TUTORIAL - 3
SIGNALS AND SYSTEMS

EXAMPLE 3.1

RESPONSE OF LTI SYSTEMS TO COMPLEX EXPONENTIALS.

$$y(t) = H(s)e^{st} \quad \text{- - - - } 3.5$$

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau \quad \text{- - - - } 3.6$$

CONSIDER

$$y(t) = x(t-3)$$

WHERE $x(t) = e^{j2t}$

THEN

$$y(t) = e^{j2(t-3)}$$

$$= \boxed{e^{-j6}} \cdot \boxed{e^{j2t}}$$

EIGENVALUE

EIGENFUNCTION

AS $h(t) = \delta(t-3)$

SUBSITUTING INTO EQ. 3.6

$$H(s) = \int_{-\infty}^{\infty} \delta(t-3)\, e^{-s\tau}\, d\tau$$

$$= e^{-3s}$$

AND

$$H(j2) = e^{-j6}$$

NOW CONSIDER

$$x(t) = \cos(4t) + \cos(7t)$$

$$y(t) = \cos(4(t-3)) + \cos(7(t-3))$$

USING EULER'S RELATION

$$x(t) = \frac{1}{2} e^{j4t} + \frac{1}{2} e^{-j4t} + \frac{1}{2} e^{j7t} + \frac{1}{2} e^{-j7t}$$

~~x(t) = a, s~~

$$x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t} \quad \text{---- } 3.11$$

THEN

$$y(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + a_3 H(s_3) e^{s_3 t} \quad - \ 3.12$$

$$y(t) = \tfrac{1}{2} e^{-j12} e^{j4t} + \tfrac{1}{2} e^{j12} e^{-j4t} + \tfrac{1}{2} e^{-j21} e^{j7t} + \tfrac{1}{2} e^{j21} e^{-j7t}$$

$$y(t) = \tfrac{1}{2} e^{j4(t-3)} + \tfrac{1}{2} e^{-j4(t-3)} + \tfrac{1}{2} e^{j7(t-3)} + \tfrac{1}{2} e^{-j7(t-3)}$$

$$= \cos(4(t-3)) + \cos(7(t-3))$$

EXAMPLE 3.2

$$x(t) = \sum_{k=-3}^{3} a_k e^{jk2\pi t}$$

$\omega_0 = 2\pi$, $a_0 = 1$,

$a_1 = a_{-1} = \frac{1}{4}$

$a_2 = a_{-2} = \frac{1}{2}$

$a_3 = a_{-3} = \frac{1}{3}$

$$x(t) = 1 + \frac{1}{4}\left(e^{j2\pi t} + e^{-j2\pi t}\right) + \frac{1}{2}\left(e^{j4\pi t} + e^{-j4\pi t}\right) + \frac{1}{3}\left(e^{j6\pi t} + e^{-j6\pi t}\right)$$

USING EULER'S RELATION

$$x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t$$

FIGURE 3.4, PAGE 188 SHOWS

$x(t)$ BUILT FROM ITS HARMONIC

COMPONENTS.

EXAMPLE 3.3

$$x(t) = \sin(w_0 t)$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\Rightarrow \frac{1}{2j} e^{jw_0 t} - \frac{1}{2j} e^{-jw_0 t}$$

COMPARING WITH FOURIER SYNTHESIS EQ.

$$x(t) = a_0 + a_1 e^{jw_0 t} + a_{-1} e^{-jw_0 t}$$
$$+ a_2 e^{j2w_0 t} + a_{-2} e^{-jw_0 t} + \dots$$

THUS THE FOURIER SERIES COEFFICIENTS OF $x(t)$ are

$a_0 = 0$

$$a_1 = \left(\frac{1}{2j}\right) = \frac{1 \cdot j}{2 \cdot j \cdot j} = -\frac{1}{2} j$$

$$a_{-1} = -\frac{1}{2j} = -\frac{1 \cdot j}{2 \cdot j \cdot j} = \frac{1}{2} j$$

$a_k = 0$ for $|k| > 1$

EXAMPLE 3.4

$$x(t) = 1 + \sin(\omega_0 t) + 2\cos(\omega_0 t) + \cos\left(2\omega_0 t + \frac{\pi}{4}\right)$$

USING EULER'S RELATION

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$x(t) = 1 + \left(\cancel{1+}\frac{1}{2j}\right)\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right] + \frac{2}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] + \frac{1}{2}\left[e^{j(2\omega_0 t + \frac{\pi}{4})} + e^{-j(2\omega_0 t + \frac{\pi}{4})}\right]$$

$$= 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \left(\frac{1}{2}e^{j\frac{\pi}{4}}\right)e^{j2\omega_0 t} + \left(\frac{1}{2}e^{-j\frac{\pi}{4}}\right)e^{-j2\omega_0 t}$$

COMPARING WITH FOURIER SERIES EXPANSION

$$x(t) = a_0 + a_1 e^{j\omega_0 t} + a_{-1} e^{-j\omega_0 t} + a_2 e^{j2\omega_0 t} + a_{-2} e^{-j2\omega_0 t} + \dots$$

$a_0 = 1$

$a_1 = \left(1 + \frac{1}{2j}\right) = 1 - \frac{1}{2}j$

$a_{-1} = \left(1 - \frac{1}{2j}\right) = 1 + \frac{1}{2}j$

$$a_2 = \frac{1}{2}e^{j\frac{\pi}{4}} = \frac{1}{2}\left(\cos\left(\frac{\pi}{4}\right) + j\sin\left(\frac{\pi}{4}\right)\right)$$

$$= \frac{1}{2}\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{2}\left(\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}\right) = \left(\frac{\sqrt{2}}{4} + j\frac{\sqrt{2}}{4}\right)$$

$$= \frac{\sqrt{2}}{4}(1 + j)$$

$$a_{-2} = \frac{1}{2}e^{-j\frac{\pi}{4}} = \frac{1}{2}\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right) = \left(\frac{\sqrt{2}}{4} - j\frac{\sqrt{2}}{4}\right)$$

$$= \frac{\sqrt{2}}{4}(1 - j)$$

$a_k = 0$ for $|k| > 2$

IF $z = x + jy$

Real $\{z\} = x$ Imaginary $\{z\} = y$.

$|z| = \sqrt{x^2 + y^2}$

magnitude
absolute

T3 - 8

$$\measuredangle z = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x > 0 \\ \arctan\left(\frac{y}{x}\right) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\ \arctan\left(\frac{y}{x}\right) - \pi & \text{if } x < 0 \text{ and } y < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \\ \text{indeterminate} & \text{if } x = 0 \text{ and } y = 0 \end{cases}$$

$a_0 = 1 + 0j$

$|a_0| = \sqrt{1^2 + 0^2} = 1$

$\measuredangle a_0 = \arctan\left(\frac{0}{1}\right) = \tan^{-1}(0) = 0$

$|a_1| = 1.1180$

$\measuredangle a_1 = -0.4636$

$|a_{-1}| = 1.1180$

$\measuredangle a_{-1} = 0.4636$

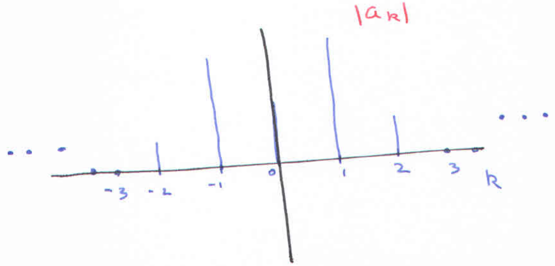
$|a_2| = 0.5$

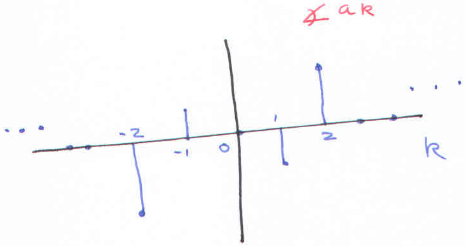
$\measuredangle a_2 = 0.7854$

$|a_{-2}| = 0.5$

$\measuredangle a_{-2} = -0.7854$

PLOTS OF THE MAGNITUDE AND PHASE OF THE FOURIER COEFFICIENTS OF $x(t)$





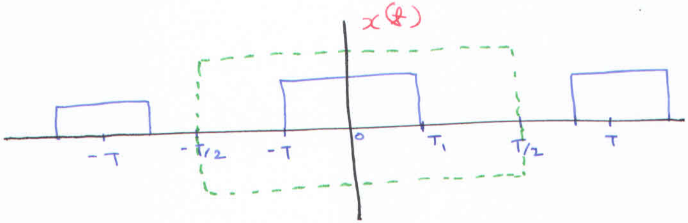
EXAMPLE 3.5

PERIODIC SQUARE WAVE

FUNDAMENTAL PERIOD T

FUNDAMENTAL FREQUENCY $\omega_0 = \frac{2\pi}{T}$

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < T/2 \end{cases}$$



WE CAN CHOOSE ANY PERIOD. AS $x(t)$ IS SYMMETRIC ABOUT $t=0$. IT IS CONVENIENT TO CHOOSE

$$-T/2 \le t \le T/2$$

USING THE FOURIER SERIES ANALYSIS EQUATION

$$a_k = \frac{1}{T} \int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T} \int_T x(t)\, e^{-jk\left(\frac{2\pi}{T}\right)t}\, dt$$

$$a_0 = \frac{1}{T} \int_T x(t)\,dt = \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, dt$$

$$= \frac{1}{T} \int_{-T_1}^{T_1} (1)\, dt = \frac{1}{T}(t)\Big|_{-T_1}^{T_1} = \frac{1}{T}\left(T_1 - (-T_1)\right)$$

$$= \frac{2T_1}{T}$$

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, e^{-jk\left(\frac{2\pi}{T}\right)t}\, dt$$

$$= \frac{1}{T} \cdot \frac{1}{-jk\left(\frac{2\pi}{T}\right)}\, e^{-jk\left(\frac{2\pi}{T}\right)t} \Bigg|_{-T_1}^{T_1}$$

$$= \frac{1}{k\pi}\left(-\frac{1}{2j}\right)\left[e^{-jk\left(\frac{2\pi}{T}\right)T_1} - e^{+jk\left(\frac{2\pi}{T}\right)T_1}\right]$$

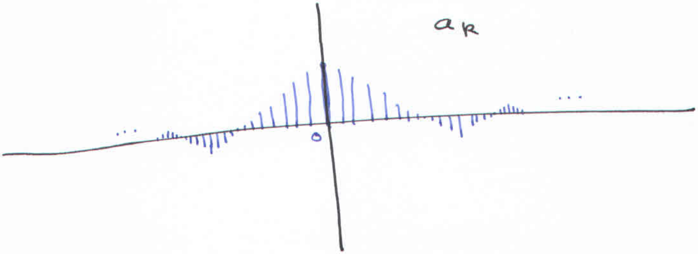
$$= \frac{1}{k\pi}\left[\frac{e^{jk\left(\frac{2\pi}{T}\right)T_1} - e^{-jk\left(\frac{2\pi}{T}\right)T_1}}{2j}\right]$$

$$\Rightarrow a_k = \frac{\sin\left(k\left(\frac{2\pi}{T}\right)T_1\right)}{k\pi}$$

$$a_k = \frac{\sin(k\omega_0 T_1)}{k\pi} \quad \text{for } k \neq 0$$

$\rightarrow$ sinc

$$a_0 = \frac{2T_1}{T}$$



PROBLEM 3.1

$$x(t) = \sum_{k=0}^{\infty} A_k \cos((w_k t) + \phi_k)$$

$$T_0 = 8. \qquad w_0 = \frac{2\pi}{8} = \frac{\pi}{4}$$

$$a_1 = a_{-1} = 2, \qquad a_3 = a_{-3}^{*} = 4j$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t}$$

$$x(t) = a_1 e^{jw_0 t} + a_{-1} e^{-jw_0 t} + a_3 e^{j3w_0 t} + a_{-3} e^{-j3w_0 t}$$

$$\Rightarrow 2e^{jw_0 t} + 2e^{-jw_0 t} + 4j e^{3w_0 t} - 4j e^{-j3w_0 t}$$

$$\Rightarrow 4\left(\frac{e^{jw_0 t} + e^{-jw_0 t}}{2}\right) + 8(-1)\left(\frac{e^{j3w_0 t} - e^{-j3w_0 t}}{2j}\right)$$

$$\Rightarrow 4\cos(w_0 t) - 8\sin(3w_0 t)$$

As $\sin(\theta) = \cos(\theta + \frac{\pi}{2})$

$$\Rightarrow x(t) = 4\cos(w_0 t + \phi) - 8\cos(3w_0 t + \frac{\pi}{2})$$

$$x(t) = 4\cos(\cancel{8}\frac{\pi}{4} t + \phi) - 8\cos(\frac{3\pi}{4} t + \frac{\pi}{2})$$

$$x(t) = \sum_{k=0}^{\infty} A_k \cos(\omega_k t + \phi_k)$$

$A_1 = 4$ $\quad \omega_1 = \frac{\pi}{4}$ $\quad \phi_1 = 0$

$A_3 = -8$ $\quad \omega_3 = \frac{3\pi}{4}$ $\quad \phi_3 = \frac{\pi}{2}$

WITH ALL OTHER

$A_k = 0, \quad \omega_k = 0 \quad \phi_k = 0.$

PROBLEM 3.3

$$x(t) = 2 + \cos\left(\frac{2\pi}{3}t\right) + 4\sin\left(\frac{5\pi}{3}t\right)$$

$w_0 = ?$ $a_k = ?$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t}$$

USING EULER'S FORMULA

$$x(t) = 2 + \frac{1}{2}\left[e^{j\frac{2\pi}{3}t} + e^{-j\frac{2\pi}{3}t}\right]$$

$$+ \frac{4}{2j}\left[e^{j\frac{5\pi}{3}t} - e^{-j\frac{5\pi}{3}t}\right]$$

$$= 2 + \left(\frac{1}{2}\right)e^{j2\left(\frac{\pi}{3}\right)t} + \left(\frac{1}{2}\right)e^{-j2\left(\frac{\pi}{3}\right)t}$$

$$+ (-2j)\,e^{j5\left(\frac{\pi}{3}\right)t} + (2j)\,e^{-j5\left(\frac{\pi}{3}\right)t}$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t}$$

$$\Rightarrow a_0 + a_1 e^{jw_0 t} + a_{-1} e^{-jw_0 t}$$

$$+ a_2 e^{j2w_0 t} + a_{-2} e^{-j2w_0 t}$$

$$+ a_3 e^{j3w_0 t} + a_3 e^{-j3w_0 t} + a_4 e^{j4w_0 t} + a_{-4} e^{-j4w_0 t}$$

$$+ a_5 e^{j5w_0 t} + a_{-5} e^{-j5w_0 t} + \cdots$$

$\therefore \quad a_0 = 2, \quad a_1 = 0, \quad a_{-1} = 0$

$a_2 = \frac{1}{2}, \quad a_{-2} = \frac{1}{2}, \quad a_3 = 0, \quad a_{-3} = 0$

$a_4 = 0, \quad a_{-4} = 0, \quad a_5 = -2j, \quad a_{-5} = 2j$

$a_k = 0 \quad \text{for} \quad |k| > 5.$

$w_0 = \frac{\pi}{3}$

PROBLEM 3.6

$$x_1(t) = \sum_{k=0}^{100} \left(\frac{1}{2}\right)^k e^{jk\left(\frac{2\pi}{50}\right)t}$$

$$x_2(t) = \sum_{k=-100}^{100} \cos(k\pi)\, e^{jk\left(\frac{2\pi}{50}\right)t}$$

$$x_3(t) = \sum_{k=-100}^{100} j\sin\left(\frac{k\pi}{2}\right) e^{jk\left(\frac{2\pi}{50}\right)t}$$

USE FOURIER SERIES PROPERTIES TO ANSWER THE FOLLOWING.

(a) WHICH OF THE THREE SIGNALS IS/ARE REAL VALUED.

(b) WHICH OF THE THREE SIGNALS IS/ARE EVEN.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

for $x_1(t)$, $\omega_0 = \frac{2\pi}{50}$

$a_k = \left(\frac{1}{2}\right)^k$ for $k = 0, 1, \dots, 100.$

and $a_k = 0$ for $k<0$ and $k>100$

FOR $x_1(t)$ TO BE REAL (FOURIER SERIES PROPERTIES)

$$a_k = a_{-k}^*$$

BUT $a_k = 0$ for $k < 0$

FOR EXAMPLE

$$a_{10} = \left(\frac{1}{2}\right)^{10}$$

HOWEVER $a_{-10} = 0$

$$\therefore a_{-10}^* = 0$$

$$\therefore \quad a_{10} \neq a_{-10}^*$$

THEREFORE $x_1(t)$ is not REAL

b) FOR $x_1(t)$ TO BE EVEN

$$x_1(t) = x_1(-t)$$

$$x_1(-t) = \sum_{k=0}^{100} \left(\frac{1}{2}\right)^k e^{-jk\left(\frac{2\pi}{50}\right)t}$$

$$= \sum_{k=-100}^{0} \left(\frac{1}{2}\right)^{-k} e^{jk\left(\frac{2\pi}{50}\right)t}$$

$$\neq \sum_{k=0}^{100} \left(\frac{1}{2}\right)^{k} e^{jk\left(\frac{2\pi}{50}\right)t}$$

$\therefore$ $x_1(t)$ IS NOT EVEN

$$x_2(t) = \sum_{k=-100}^{100} \cos(k\pi)\, e^{jk\left(\frac{2\pi}{50}\right)t}$$

$$w_0 = \frac{2\pi}{50}$$

$$a_k = \begin{cases} \cos(k\pi) & \text{for } -100 \leq k \leq 100 \\ 0 & \text{OTHERWISE} \end{cases}$$

$$a_{-k}^{*} = (\cos(-k\pi))^{*} = \cos(k\pi) = a_k$$

AS $\cos(-\theta) = \cos(\theta)$