, predator and super predator biomass respectively, r and k are prey growth rate and carrying capacity, and a_i, b_i, e_i and d_i, $i = 2, 3$ are maximum predation rate, half saturation constant, efficiency, and death rate of predator and super predator. As the super predator is exploited, its mortality d_3 is the sum of basic mortality and harvesting effort, so that food yield is proportional to super predator biomass.

The model (1) appears to have received only limited study (Turchin, 2003; Yodzis, 1989)[21, 22] as Type III functional response is generally only applicable to generalist, not specialist, predators, and is perhaps less widely applicable in typical natural environments than other models.

3   Chaos and Stability Analysis

In this section the chaotic behavior of the food chain model will be investigated. Also, we will discuss the linear stability analysis this model. Further, we examine the behavior of the trajectories of the subsystems of the food chain model near the equilibrium points.

For a biological food web model to be logically credible, it must satisfy the following conditions:

1. The equations should be invariant under identification of identical species.

2. The system of equations for a food web must be separate into independent subsystems if the community splits into disconnected sub-webs.

The interaction of two competing predators and one prey given by (1) satisfy the above conditions. So the system (1) can be separated into two independent subsystems. The first subsystem is obtained by assuming the absence of the predator prey x_2 and the second is obtained by assuming the absence of the super predator prey x_3.

The first subsystem of (1) is obtained by setting the super predator is absent:

$$\left.\begin{aligned} \dot{x}_1 &= x_1\left[r(1 - k^{-1}x_1) - \frac{a_2 x_2}{b_2 + x_1}\right], \\ \dot{x}_2 &= x_2\left[\frac{e_2 a_2 x_1}{b_2 + x_1} - d_2\right], \end{aligned}\right\} \tag{2}$$

The second subsystem of (1) is obtained by setting the predator is absent:

$$\left.\begin{aligned} \dot{x}_1 &= rx_1(1 - k^{-1}x_1), \\ \dot{x}_3 &= -d_3 x_3, \end{aligned}\right\} \tag{3}$$

It is easy to show that all solutions of (2) and (3) are bounded in the future and remain in the regions $\{(x_1, x_2) : x_1 > 0 \text{ and } x_2 > 0\}$ for (2) and in the regions $\{(x_1, x_3) : x_1 > 0 \text{ and } x_3 > 0\}$ for (3).

In what follows we examine the behavior of the trajectories of the subsystems (2) and (3) near the equilibrium points. The first subsystem (2) has thee equilibrium points which are given by

$$E_{11} = (0,0), \quad E_{12} = (k,0), \quad E_{13} = (\bar{x}_1, \bar{x}_3), \quad \bar{x}_1 = b_2 d_2/(e_2 a_2 - d_2), \quad \bar{x}_2 = r(k - \bar{x}_1)(b_2 + \bar{x}_1)/r \tag{4}$$

where

$$\bar{x}_1 = b_2 d_2/(e_2 a_2 - d_2), \quad \bar{x}_2 = r(k - \bar{x}_1)(b_2 + \bar{x}_1)/a_2 k \tag{5}$$

The characteristic equation of the first equilibrium point has the eigenvalues $\lambda_{11} = r$ and $\lambda_{12} = -d_2 < 0$ which are real and λ_{12} lies in the left-half plane and λ_{11} lies in the right-half plane. Therefore, the first equilibrium point of the subsystem (2) is unstable saddle point. The necessary condition for linear stability of E_{12} is

$$d_2 > \frac{ka_2\,e_2}{b_2 + k} \tag{6}$$

and the necessary condition for linear stability of E_{13} is

$$d_2 > \frac{e_2 a_2(k - b_2)}{k + b_2} \tag{7}$$

Therefore E_{13} is the only stable equilibrium point of the subsystem (2) if the system parameters satisfy the condition:

$$\frac{e_2 a_2 (k - b_2)}{k + b_2} < d_2 < \frac{k a_2 e_2}{b_2 + k} \tag{8}$$

The steady-states of the food chain dynamical system (1) are given by:

$$E_1 = (0, 0, 0), \quad E_2 = (k, 0, 0), \quad E_3 = (\nu_1, \nu_2, 0), \quad E_4 = (\bar{x}_1, \bar{x}_2, \bar{x}_3), \tag{9}$$

where

$$\nu_1 = b_2 d_2 / (e_2 a_2 - d_2), \quad \nu_2 = r(k - \bar{x}_1)(b_2 + \bar{x}_1)/a_2 k \tag{10}$$

and

$$\left.\begin{array}{l} \bar{x}_1 = \left[(k - b_2)r + \sqrt{(k - b_2)^2 \, r^2 - 4r(a_2 k \bar{x}_2 - r k b_2)} \right]/2r, \\[2mm] \bar{x}_2 = b_3 d_3 / (a_3 e_3 - d_3), \quad \bar{x}_3 = (b_3 + \bar{x}_2)[a_2 e_2 \bar{x}_1 - d_2(b_2 + \bar{x}_1)]/a_2(b_2 + \bar{x}_1) \end{array}\right\} \tag{11}$$

The linear stability analysis of the food chain dynamical system (1) indicates that this system has the following equilibrium states. The first steady-state E_1 is absolutely unstable, while the second steady-state E_2 is linear stable if the condition (6) is satisfied. The other steady-states need further stability analysis.

The following figures display the numerical solution of the uncontrolled food chain dynamical system with different values of the initial densities of prey, predator and supper-predator and system parameters.
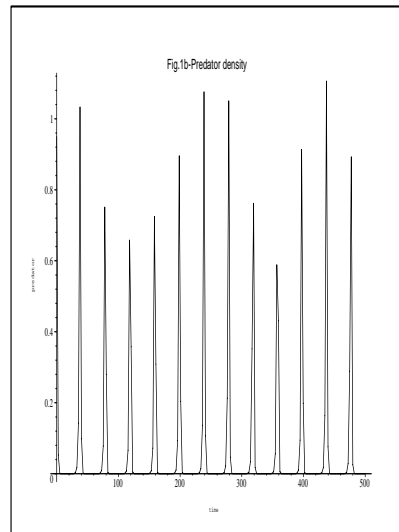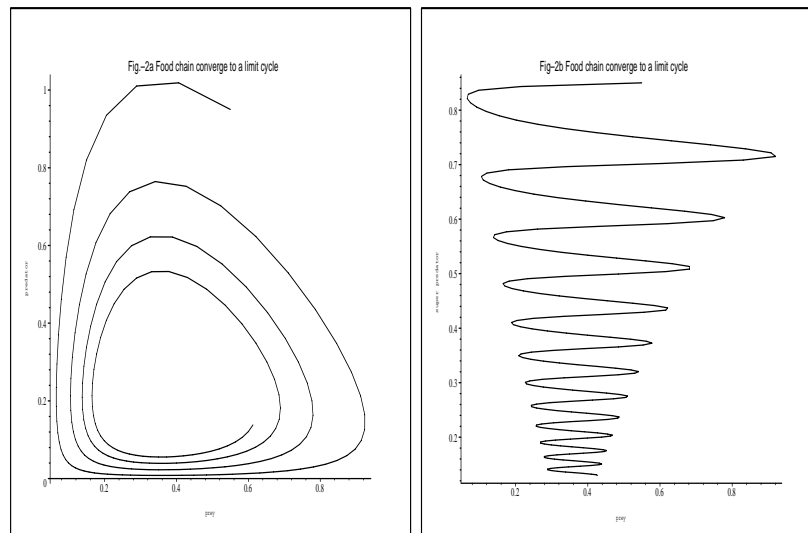

Fig.1a-prey density   Fig.1b-Predator density
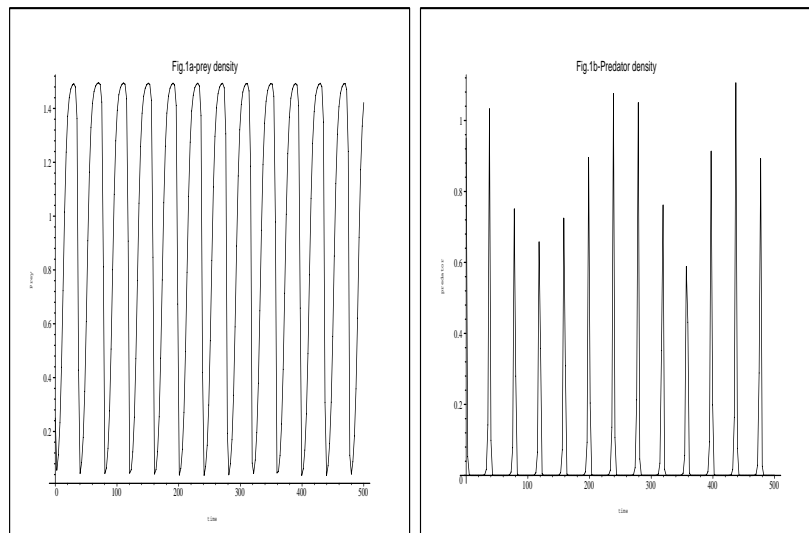
Figure 1. Densities of prey, predator and supper-predator for a set values of the system parameters and initial densities $r = 0.32$, $k = 1.5$, $a_2 = 1.6$, $b_2 = 0.65$, $e_2 = 2.1$, $a_3 = 0.5$, $b_3 = 0.39$, $d_2 = 0.84$, $e_3 = 0.5$, $d_3 = 0.02$ and $x_1(0) = 0.25$, $x_2(0) = 0.95$, $x_3(0) = 0.85$..
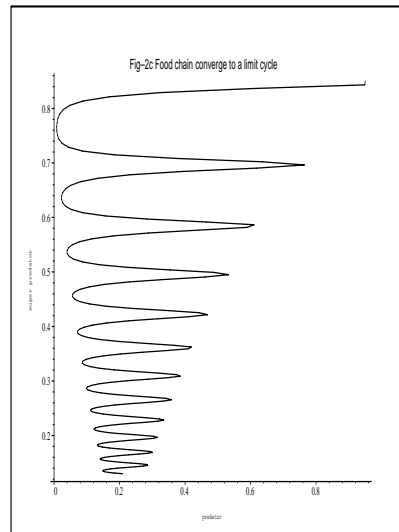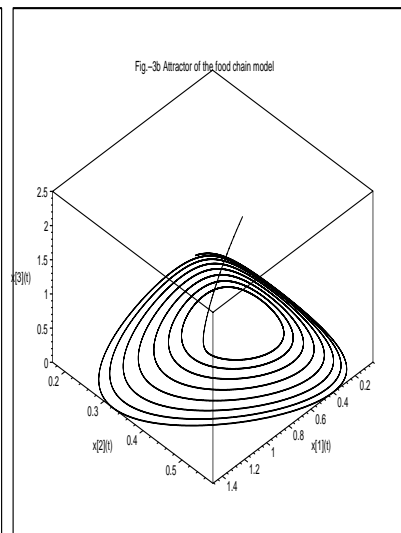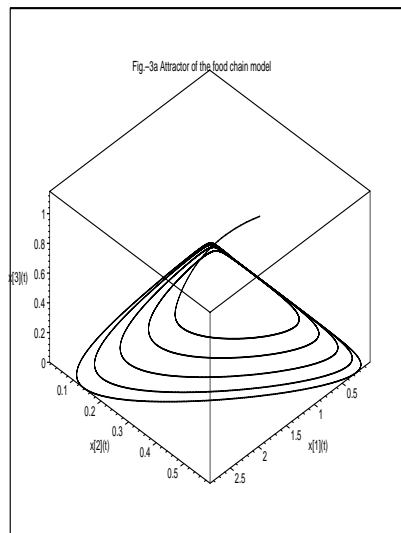
Figure 2. Three high frequency limit cycles correspond to the values of the system parameters $r = 6.3, k = 2.6, a_2 = 1.5, b_2 = 1.4, e_2 = 1.5, d_2 = 2.9, a_3 = 2.1, b_3 = 2.3, e_3 = 0.2, d_3 = 3.9$ respectively, and initial densities $x_1(0) = 0.3, x_2(0) = 1.5, x_3(0) = 0.5$.
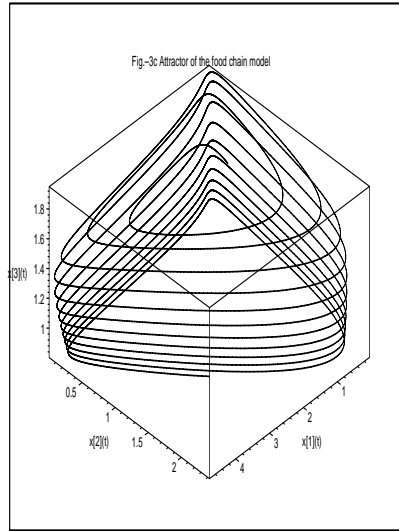
Figure 3. Three attractors and quasi-attractors of the food chain model correspond to the values of the system parameters $r = 6.3, k = 2.6, a_2 = 1.5, b_2 = 1.4, e_2 = 1.5, d_2 = 2.9, a_3 = 2.1, b_3 = 2.3, e_3 = 0.2, d_3 = 3.9$ respectively, and initial densities $x_1(0) = 0.3, x_2(0) = 1.5, x_3(0) = 0.5$.

The numerical solution of the food chain model indicates that the system has have a distinct geometric configuration limit cycles and attractors occur for some values of the system parameters. Such that limit cycles represent a steady-state oscillation, from which all trajectories nearby will converge or diverge. A limit cycle in a nonlinear system describes the amplitude and period of a self-sustained oscillation.

In the next section, we discuss the problem of adaptive control of the food chain model using nonlinear feedback approach.

4 Adaptive Control Problem

This section devoted to study the problem of a adaptive control of the food chain model. To discuss the adaptive control of the food chain model using nonlinear feedback control approach, we start by assuming that the system (1) can be written in the following suitable form

$$
\left.
\begin{aligned}
\dot{x}_1 &= x_1\Big[r(1 - k^{-1}x_1) - \frac{a_2 x_2}{b_2 + x_1} + u_1\Big], \\
\dot{x}_2 &= x_2\Big[\frac{e_2 a_2 x_1}{b_2 + x_1} - \frac{a_2 x_3}{b_3 + x_2} - d_2 + u_2\Big], \\
\dot{x}_3 &= x_3\Big[\frac{e_3 a_3 x_2}{b_3 + x_2} - d_3 + u_3\Big],
\end{aligned}
\right\}
\tag{12}
$$

where u_1, u_2 and u_3 are control inputs that will be suitably choice to make the trajectory of the whole system (1) that specified by the equilibrium states $E_1(0,0,0)$, $E_2(k,0,0)$ $E_3(\nu_1,\nu_2,0)$ and $E_4(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ to be asymptotically stable about these equilibrium states of the uncontrolled system. Note the form (12) is more suitable for study the adaptive control of the system (1).

If $u_i = 0$, $(i = 1,2,3)$ then the system (1) and (12) have absolutely an unstable special solution

$$x_1 = x_2 = x_3 = 0 \tag{13}$$

where $x_i = 0$, $(i = 1,2,3)$ is the trivial equilibrium point of the uncontrolled system (1). The eigenvalues of the characteristic equation for the trivial equilibrium point (12) has the following real values

$$\lambda_1 = r > 0, \quad ,\lambda_2 = -d_2 < 0 \quad ,\lambda_3 = -d_3 < 0 \tag{14}$$

Hence this trivial equilibrium point is absolutely unstable. In this study we will asymptotically stabilized this point using the control inputs u_1, u_2 and u_3.

Adaptive control is one of those research topics that have received much attention from the systems and control theory and engineering societies, but even so it has always been a controversial. A possible definition of adaptive control is a system that adapts itself to change in the process. Adaptation is a fundamental characteristic of living organisms such as food chain systems and host parasite model and many other biological systems since they attempt to maintain physiological equilibrium in the midst of changing environmental conditions. An approach to the design of adaptive systems is then to consider the adaptive aspects of human or animal behavior and to develop systems which behave somewhat analogously.

The following theorem provides us the adaptive control inputs u_1, u_2 and u_3 that asymptotically stabilized the food chain system (12) trivial its equilibrium state.

Theorem 4.1 *Using the nonlinear feedback controllers*

$$\left.\begin{array}{l} u_1 = a_2 x_2 - r(1 - x_1/k)(b_2 + x_1) - \alpha_1, \\[2mm] u_2 = a_2 x_3(b_2 + x_1) + (b_3 + x_2)[d_2(b_2 + x_1) - a_2 e_2 x_1] - \alpha_2, \\[2mm] u_3 = d_3(b_3 + x_2) - a_3 e_3 x_2 - \alpha_3, \end{array}\right\} \tag{15}$$

the system (12) will be asymptotically stable in the Liapunov sense about its equilibrium state (13).

Proof. The proof of this theorem can be reached by using Liapunov stability theorem which gives sufficient conditions for asymptotic stability. Substituting (15) into (12) one can get the following nonlinear system of differential

equations:

$$\begin{aligned}
\dot{x}_1 &= -\frac{\alpha_1 x_1}{b_2 + x_1}, \\[2mm]
\dot{x}_2 &= -\frac{\alpha_2 x_2}{(b_2 + x_1)(b_3 + x_2)}, \\[2mm]
\dot{x}_3 &= -\frac{\alpha_3 x_3}{b_3 + x_2},
\end{aligned} \right\} \tag{16}$$

where α_1, α_2 and α_3 positive control gains constants.

Let us consider the Liapunov function for the (16) in the form

$$2V(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 \tag{17}$$

Obviously, the functions (17) is a positive definite form with respect to the variables x_1, x_2 and x_3 and its time derivative along the trajectory of the system (16) is given by

$$\dot{V} = -\frac{\left[\alpha_1(b_3 + x_2)x_1^2 + \alpha_2 x_2^2 + \alpha_3(b_2 + x_1)x_3^2\right]}{(b_2 + x_1)(b_3 + x_2)}, \tag{18}$$

Since the densities $x_1(t), x_2(t)$ and $x_3(t)$ are usually positive and the parameters $\alpha_1, \alpha_2, \alpha_3, b_2$ and b_3 are also take only positive values then

$$b_2 + x_1(t) > 0, \quad b_3 + x_2(t) > 0, \; \forall \, x_1(t), x_2(t), x_3(t) > 0 \tag{19}$$

Using the variable gradient method we find that

$$\nabla V \dot{\vec{x}} = -\left(\frac{\alpha_1}{b_3 + x_2}\right) x_1^2 - \left(\frac{\alpha_2}{(b_2 + x_1)(b_3 + x_2)}\right) x_2^2 - \left(\frac{\alpha_3}{b_3 + x_2}\right) x_3^2 \tag{20}$$

Using the inequalities (19) we can verify that the function in (20) and so (18) are negative definite forms which proves the asymptotic stability of the system (16) in the Liapunov sense. Therefore the coupled system (12) is asymptotically stable with the nonlinear feedback controllers (15), which completes the proof [15].

Now we can easily conclude that the food chain model populations can be asymptotically stabilized using nonlinear feedback controllers about its trivial equilibrium point.

5   Analysis and Numerical Simulation

The main objective of the numerical simulation is to obtain the numerical solution of the non-linear system (16). Some numerical examples for the controlled

food chain model were carried out for various parameters values and different initial densities.
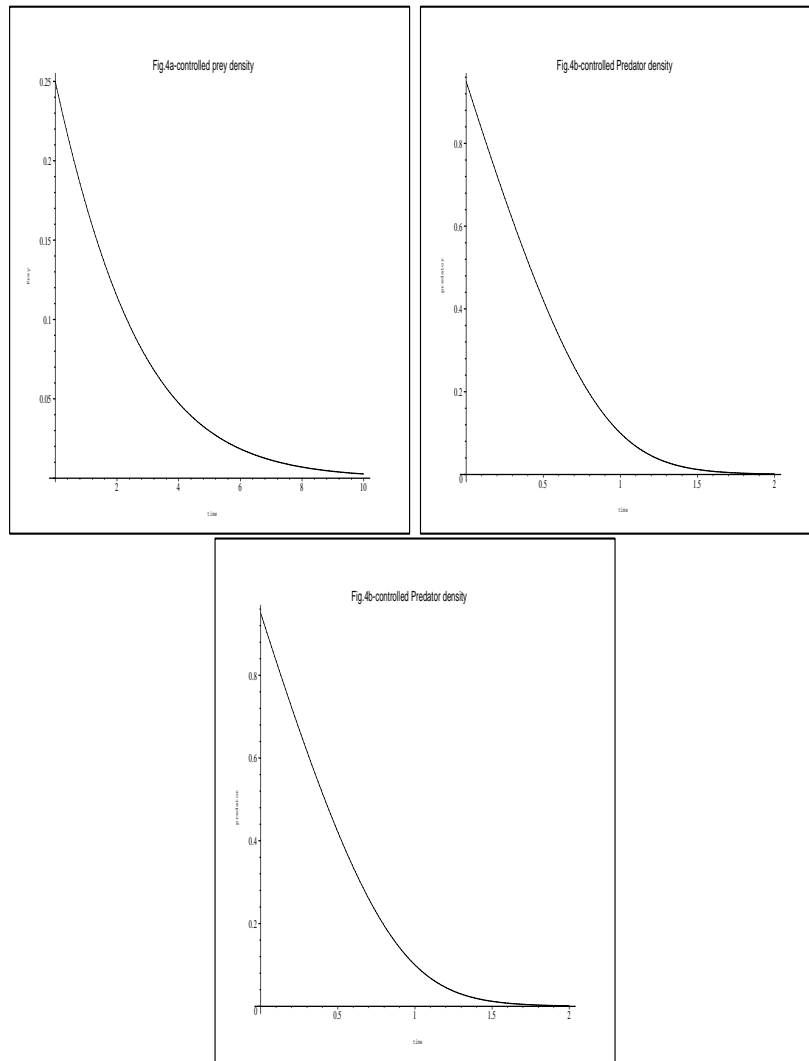

Figure 4. Densities of the controlled prey, predator and supper-predator for the set values of the system parameters and initial densities $r = 0.32$, $k = 1.5$, $a_2 = 1.6$, $b_2 = 0.65$, $e_2 = 2.1$, $a_3 = 0.5$, $b_3 = 0.39$, $d_2 = 0.84$, $e_3 = 0.5$, $d_3 = 0.02$ and $x_1(0) = 0.25$, $x_2(0) = 0.95$, $x_3(0) = 0.85$. Note that there is an exponentially convergence rate of the food chain densities.
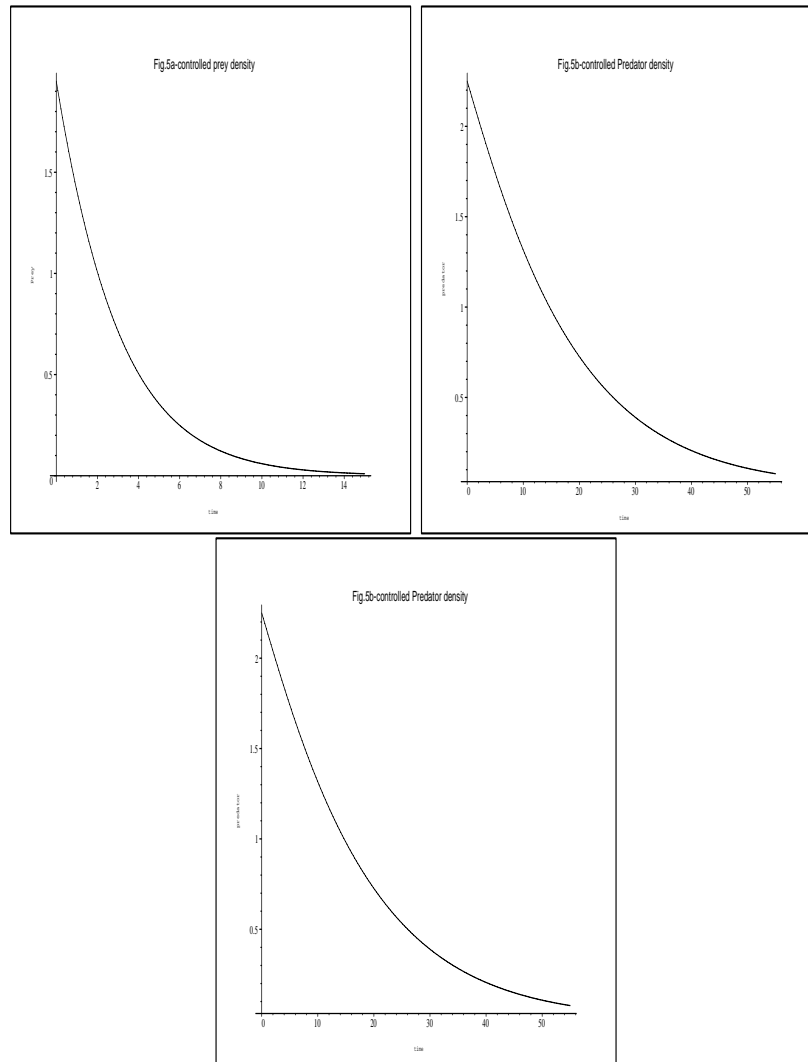
Figure 5. Densities of the controlled prey, predator and supper-predator for the set values of the system parameters and initial densities $r = 5.5$, $k = 9.5$, $a_2 = 8.6$, $b_2 = 15.65$, $e_2 = 2.1$, $a_3 = 6.5$, $b_3 = 9.9$, $d_2 = 8.4$, $e_3 = 3.5$, $d_3 = 5.2$ and $x_1(0) = 1.95$, $x_2(0) = 2.25$, $x_3(0) = 1$. Note that there is an exponentially convergence rate of the food chain densities.

From the numerical study, we can conclude that the controlled food chain has exponential damping with damping rates depends upon the parameters value. Therefore, the food chain system that consists of one prey, one predator and super-predator can be asymptotically stabilized using nonlinear feedback control law.

6   Conclusion

The chaotic behavior of continuous time food chain system is investigated. The ranges of the system parameters for which subsystems converge to limit cycles are determined. The problem of adaptive control of the food chain model is studied. The asymptotic stability of the controlled system is proved using the Liapunov function. The necessary control inputs for this asymptotic stability is obtained as nonlinear feedback. Finally, extensive numerical examples and simulation are introduced.

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