Denker **Spring 2010**

416 Stochastic Modeling - Assignment 5

Solution

NOTE: All problems from assignemnts 1–5 may appear in the first exam on Feb 26!!!

Problem 1: (Example 4.6) A gambler beds at each play one third of his present fortune rounded to the next full \$-amount. For example, if his fortune is \$4, he beds \$2. At each play he wins twice the amount of his bed or he loses it. If he is broken he stops playing and if he owns a fortune of \$$N$, he stops as well. Model the gambler's fortune by a Markov chain. Classify the states and find absorbing states.

Solution:

Let Z_n be 1 or 0 according whether the gambler wins or loses the n-th game. Assume that the random variables are i.i.d. with winning probability p. Let X_n denote the fortune of the gambler after the n-th game.

To model the fortune as a Markov chain, let $g(z)$ denote the integer part of $z/3$. For example, if $z = 1$, $g(z) = 0$ and if $z = 3$, $g(z) = 1$. Then the gambler beds $g(X_n) + 1$ in the n+1st game unless he is broke, when the fortune falls below 1 \$. If he wins the n+1st game his fortune is $X_n + g(X_n) + 1$ and if he loses it is $X_n - g(X_n) - 1$. This can be expressed as

$$X_{n+1} = X_n + (2Z_{n+1} - 1)(g(X_n) + 1).$$

Therefore, X_n is a Markov chain with transition probabilities $p_{ij} = p$ if $j = i + (g(i) + 1)$ and $p_{ij} = 1 - p$ if $j = i - (g(i) + 1)$. All other transitions between states in $\mathbb{Z}$ are zero.

A loosing streak always will keep the player in the game until his fortune drops below 1\$. This follows from the fact that $g(z)+1 \leq g(3z) = z$ for all $z \geq 1$. Moreover, his fortune decreases by at least 1 \$ in each loss. Thus there is a positive probability to reach the state 1 from any other state (e.g. from state 3 he beds 2 \$ and losing leaves him with 1\$, or if his state is 2\$, he beds 1\$ and loses 1 to reach 1\$). It follows that the class containing 1 consists of all states wich are accessible from 1. This is the only class and the chain is irreducible if every state is accessible from 1. To decide which states are accessible from 1 is difficult and is not needed here.

Problem 2: (Example 4.13) Consider the matrix

$$\begin{pmatrix} 0 & 0 & 1/3 & 0 & 1/3 & 1/3 \\ 0 & 3/4 & 0 & 0 & 0 & 1/4 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 3/4 & 1/4 \\ 0 & 1/6 & 0 & 0 & 0 & 5/6 \end{pmatrix}$$

Which states are transient and which are recurrent?

Solution: Let 1,2,...,6 denote the states (row numbers of the matrix). State 4 goes to 6 and state 4 is not accessible from any other state. Thus the probability that state 4 returns is zero and it is a transient state. The state 1 is only accessible from 3, and 3 is only accessible from 1. Hence the return to state 1, $P(X_n = 0 \text{ for some } n \geq 1 | X_0 = 0) = P(X_2 = 0 | X_0 = 0) = 1/3 < 1$, so 0 is a transient state as well, and so is 3 since it communicates with 0. The state 5 can only go to itself and to state 6 and is accessible only from itself and state 1. Hence the probability being in state 5 at some time $n \geq 1$ for the first time given that we start in 5, can only happen for $n = 1$. But this conditional probability is $3/4 < 1$, hence 5 is transient as well. We are left with states 2 and 6 which form a class. Since the chain must have a recurrent state it must be 2 or 6 and since recurrence is a class property both states are recurrent.

Problem 3: (Problem 6) Let

$$P = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

Find (by mathematical induction) P^n.

Solution:

Let $a_n = \frac{1}{2}(1 + (2p-1)^n)$ and $b_n = \frac{1}{2}(1 - (2p-1)^n)$. We claim that

$$P^n = \begin{pmatrix} a_n & b_n \\ b_n & a_n \end{pmatrix}$$

We show this by induction. For $n = 1$ the assertion is obviously true. Suppose it is true for P_n. Then the entries of P^{n+1} satisfy

$$\begin{aligned} a_n p + b_n(1-p) &= \frac{1}{2}[(1 + (2p-1)^n)p + (1 - (2p-1)^n)(1-p) = \frac{1}{2}(1 + (2p-1)^{n+1}) \\ a_n(1-p) + b_n p &= \frac{1}{2}(1 - (2p-1)^{n+1}) \\ b_n p + a_n(1-p) &= \frac{1}{2}(1 - (2p-1)^{n+1}) \\ b_n(1-p) + a_n p &= \frac{1}{2}(1 + (2p-1)^{n+1}) \end{aligned}$$

proving the claim.

Problem 4: (Problem 1) Three white and three black balls are distributed in two urns in such a way that each contains three balls. We say that the system is in state i, if the first urn contains i white balls. At each step, we draw one ball from each urn and place the ball drawn from the first urn into the second, and conversely with the ball drawn from the second urn. Let X_n denote the state of the system after the nth step. Explain why X_n is a Markov chain and calculate its transition probability. Classify the states, determine transient and recurrent states, and decide whether the chain is irreducible or aperiod.

Solution:

let $E = \{0, 1, 2, 3\}$ denote a state space of four symbols which represents the states k that the first urn contains k white balls. Let X_n denote the state of the first urn before the n-th drawing procedure. In the n-th drawing procedure let Y_n be 1 if the ball drawn

from the first urn is white (otherwise 0) and denote by Z_n the corresponding variable describing the drawing from the second urn. Then

$$\begin{aligned} P(Y_n = 1|X_n = 0) &= 0 & \qquad P(Z_n = 1|X_n = 0) &= 1 \\ P(Y_n = 1|X_n = 1) &= \frac{1}{3} & \qquad P(Z_n = 1|X_n = 1) &= \frac{2}{3} \\ P(Y_n = 1|X_n = 2) &= \frac{2}{3} & \qquad P(Z_n = 1|X_n = 2) &= \frac{1}{3} \\ P(Y_n = 1|X_n = 3) &= 1 & \qquad P(Z_n = 1|X_n = 3) &= 0 \end{aligned}$$

From these equations one easily computes the transition matrix of the Markov chain. For example, p_{12} is obtained as follows: $p_{12} = P(X_2 = 2|X_1 = 1)$ is the probability that the first urn has two white balls after the first round of drawings when originally there was exactly one ball in the first urn. In order to describe this event we use Y_1 and Z_1. The event can only happen if $Y_1 = 0$ and $Z_1 = 1$, hence the probability in question is 4/9.

This leads to the transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{9} & \frac{4}{9} & \frac{4}{9} & 0 \\ 0 & \frac{4}{9} & \frac{4}{9} & \frac{1}{9} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The chain is irreducible, recurrent and aperiodic, so no transient state.

Problem 5: Let X_n be a sequence of random variables with values in $\mathbb{Z}$, such that $X_n = X_{n-1}^2$. Is X_n a Markov chain? If so, determine the transition matrix and find all recurrent states.

Solution:

Since $X_{n+1} = X_n^2$, the process is a Markov chain. We have $P(X_{n+1} = k|X_n = j) = 1$ for $k = j^2$ (all other trnsitions from j are zero). Thus 0 and 1 are recurrent states, all other states are transient, and one has two classes.