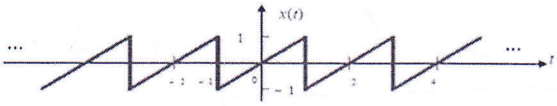
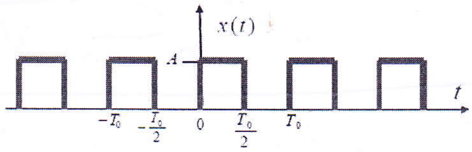
Q1) Compute the FS representation for the signals shown, and determine the DC part and the 1^{st} harmonic



Q2) (i) Compute the complex Fourier series (FS) representation for the periodic signal $x(t)$ shown.
(ii) Determine the DC part and the first harmonics of the signal $x(t)$.
(iii) Write an expression for the FS coefficients of the signal $x(t - T_o/4)$
(iv) Write an expression for the FS coefficients of the signal $dx(t)/dt$



Q3) Determine and sketch the FS coeffecients for $x(t) = \sin(2\pi t + \pi/4)$

Q1) $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$; $T_o = 2$

• $a_o = \frac{1}{T_o}\int_{T_o} x(t)\,dt = \frac{1}{2}\int_{-1}^{1} t\,dt = \frac{1}{2}\left(\frac{t^2}{2}\right)\Big|_{-1}^{1} = \frac{1}{4}\cdot 0 = 0$

• $a_k = \frac{1}{T_o}\int_{-1}^{1} x(t)\, e^{-jk\omega_o t}\,dt = \frac{1}{2}\int_{-1}^{1} t\, e^{-jk\omega_o t}\,dt$ let $u = t$, $u' = 1$, $v' = e^{-jk\omega_o t}$, $v = \frac{e^{-jk\omega_o t}}{-jk\omega_o}$

$\therefore a_k = \left(uv - \int u' v\,dt\right) * \frac{1}{2}$

$= \frac{1}{2}\left[t\,\frac{e^{-jk\omega_o t}}{-jk\omega_o}\Big|_{-1}^{1} - \int_{-1}^{1} \frac{e^{-jk\omega_o t}}{-jk\omega_o}\,dt \right]$ $\omega_o = \frac{2\pi}{T_o} = \pi$

$= \frac{1}{2}\left[\frac{e^{-jk\pi}}{-jk\pi} + \frac{e^{+jk\pi}}{-jk\pi} + \frac{1}{k^2\pi^2}\left(e^{-jk\pi} - e^{+jk\pi}\right)\right]$ $\Big|\ e^{\pm jk\pi} = (-1)^k$

$= \frac{(-1)^k}{-jk\pi}$; $DC = a_o = 0$; 1st $H = a_{\pm 1} = \frac{\mp 1}{j\pi}$

$\therefore$ $\boxed{x(t) = a_o + \sum_k a_k e^{jk\omega_o t}}$ $\forall\ k \neq 0$

Q2] $a_0 = \frac{1}{T_0}\int_0^{T_0/2} A\,dt = \frac{A}{T_0}\cdot\frac{T_0}{2} = A/2$

$a_k = \frac{1}{T}\int_0^{T_0/2} A\,e^{-jk w_0 t}\,dt = \frac{A}{-T_0 jk w_0}\,e^{-jk w_0 t}\Big|_0^{T_0/2} = \frac{-A}{j2\pi k}\left[e^{-jk\pi} - 1\right] = \frac{A}{j2\pi k}\left[1-(-1)^k\right]$

$\therefore\ a_k = \begin{cases} 0 & \text{if } k \text{ is even} \\ \frac{A}{j2\pi k} & \text{if } k \text{ is odd} \end{cases}$ (for k to be odd ; $k = 2m+1$)

$$\boxed{\therefore\ x(t) = \frac{A}{2} + \sum_{m=-\infty}^{\infty} \frac{A\,e^{-j(2m+1)w_0 t}}{j2\pi(2m+1)}}$$

ii) DC $= a_0 = A/2$

1st H; $a_{\pm 1} = \frac{\pm A}{j2\pi k}$

iii) FS coeff of $x(t - \frac{T_0}{4}) \longrightarrow a_k\,e^{-jk w_0 \frac{T_0}{4}} = a_k\,e^{-jk\frac{\pi}{2}}$

iv) " " " $\frac{d\,x(t)}{dt} \longrightarrow a_k\,jk w_0$

Q3] $x(t) = \sin(2\pi t + \frac{\pi}{4})$

$= \frac{1}{2j}\left[e^{j(2\pi t + \pi/4)} - e^{-j(2\pi t + \frac{\pi}{4})}\right]$

$= \frac{e^{j\frac{\pi}{4}}}{2j}\left[e^{jw_0 t}\right] + \frac{-e^{-j\frac{\pi}{4}}}{2j}\left[e^{-jw_0 t}\right]$

$= \underline{\underline{a_1}}\left[e^{jw_0 t}\right] + \underline{\underline{a_{-1}}}\left[e^{-jw_0 t}\right]$

$w = 2\pi \longrightarrow T = 1$

$\therefore\ a_k = \begin{cases} \frac{e^{j\pi/4}}{2j} & k = 1 \\ \frac{-e^{-j\frac{\pi}{4}}}{2j} & k = -1 \\ 0 & \text{O.W} \end{cases}$

compare

Note:- from FS:- $\left\{ x(t) = \sum a_k e^{jk w_0 t} = \cdots + a_2 e^{2jw_0 t} + a_1 e^{jw_0 t} + a_0 + a_{-1} e^{-jw_0 t} + \cdots \right\}$

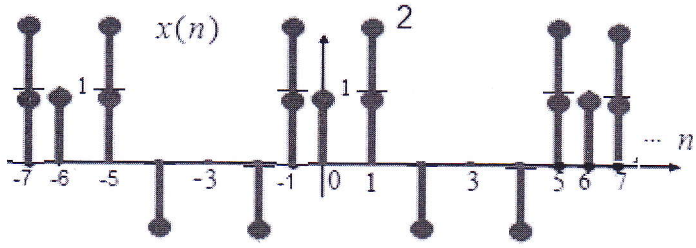
- **Fourier Series Representation (in DT):-**

The main difference between DTFS & CTFS is that in DT the FS coefficients are periodic ($a_k = a_{k+N}$)
(Note that if $\Phi_k[n] = e^{jk\omega_o n}$; then $\Phi_{k+N}[n] = e^{j(k+N)\omega_o n} = e^{j(k+N)(2\pi/N)n} = e^{j(k)(2\pi/N)n} = \Phi_k[n]$)

→ $\boxed{x[n] = \sum_{k=\langle N\rangle} a_k\, e^{jk(2\pi/N)n}}$; FS coefficients $\boxed{a_k = \frac{1}{N}\sum_{k=\langle N\rangle} x[n]\, e^{-jk(2\pi/N)n}}$;

The DTFS properties are shown in Table3.2 in attached Fourier tables pdf file.

Q1) Find the FS representation of the following signal



Q2) Determine the FS coefficient of the signal x(n) = sin(πn). Sketch a_k from [-3,3]

Q1 $x[n] = \sum_{k=\langle N\rangle} a_k e^{+jk\omega_0 n} = \sum_{k=\langle 6\rangle} a_k e^{jk\frac{2\pi}{6}n}$

$a_k = \frac{1}{N_0}\sum_{n=\langle N\rangle} x[n] e^{-jk\omega_0 n} = \frac{1}{6}\sum_{n=-2}^{3} x[n] e^{-jk\frac{\pi}{3}n}$

$= \frac{1}{6}\left[x[0]e^{-jk\frac{\pi}{3}(0)} + x[1]e^{-jk\frac{\pi}{3}(1)} + x[2]e^{-jk\frac{\pi}{3}(2)} + x[3]e^{-jk\frac{\pi}{3}(3)} + x[-1]e^{-jk\frac{\pi}{3}(-1)} + x[-2]e^{-jk\frac{\pi}{3}(-2)}\right]$

$= \frac{1}{6}\left[1 + 2e^{-jk\frac{\pi}{3}} - 1e^{-jk\frac{2\pi}{3}} + 0 + 2e^{+jk\frac{\pi}{3}} - e^{+jk\frac{2\pi}{3}}\right]$

$= \frac{1}{6}\left[1 + 2\cdot(2)\frac{\left(e^{-jk\frac{\pi}{3}} + e^{jk\frac{\pi}{3}}\right)}{2} + (-2)\frac{\left(e^{-jk\frac{\pi 2}{3}} + e^{jk\frac{2\pi}{3}}\right)}{2}\right]$

$= \frac{1}{6}\left[1 + 4\cos(k\frac{\pi}{3}) - 2\cos(\frac{2\pi}{3})\right]$

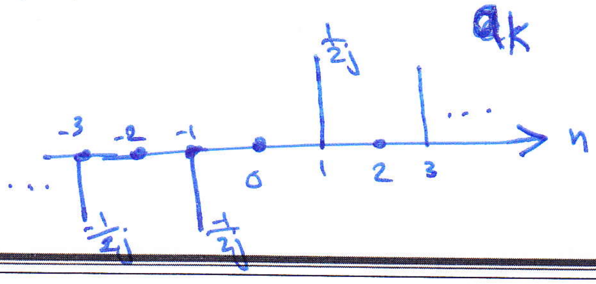
Q2 $x(n) = \frac{1}{2j}\left(e^{j\pi n} - e^{-j\pi n}\right) = \frac{1}{2j}\left(e^{j\omega_0 n} - e^{-j\omega_0 n}\right)$ $N = \frac{2\pi}{\omega_0} = 2$

FS ⇒ $x(n) = \sum_{n=\langle N\rangle} a_k e^{+jk\omega_0 n} = \sum_{n=0}^{1} a_k e^{jk\omega_0 n} = a_0 e^{jk\omega_0(0)} + a_1 e^{jk\omega_0(1)} + \dots$

by comparing ⇒ $a_0 = 0$ $a_1 = \frac{1}{2j}$

Using $a_{k+n} = a_k$

& $a_k = a_k^*$ (for real sg.)



Q1 $y(n) = h(n) * x(n) \xleftrightarrow{FT} Y(e^{jw}) = H(e^{jw})\, X(e^{jw}) \Rightarrow H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})}$

& from $x(n-n_0) \xrightarrow{FT} e^{-jwn_0} X(e^{jw})$

$$Y(e^{jw}) = 2X(e^{jw}) + \frac{1}{3} e^{-jw(1)} Y(e^{jw})$$

$$Y(e^{jw})\left(1 - \frac{1}{3}e^{-jw}\right) = 2X(e^{jw})$$

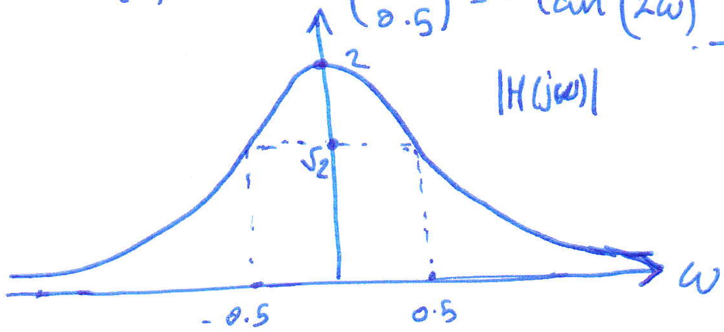
$$\Rightarrow H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \frac{2}{1 - \frac{1}{3}e^{-jw}}$$

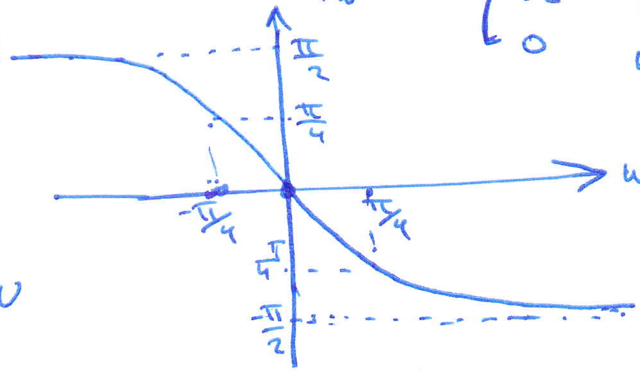
Q2 a) i) $H(jw) = \int_{-\infty}^{\infty} h(t)\, e^{-jwt}\, dt$

$$= \int_0^{\infty} e^{-0.5t} e^{-jwt}\, dt = \int_0^{\infty} e^{-t(0.5+jw)}\, dt = \frac{-1}{(0.5+jw)} e^{-t(+)}\Big|_0^{\infty} = \frac{1}{0.5+jw}$$

ii) $|a+jb| = \sqrt{a^2+b^2}$; $\left|\frac{1}{0.5+jw}\cdot\frac{0.5-jw}{0.5-jw}\right| = \left|\frac{0.5-jw}{0.5^2+w^2}\right| = \frac{\sqrt{0.5^2+w^2}}{0.5^2+w^2} = \begin{cases} 2 & w=0 \\ \sqrt{2} & w=\pm 0.5 \\ 0 & w=\pm\infty \end{cases}$

$\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{-w}{0.5}\right) = -\tan^{-1}(2w)$



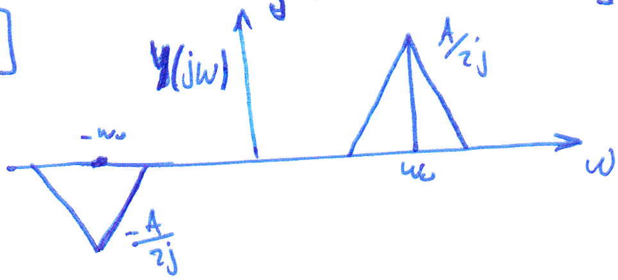


b) i] $FT\{y(t)\} = FT[x(t)\sin(2\pi t)]$

from $x(t)\,y(t) \xrightarrow{FT} \frac{1}{2\pi}[X(jw) * Y(jw)]$ & $\sin(w_0 t) \xrightarrow{FT} \frac{\pi}{j}[\delta(w-w_0) - \delta(w+w_0)]$

$$\Rightarrow Y(jw) = \frac{1}{2\pi}\left[X(jw) * \frac{\pi}{j}(\delta(w-w_0) - \delta(w+w_0))\right]$$

$$= \frac{1}{2j}\left[X(j(w-w_0)) - X(j(w+w_0))\right]$$



ii) $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)\, e^{jwt}\, dw$

$$= \frac{1}{2\pi}\int_{-5}^{5} 2e^{jwt}\, dw = \frac{2}{2\pi jt} e^{jwt}\Big|_{-5}^{5} = \frac{2}{\pi t}\cdot\sin(5t)$$

iii) $* \; F\mathcal{T}^{-1}\{P_a(w)\} \Rightarrow \frac{\sin at}{\pi t} = P_a(t)$

$\&$ <u>Freq-shift</u>: $X(j(w-w_o)) \longrightarrow e^{jw_o t} x(t)$

$\therefore x(t) = 0.5\, e^{jw_o t} P(t) + 0.5\, e^{-jw_o t} P(t)$

$= 0.5 P(t) \left[e^{jw_o t} + e^{-jw_o t} \right]$

$= \frac{\sin at}{\pi t} \cos(w_o t)$

c) i] $* \; t x(t) \xrightarrow{F\mathcal{T}} j \frac{d}{dw} X(jw)$

$\therefore t e^{-|t|} \xrightarrow{F\mathcal{T}} j \frac{d}{dw}\left(\frac{2}{1+w^2}\right) = j \frac{-2(2w)}{(1+w^2)^2} = j \frac{-4w}{(1+w^2)^2}$

ii] from duality $x(t) \xleftrightarrow{F\mathcal{T}} X(jw)$

$X(t) \xleftrightarrow{F\mathcal{T}} 2\pi\, x(-w)$

$\therefore \frac{1}{(-j)} \frac{-j4t}{(1+t^2)^2} \xleftrightarrow{F\mathcal{T}} \frac{1}{(-j)} 2\pi(-w) e^{-|-w|} = -j2\pi w\, e^{-|w|}$

Q3] a) i] From $\delta[n] \xrightarrow{F\mathcal{T}} 1$ $\&$ $x[n-n_o] \longrightarrow e^{-jwn_o} X[e^{jw}]$

$\Rightarrow F\mathcal{T}\{x[n]\} = 2 + 2e^{-jw} - 2e^{-jw2} - 2e^{-jw3} = X(e^{jw})$

ii] $F\mathcal{T}[x[-(n-2)]] = e^{-jw2} X(e^{-jw})$

b) $X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn}$

$= \sum_{2}^{\infty} a^n (e^{-jw})^n = \sum_{2}^{\infty} (a e^{-jw})^n = \frac{(ae^{-jw})^2 - 0}{1 - ae^{-jw}} = \frac{a^2 e^{-jw2}}{1 - ae^{-jw}}$

c) $X(e^{jw}) = \cos^2 w$ $\quad \left(* \; \delta[n-n_o] \xrightarrow{F\mathcal{T}} e^{-jw_o n_o} \right)$

$= \left[\frac{e^{-jw} + e^{jw}}{2} \right]^2$

$= \frac{1}{4}\left(e^{-2jw} + 2e^{0} + e^{2jw} \right) \xrightarrow{F\mathcal{T}^{-1}} \frac{1}{4}\left[\delta(n-2) + 2\delta[n] + \delta[n+2] \right]$