THE TENSOR PRODUCT OF SEMILOCAL ALGEBRAS

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Abstract

M. Sweedler [5] proved that if a tensor product of two commutative algebras over a field is local then each of the algebras is local and that the tensor product of the residue field is local. Moreover, one of the algebras most be algebraic over the ground field. But the tensor product of two Artinian algebras over a field in general need not be Artinian. In this paper we generalize M. Sweedler theorem and shows that tensor product of two algebras over a field K is semilocal if and only if each of these algebras is semilocal.

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0 Introduction

Semilocal algebras has many approaches, either algebraic or geometric. Semilocal algebra is an important subject in algebra as it is related to the concept of semisimple Artinian algebra over a field K which is characterize as a direct product of a finite number of simple algebra ideals each of which is isomorphic to square matrix over a division K-algebra (a generalization of Wedderburn and

Artian theorem) see Hungerford[7]page 425. Moreover any semilocal algebra has finitely many maximal algebra ideals, so this concept is a generalization of local algebra. G. Bergman [2] shows that the tensor product $A \otimes_K B$ of two Artinian modules over a commutative ring K has a finite length. R. H. Zhang [4] gave several characterization of semilocal rings. M. Sweedler [5] proved that if a tensor product of two commutative algebras over a field is local then each of the algebras is local and that the tensor product of the residue field is local. Moreover, one of the algebras most be algebraic over the ground field.On the other hand Exercise IX.6.6 in Hungerford [7] asserts that the tensor product of two Artinian, algebras over a field need not be Artinian.It is well known that the tensor product of two local algebras over a field need not to be a local algebra. However, if the field is an algebraically closed field we have the tensor product of two local algebras is a local algebra. In this paper we generalize M. Sweedler theorem and shows that tensor product of two algebras over an algebraically closed field K is semilocal if and only if each of these algebras is semilocal.

All rings and algebras will be associative and unital. The Jacopson Radical of a ring R will be denoted by J(R). Let K be an algebraically closed field and A be an algebra over K. This means that A is a ring as well as a vector space over K such that $\lambda(ab) = (\lambda a)b = a(\lambda b)$, for all $\lambda \in K$ and $a, b \in A$.

1 Preliminaries and Definitions

In this section, we recast some definitions and some standard facts which can be used to reach our main results in this paper. There are many characterizations of the notion of Semilocal rings and modules(see [2]and [1] and the references their for such characterizations.But we shall use the following definition which is very amenable to get the main result.

Definition 1.1 *A ring R is said to be semisimple if its Jacobson radical J(R) is zero.*

Definition 1.2 *A module A over a ring R is said to be semisimple if it can be written as a sum of a family of simple R-submodules.*

In particular every module over a (left)Artinian semisimple ring is semisimple. In the following we will give a definition of a samilocal algebra A, Algebraist define it in two ways either equivalent to $A/J(A)$,be Artinian algebra or $A/J(A)$,be semisimple Artinian algebra. In our case we will take the first definition since we will assume those algebras are over a field K and hence semisimple Artinian algebra is equal to Artinian algebra.

Definition 1.3 *An algebra A over a field K is called a* ***semilocal*** *algebra if $A/J(A)$,is Artinian K-algebra. where $J(A)$ is the Jacobson radical of A.*

Note that: Artinian, Noetherian, finitely generated, are all equivalent for a semisimple module, we will use this fact in the prove of theorems in the following section.

2 Tensor product of semilocal algebras

The tensor product is an important tool in the so-called Multilinear algebra. It is an operation for which we reform and construct certain algebraic structures.Let A and B be two algebras over the same field K. We shall assume that the field K is an algebraically closed field. It is well known that the tensor product $A \otimes_K B$ is an algebra over K.

Theorem 2.1 *The Tensor product $A \otimes_K B$ of two Artinian modules over a field K has finite length.*

However in general the tensor product of two Artinian algebras over a field need not be Artinian. For if A is a division algebra with center K and maximal subfield B such that A is infinite dimensional as a lift B module then the tensor will not be Artinion. But if we add the condition that B and A are semisimple then the tensor product will be also Artinion as we will show in the following.

Theorem 2.2 *the Tensor product $A \otimes_K B$ of two semisimple Artinian K-algebras over a field K is Artinian.*

Proof. $A \otimes_K B$ is an A-module with the action of a in A on a generator $a_1 \otimes_K b$ of $A \otimes_K B$ given by$a(a_1 \otimes_K b) = aa_1 \otimes_K b = (a \otimes_K 1_B\ )(a_1 \otimes_K b\ )$. Consequently every left ideal of $A \otimes_K B$ is also an A-submodul of $A \otimes_K B$. As B is semisimple Artinian K module, so it is finitely generated as a K-module. But A has a copy of K, so if X is a finite set of B generator, then $1 \otimes_K X$ will generate $A \otimes_K B$ as an A-module, and hence $A \otimes_K B$ is finitely generated as an A-module. But A is also semisimple Artinian as K-module, and hence it is finitely generated as a K-module. Hence $A \otimes_K B$ is finitely generated as a K-module. Then any descending chain of K-algebra ideals of $A \otimes_K B$ is stationary. And $A \otimes_K B$ is Artinian K-algebra

Theorem 2.3 *the Tensor product $A \otimes_K B$ of two semilocal K-algebras over a field K is semilocal.*

Proof.Let A,B be two semilocal K-algebras where K is a field. Then by definition $A/J(A), B/J(B)$ are Artinian K-algebras, also $A/J(A), B/J(B)$ are semisimple over a field K. Then by Theorem 3 we have $A/J(A) \otimes_K B/J(B)$ is Artinian K-algebra. But

$$A \otimes_K B/J(A \otimes_K B) \cong A \otimes_K B/(J(A) \otimes_K B + A \otimes_K J(B)) \cong A/J(A) \otimes_K B/J(B).$$

Hence $A \otimes_K B/J(A \otimes_K B)$ is Artinian K-algebra and hence $A \otimes_K B$ is semilocal K-algebra.

3 Main Results

In this section we will give the main result of this paper which is a generalization of M. Sweedler result on local algebras. We show that the tensor product of two algebras over an algebraically closed field K is semilocal algebra if and only if both algebras are semilocal over K.

Theorem 3.1 *[4] Let A and B be K-algebras. If $A \otimes_K B$ is semilocal, then A is semilocal.*

Theorem 3.2 *Let A and B be K-algebras. If $A \otimes_K B$ is semilocal, then both A, B are semilocal.*

Proof. Let A and B be K-algebras and assume that $A \otimes_K B$ is Semilocal, then by above Theorem, A is semilocal. But $A \otimes_K B \cong B \otimes_K A$,and hence B is also semilocal.

From Theorem 7 and Theore 8 we can get the following result:

Theorem 3.3 *Let A and B be K-algebras. Then $A \otimes_K B$ is semilocal if and only if both A, B are semilocal K-algebras.*

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