TUTORIAL 4

SIGNALS AND SYSTEMS

EXAMPLE 4.1

$$x(t) = e^{-at}\,u(t) \qquad a > 0$$

FIND OUT $X(j\omega)$, THE FOURIER TRANSFORM

USING $X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

PUTTING THE VALUE OF $x(t)$

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at}\, u(t)\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-at} \cdot e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

$$= \left. \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right|_{0}^{\infty}$$

$$= \frac{1}{-(a+j\omega)} \left(e^{-\infty} - e^{-0} \right)$$

$$= \frac{1}{-(a+j\omega)} \left(\frac{1}{e^{\infty}} - 1 \right)$$

$$= \frac{1}{(-a+j\omega)}(0-1)$$

$$= \frac{1}{a+j\omega}$$

MAGNITUDE AND PHASE

$$|X(j\omega)| = \left|\frac{1}{a+j\omega}\right|$$

$$= \left|\frac{1(a-j\omega)}{(a+j\omega)(a-j\omega)}\right|$$

$$= \left|\frac{a-j\omega}{a^2+\omega^2}\right| = \left|\frac{a}{a^2+\omega^2} - \frac{j\omega}{a^2+\omega^2}\right|$$

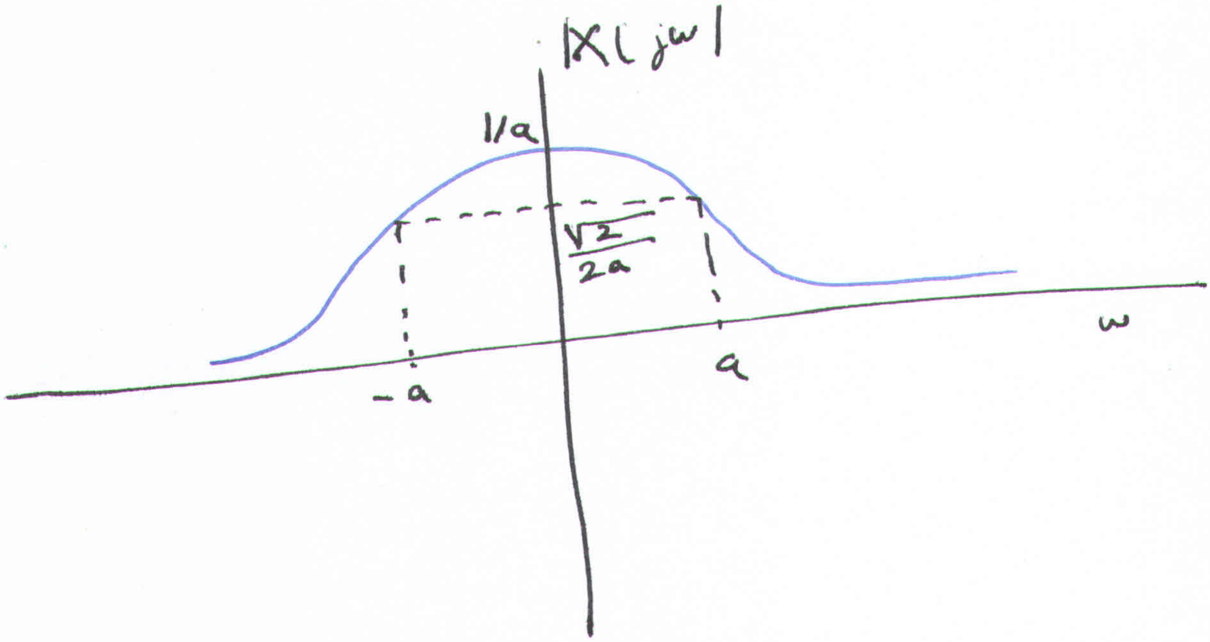
$$= \sqrt{\frac{a^2}{(a^2+\omega^2)^2} + \frac{\omega^2}{(a^2+\omega^2)^2}}$$

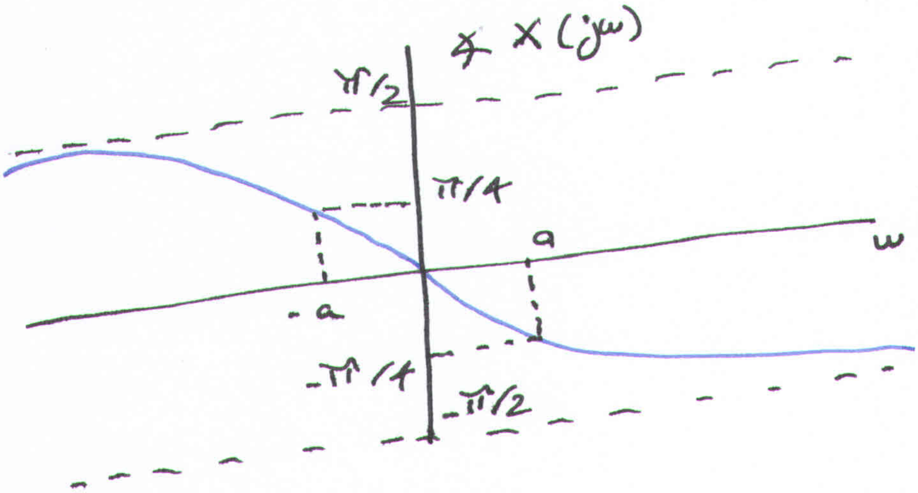
$$= \sqrt{\frac{a^2+\omega^2}{(a^2+\omega^2)^2}} = \frac{1}{\sqrt{a^2+\omega^2}}$$

$$\measuredangle X(j\omega) = \tan^{-1} - \frac{\omega/(a^2+\omega^2)}{a/(a^2+\omega^2)} = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

THE PLOTS FOR MAGNITUDE AND PHASE

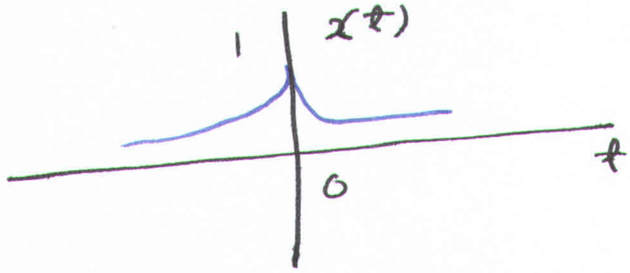
TA ③





EXAMPLE 4.2

$$x(t) = e^{-a|t|} \quad \text{for} \quad a>0$$



TIME DOMAIN

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$|t| = \begin{cases} t & t>0 \\ -t & t<0 \end{cases}$$

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-a|t|}\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{at}\, e^{-j\omega t} + \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt$$

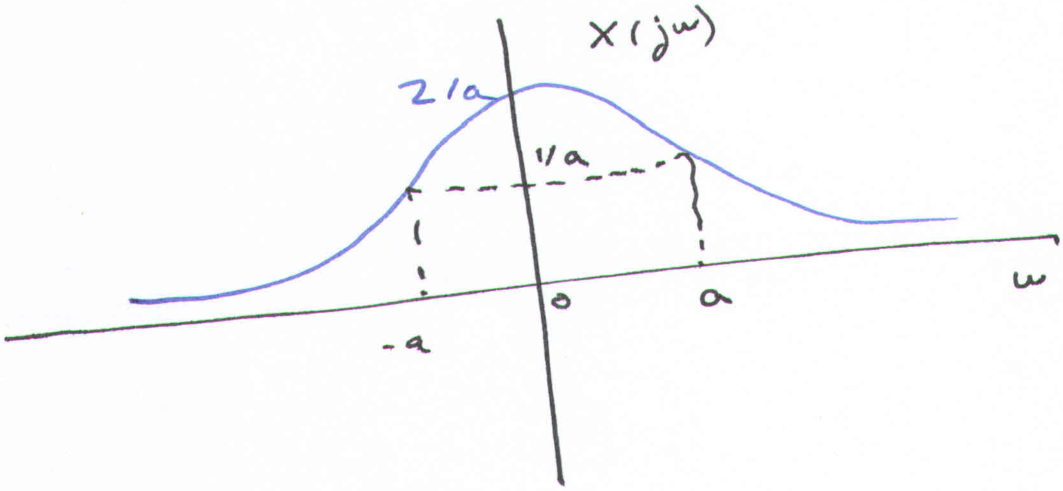
$$= \left.\frac{e^{(a-j\omega)t}}{a-j\omega}\right|_{-\infty}^{0} + \left.\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right|_{0}^{\infty}$$

$$= \frac{e^{0} - \frac{1}{e^{\infty}}}{a-j\omega} + \frac{\frac{1}{e^{\infty}} - \frac{1}{e^{0}}}{-(a+j\omega)}$$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega}$$

$$= \frac{a + jw + a - jw}{a^2 + w^2}$$

$$X(jw) = \frac{2a}{a^2+w^2} \quad \text{(PURE REAL)}$$



EXAMPLE 4.3

$$x(t) = \delta(t)$$

$$X(jw) = \int_{-\infty}^{\infty} x(t)\, e^{-jwt}\, dt$$

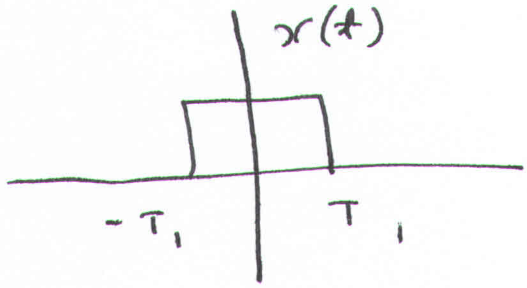
$$= \int_{-\infty}^{\infty} \delta(t)\, e^{-jwt}\, dt$$

$$= e^{0} = 1 .$$

THE UNIT IMPULSE HAS A FOURIER TRANSFORM CONSISTING OF EQUAL CONTRIBUTIONS AL ALL FREQUENCIES.

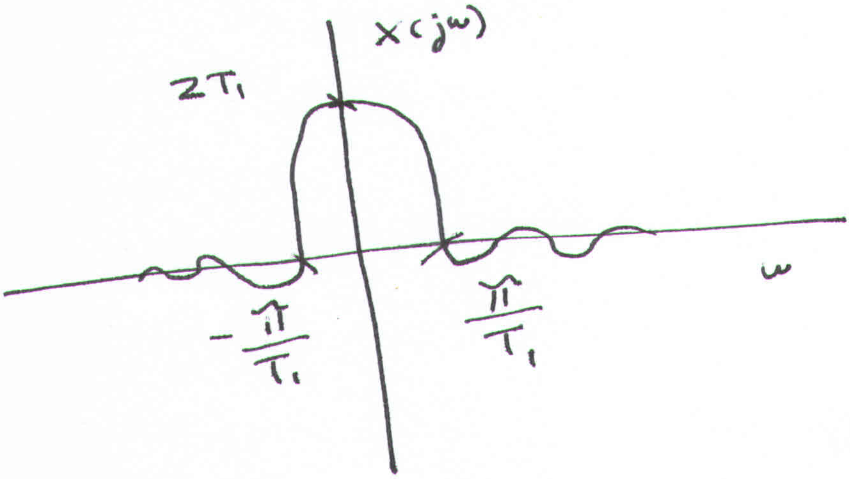
EXAMPLE 4.4

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$



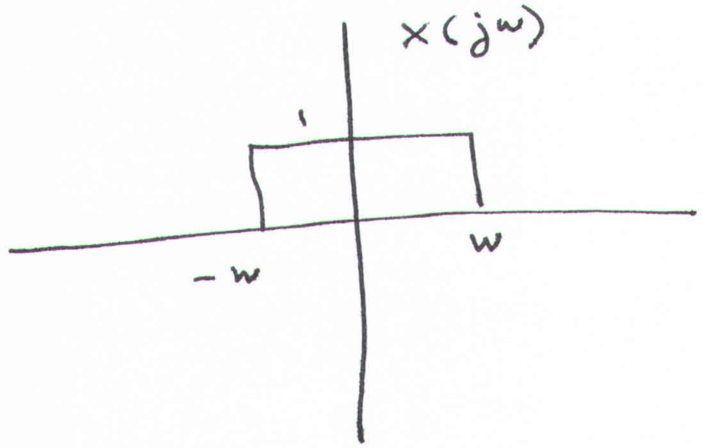
$$X(j\omega) = \int_{-T_1}^{T_1} 1 \cdot e^{-j\omega t}\, dt$$

$$\Rightarrow \frac{2 \sin \omega T_1}{\omega}$$



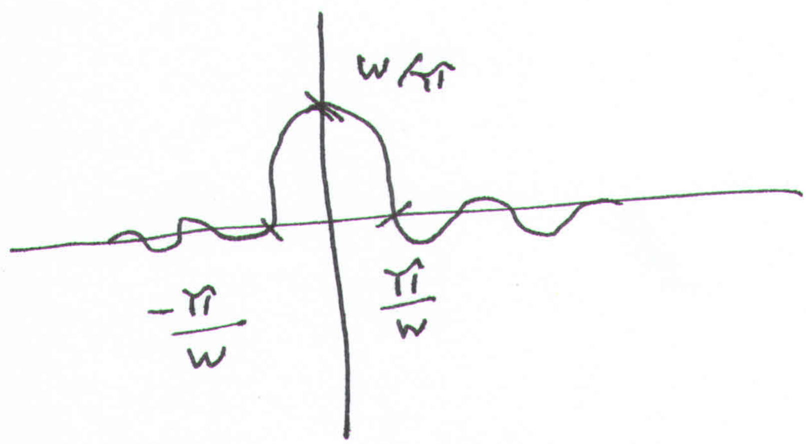
EXAMPLE 4.5

$$X(j\omega) = \begin{cases} 1 & |\omega| \leq W \\ 0 & |\omega| > W \end{cases}$$



$$x(t) = \frac{1}{2\pi} \int_{-W}^{W} 1 \cdot e^{j\omega t} \, d\omega$$

$$= \frac{\sin Wt}{\pi t}$$



PROBLEM 4.1

a) $e^{-2(t-1)} u(t-1)$

b) $e^{-2|t-1|}$

$X(jw) = ?$

$$X(jw) = \int_{-\infty}^{\infty} x(t) e^{-jwt} \, dt$$

a) $$X(jw) = \int_{-\infty}^{\infty} e^{-2(t-1)} u(t-1) e^{-jwt} \, dt$$

$\tau = t - 1 \qquad d\tau = dt$

$t = \tau + 1$

$$X(jw) = \int_{-\infty}^{\infty} e^{-2\tau} u(\tau) e^{-jw(\tau+1)} \, d\tau$$

$$X(jw) = \int_{0}^{\infty} e^{-2\tau} e^{-jw(\tau+1)} \, d\tau$$

$$= \int_{0}^{\infty} e^{-2\tau} e^{-jw\tau} e^{-jw} \, d\tau$$

$$= e^{-jw} \int_{0}^{\infty} e^{-2\tau} e^{-jw\tau} \, d\tau$$

$$\cancel{= e^{-jw} \int_{-\infty}}$$

$$= e^{-jw} \int_0^{\infty} e^{-(2+jw)\tau} \, d\tau$$

$$= e^{-jw} \cdot \frac{1}{-(2+jw)} \cdot e^{-(2+jw)\tau} \Big|_0^{\infty}$$

$$= \frac{e^{-jw}}{-(2+jw)} \cdot \left(e^{-\infty} - e^{-0}\right)$$

$$= \frac{e^{-jw}}{-(2+jw)} (0-1)$$

$$= \frac{e^{-jw}}{2+jw}$$

b) $\cancel{e^{-2|t+1|}}$ $\quad e^{2|t-1|}$

$$X(jw) = \int_{-\infty}^{\infty} e^{-2|t-1|} \, e^{-jwt} \, dt$$

$t = \tau + 1 \qquad \tau = t-1 \qquad d\tau = dt$

$$X(jw) = \int_{-\infty}^{\infty} e^{-2|\tau|} \, e^{-jw(\tau+1)} \, d\tau$$

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-2|\tau|} e^{-j\omega\tau} e^{-j\omega} \, d\tau$$

$$= e^{-j\omega} \int_{-\infty}^{\infty} e^{-2|\tau|} e^{-j\omega\tau} \, d\tau$$

$$= e^{-j\omega} \left[\int_{-\infty}^{0 \,\cancel{\infty}} e^{2\tau} . e^{-j\omega\tau} \, d\tau + \int_{0}^{\infty} e^{-2\tau} . e^{-j\omega\tau} \, d\tau \right]$$

$$= e^{-j\omega} \left[\int_{-\infty}^{0} e^{(2-j\omega)\tau} \, d\tau + \int_{0}^{\infty} e^{-(2+j\omega)\tau} \, d\tau \right]$$

$$= e^{-j\omega} \left[\frac{e^{(2-j\omega)\tau}}{2-j\omega} \Bigg|_{-\infty}^{0} + \frac{e^{-(2+j\omega)\tau}}{-(2+j\omega)} \Bigg|_{0}^{\infty} \right]$$

$$= e^{-j\omega} \left[\frac{e^{0} - e^{-\infty}}{2-j\omega} + \frac{e^{-\infty} - e^{0}}{-(2+j\omega)} \right]$$

$$= e^{-j\omega} \left[\frac{1}{2-j\omega} + \frac{0-1}{-(2+j\omega)} \right]$$

$$= e^{-j\omega} \left[\frac{1}{2-j\omega} + \frac{1}{2+j\omega} \right]$$

$$= e^{-j\omega} \left[\frac{2+j\omega + 2-j\omega}{(2-j\omega)(2+j\omega)} \right]$$

$$= \frac{4 e^{-j\omega}}{4+\omega^2}$$

PROBLEM 4.2 a

$\delta(t+1) + \delta(t-1)$

$|X(jw)| = ?$

$$X(jw) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt$$

$$\Rightarrow X(jw) = \int_{-\infty}^{\infty} [\delta(t+1) + \delta(t-1)] e^{-jwt} dt$$

$$= \int_{-\infty}^{\infty} \delta(t+1) e^{-jwt} dt + \int_{-\infty}^{\infty} \delta(t-1) e^{-jwt} dt$$

$$= e^{-jw(-1)} + e^{-jw(1)}$$

$$= e^{jw} + e^{-jw}$$

$$= 2 \left(\frac{e^{jw} + e^{-jw}}{2} \right) = 2\cos w$$

$$|X(jw)| = 2 |\cos(w)|$$

PROBLEM 4.2 b

$$\frac{d}{dt}\left\{u(-2-t)+u(t-2)\right\}$$

$$X(j\omega)=\int_{-\infty}^{\infty}\frac{d}{dt}\left\{u(-2-t)+u(t-2)\right\}e^{-j\omega t}\,dt$$

$$=\int_{-\infty}^{\infty}-\delta(-2-t)\,e^{-j\omega t}\,dt+\int_{-\infty}^{\infty}\delta(t-2)\,e^{-j\omega t}\,dt$$

$$=-I_1+I_2$$

where

$$I_1=\int_{-\infty}^{\infty}\delta(-2-t)\,e^{-j\omega t}\,dt \qquad \tau=-2-t,\quad d\tau=-dt$$

$$=\int_{\infty}^{-\infty}\delta(+\tau)\,e^{j\omega(2+\tau)}(-d\tau)$$

$$=e^{2j\omega}\int_{-\infty}^{\infty}\delta(\tau)\,e^{j\omega\tau}\,d\tau = e^{2j\omega}\cdot e^{0}=e^{2j\omega}$$

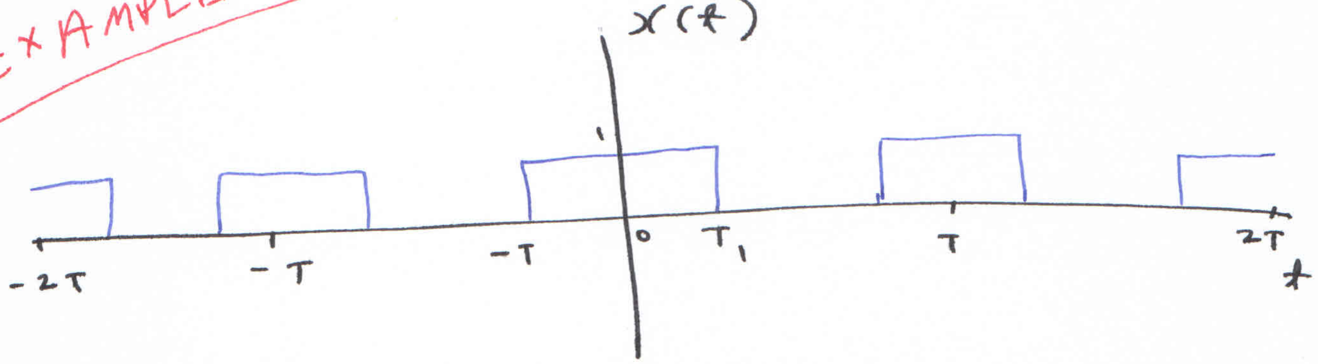
$$I_2=\int_{-\infty}^{\infty}\delta(t-2)\,e^{-j\omega t}\,dt=e^{-2j\omega}$$

$$X(j\omega)=-e^{2j\omega}+e^{-2j\omega}=-2j\left(\frac{e^{2j\omega}-e^{-2j\omega}}{2j}\right)$$

$$=-2j\sin(2\omega)$$

$$|X(j\omega)|=2|\sin(2\omega)|$$

EXAMPLE 4.6



$$a_k = \frac{\sin(k\omega_0 T_1)}{\pi k}$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\boxed{X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)}$$

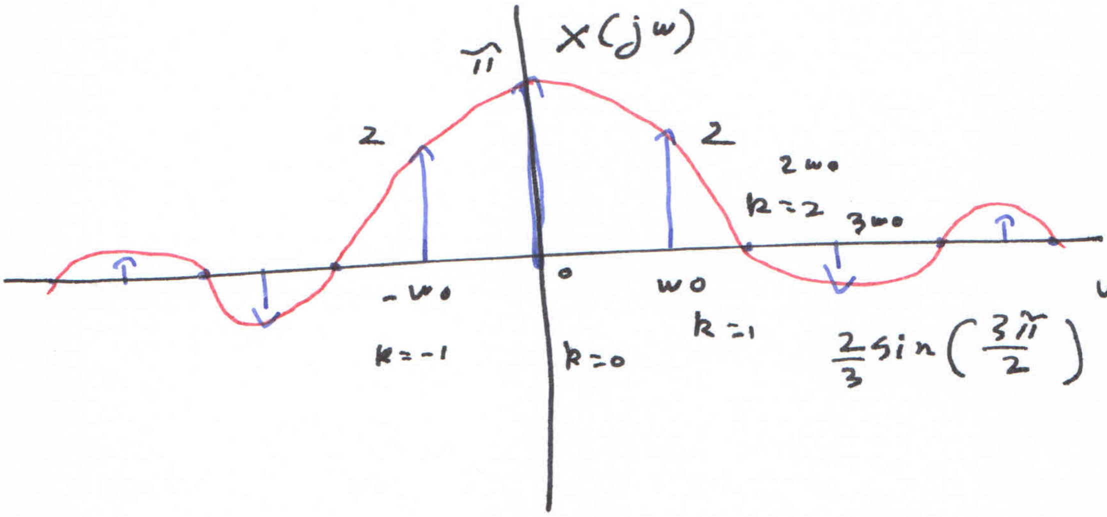
FOURIER TRANSFORM OF PERIODIC SIGNALS USING THEIR FOURIER SERIES COEFFICIENTS.

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

$$= \sum_{k=-\infty}^{\infty} 2\pi \frac{\sin(k\omega_0 T_1)}{\pi k} \delta(\omega - k\omega_0)$$

$$= \sum_{k=-\infty}^{\infty} \frac{2\sin(k w_0 T_1)}{k} \delta(w - k w_0)$$

USING $T = 4T_1$



$$\frac{2}{3}\sin\left(\frac{3\pi}{2}\right) = -\frac{2}{3}$$

EXAMPLE 4.7

$x(t) = \sin(w_0 t)$

FOURIER SERIES COEFFICIENTS

$$a_1 = \frac{1}{2j} \qquad a_{-1} = -\frac{1}{2j}$$

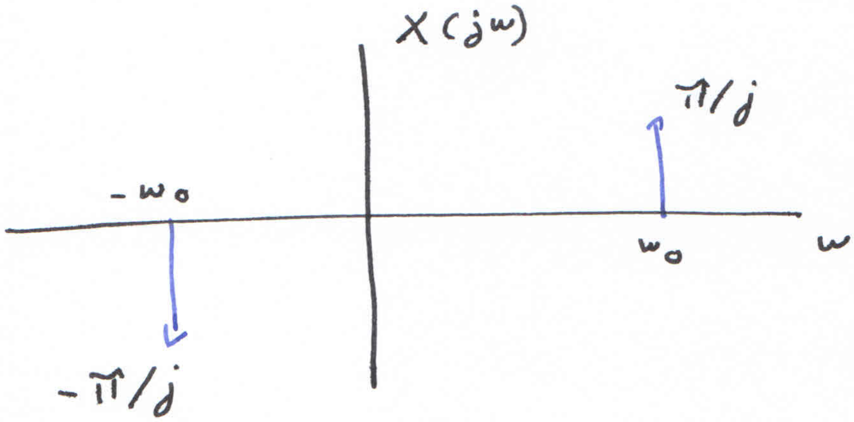
$$a_k = 0 \quad \text{for} \quad |k| \neq 1$$

$X(jw) = ?$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \, \delta(\omega - k\omega_0)$$

$$X(j\omega) = \frac{2\pi}{2j} \delta(\omega - \omega_0) + \frac{2\pi}{2j} (-1) \, \delta(\omega + \omega_0)$$

$$X(j\omega) = \frac{\pi}{j} \delta(\omega - \omega_0) - \frac{\pi}{j} \delta(\omega + \omega_0)$$

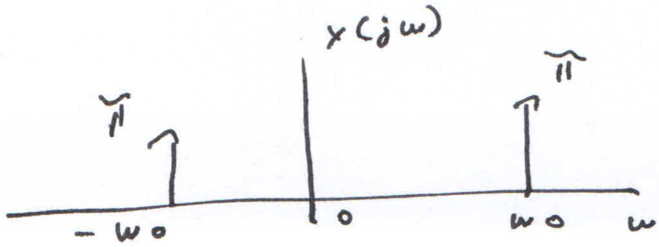


$$x(t) = \cos(\omega_0 t)$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \, \delta(\omega - k\omega_0)$$

$a_1 = \frac{1}{2}$ $\quad a_{-1} = \frac{1}{2}$ $\quad a_k = 0$ for $|k| \neq 1$

$$X(j\omega) = \frac{2\pi}{2} \delta(\omega - \omega_0) + \frac{2\pi}{2} \delta(\omega + \omega_0)$$



EXAMPLE 4.9

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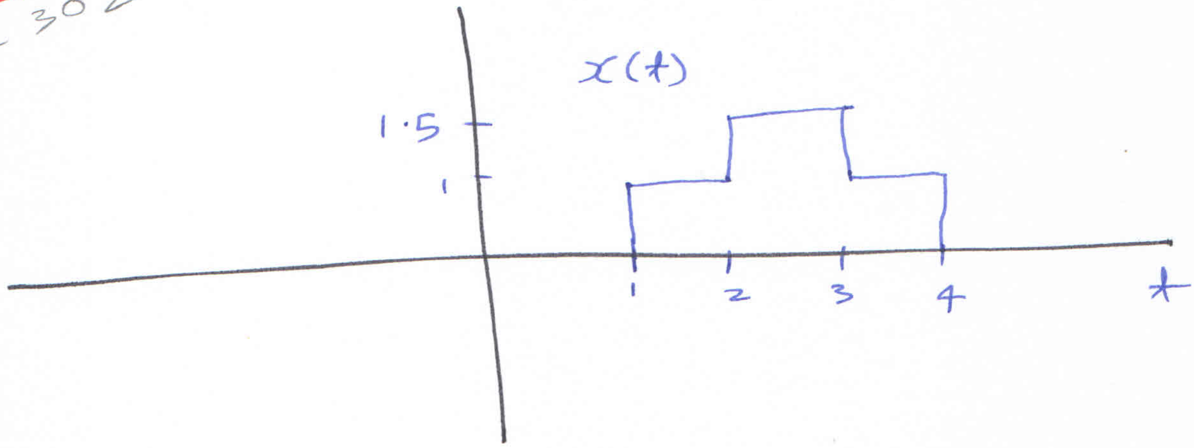
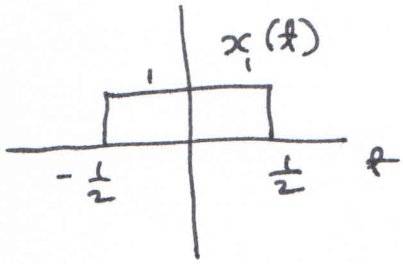
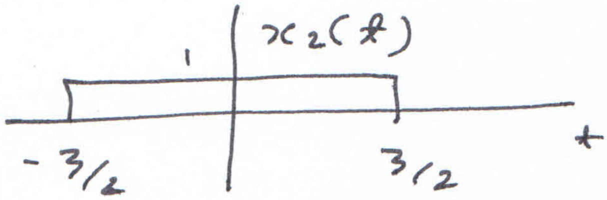
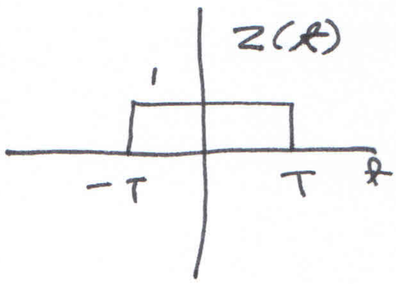


$$x(t) = \frac{1}{2} x_1(t-2.5) + x_2(t-2.5)$$





FROM EXAMPLE 4.4 (PAGE 293)



$$Z(j\omega) = \int_{-\infty}^{\infty} z(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-T}^{T} (1)\, e^{-j\omega t}\, dt \qquad = \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{-T}^{T}$$

$$= \frac{e^{j\omega T} - e^{-j\omega T}}{j\omega} = \frac{2\sin(\omega T)}{\omega}$$

$\therefore X_1(j\omega) = F\{x_1(t)\} = \dfrac{2\sin(\omega/2)}{\omega}$

$X_2(j\omega) = F\{x_2(t)\} = \dfrac{2\sin(3\omega/2)}{\omega}$

As $x(t) = \frac{1}{2} x_1(t-2.5) + x_2(t-2.5)$

$X(j\omega) = F\{x(t)\}$

$= \frac{1}{2} F\{x_1(t-2.5)\} + F\{x_2(t-2.5)\}$

LINEARITY

$= \frac{1}{2} e^{-2.5j\omega} F\{x_1(t)\} + e^{-2.5j\omega} F\{x_2(t)\}$

TIME SHIFTING

$= \frac{1}{2} e^{-2.5j\omega} \cdot \dfrac{2\sin(\omega/2)}{\omega} + e^{-2.5j\omega} \cdot \dfrac{2\sin(3\omega/2)}{\omega}$

$X(j\omega) = e^{-2.5j\omega}\left(\dfrac{\sin(\omega/2) + 2\sin(3\omega/2)}{\omega}\right)$

EXAMPLE 4.10
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CONSIDER EXAMPLE 4.2 FOR (PAGE 291)

$$x(t) = e^{-a|t|} \quad \text{for} \quad a>0$$

FROM EXAMPLE 4.1 (PAGE 290)

$$e^{-at}u(t) \xleftrightarrow{F} \frac{1}{a+j\omega}$$

for $t>0 \Rightarrow x(t) \Rightarrow e^{-at}u(t)$ MIRROR

for $t<0 \Rightarrow x(t) \Rightarrow e^{at}u(-t)$ IMAGE

$$x(t) = e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t)$$

$$= 2\left[\frac{e^{-at}u(t) + e^{at}u(-t)}{2}\right]$$

$$= 2\,\text{EVEN}\{e^{-at}u(t)\}$$

AS $e^{-at}u(t)$ IS REAL, FROM SYMMETRY PROPERTY

$$2\,\text{EVEN}\{e^{-at}u(t)\} \xleftrightarrow{F} 2\,\text{REAL}\{F\{e^{-at}u(t)\}\}$$

$$X(j\omega) = 2\,\text{REAL}\left\{ F\left\{ e^{-at}u(t) \right\} \right\}$$

$$= 2\,\text{REAL}\left\{ \frac{1}{a+j\omega} \right\}$$

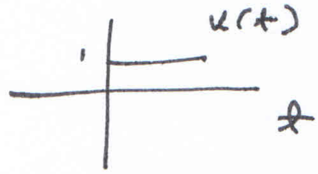
$$= 2\,\text{REAL}\left\{ \frac{a-j\omega}{a^2+\omega^2} \right\}$$

$$= \frac{2a}{a^2+\omega^2}$$

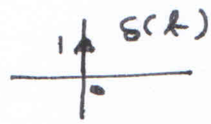
SAME AS ANSWER FOR EXAMPLE 4.2.

EXAMPLE 4.11

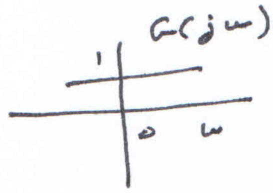
$$x(t) = u(t)$$



$$\frac{du}{dt} = \delta(t)$$



$$g(t) = \delta(t) \overset{F}{\longleftrightarrow} G(j\omega) = 1$$



NOW $\quad x(t) = \int_{-\infty}^{t} g(\tau)\,d\tau$

TAKING FOURIER TRANSFORM ON BOTH SIDES

$$X(j\omega) = \frac{G(j\omega)}{j\omega} + \pi G(0)\,\delta(\omega)$$

$$G(j\omega) = 1, \quad G(0) = 1.$$

$$\therefore\ X(j\omega) = \frac{1}{j\omega} + \pi\,\delta(\omega)$$

AND $\quad \delta(t) = \frac{du}{dt} \overset{F}{\longleftrightarrow} j\omega\left\{ \frac{1}{j\omega} + \pi\,\delta(\omega) \right\}$ $\cancel{= 1 + \pi j\,\delta(\omega)}$

$$= 1 + \pi j\omega\,\delta(\omega)^{0}$$

$$= 1$$

EXAMPLE 4.13
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$$g(t) = \frac{2}{1+t^2}$$

$G(j\omega)$?

FROM EXAMPLE 4.2

$$x(t) = e^{-a|t|} \quad , a > 0$$

$$X(j\omega) = \frac{2a}{a^2+\omega^2}$$

FOR $a=1$, we can write

$$x(t) = e^{-|t|} \xleftrightarrow{F} X(j\omega) = \frac{2}{1+\omega^2}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

USING INVERSE FOURIER TRANSFORM

$$e^{-|t|} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{2}{1+\omega^2}\right) e^{j\omega t}\, d\omega$$

$$2\pi\, e^{-|t|} = \int_{-\infty}^{\infty} \left(\frac{2}{1+\omega^2}\right) e^{j\omega t}\, d\omega$$

INTERCHANGING t and ω

$$2\pi e^{-|\omega|} = \int_{-\infty}^{\infty} \left(\frac{2}{1+t^2}\right) e^{jt\omega}\, dt$$

SUBSITUTING t by $-t$

$$2\pi e^{-|\omega|} = \int_{\infty}^{-\infty} \left(\frac{2}{1+(-t)^2}\right) e^{-j\omega t} (-dt)$$

$$= \int_{-\infty}^{\infty} \left(\frac{2}{1+t^2}\right) e^{-j\omega t}\, dt$$

AS $X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

$$2\pi e^{-|\omega|} = F\left\{\frac{2}{1+t^2}\right\}$$

$$\therefore\ g(t) = \frac{2}{1+t^2} \overset{F}{\longleftrightarrow} 2\pi e^{-|\omega|} = G(j\omega)$$

EXAMPLE 4.15

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$$h(t) = \delta(t - t_0)$$ (CONTINUOUS TIME LTI SYSTEM)

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$$ (FREQUENCY RESPONSE OF THE SYSTEM)

$$= \int_{-\infty}^{\infty} \delta(t - t_0)\, e^{-j\omega t}\, dt$$

$$= e^{-j\omega t_0}$$

$$\therefore \quad Y(j\omega) = H(j\omega)\, X(j\omega)$$

$$= e^{-j\omega t_0}\, X(j\omega)$$

THIS IS CONSISTENT WITH THE TIME SHIFTING PROPERTY.

$$y(t) = x(t - t_0) \quad \Rightarrow \quad h(t) = \delta(t - t_0)$$

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t - t_0)\, e^{-j\omega t}\, dt$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega t}\, d\tau$$

$$= e^{-j\omega t_0}\, X(j\omega)$$

EXAMPLE 4.16

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$$x(t) \longrightarrow \boxed{\text{LTI SYSTEM}} \longrightarrow y(t)$$

$$y(t) = \frac{dx(t)}{dt} \quad \text{(LTI SYSTEM)}$$

FROM DIFFERENTIATION PROPERTY

$$Y(jw) = jw\, X(jw)$$

THIS IMPLIES THAT

$$H(jw) = jw$$

FREQUENCY RESPONSE OF A DIFFERENTIATOR.

EXAMPLE 4.17

$$x(t) \longrightarrow \boxed{\text{LTI SYSTEM}} \longrightarrow y(t)$$

CONSIDER AN INTEGRATOR

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

THE IMPULSE RESPONSE OF THIS SYSTEM IS THE UNIT STEP $u(t)$

$$h(t) = u(t)$$

$$H(j\omega) = \int_{-\infty}^{\infty} u(t) e^{-j\omega t} dt$$

FROM EXAMPLE 4.11,

$$H(j\omega) = \frac{1}{j\omega} + \pi \delta(\omega)$$

$$\therefore \quad Y(j\omega) = H(j\omega) X(j\omega)$$

$$= \left(\frac{1}{j\omega} + \pi \delta(\omega)\right) X(j\omega)$$

$$= \frac{1}{j\omega} X(j\omega) + \pi \delta(\omega) X(j\omega)$$

$$Y(j\omega) = \frac{1}{j\omega} X(j\omega) + \pi X(0) \delta(\omega)$$

EXAMPLE 4.19
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LTI SYSTEM

$$h(t) = e^{-at} u(t) \quad , \quad a > 0$$

$$x(t) = e^{-bt} u(t) \quad , \quad b > 0$$

$$y(t) = h(t) * x(t)$$

TAKING THE PROBLEM INTO THE FREQUENCY DOMAIN

$$\underline{Y(jw) = H(jw)\, X(jw)}$$

USING THE CONVOLUTION PROPERTY

$$H(jw) = \int_{-\infty}^{\infty} h(t)\, e^{-jwt}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-at}\, u(t)\, e^{-jwt}\, dt$$

$$= \int_{0}^{\infty} e^{-(a+jw)t}\, dt$$

$$= \frac{1}{-(a+jw)}\, e^{-(a+jw)t} \Big|_{0}^{\infty}$$

$$= \frac{1}{-(a+jw)} \left(e^{-\infty} - e^{0} \right) = \frac{1}{-(a+jw)} (-1) = \frac{1}{a+jw}$$

$$X(jw) = \int_{-\infty}^{\infty} x(t)\, e^{-jwt}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-bt} u(t)\, e^{-jwt}\, dt$$

$$= \frac{1}{b+jw}$$

$$Y(jw) = H(jw)\, X(jw)$$

$$= \left(\frac{1}{a+jw}\right)\left(\frac{1}{b+jw}\right)$$

TO FIND $y(t) \rightarrow$ INVERSE FOURIER TRANSFORM OF $Y(jw)$

EXPANDING $Y(jw)$ INTO ITS PARTIAL FRACTIONS

$$\frac{1}{(a+jw)(b+jw)} = \frac{A}{a+jw} + \frac{B}{b+jw}$$

$$1 = A(b+jw) + B(a+jw)$$

$$1 = Ab + Ajw + Ba + Bjw$$

$$Ab + Ba + jw(A+B) = 1$$

$A + B = 0$ (REAL COMPONENTS)

$A = -B$

$Ab + Ba = 1$ (IMAGINARY COMPONENTS)

$Ab + (-A)a = 1$

$A(b-a) = 1$

$A = \frac{1}{b-a} = -B$

IF $b = a$

$$Y(j\omega) = \frac{1}{(a+j\omega)(b+j\omega)}$$

$$= \frac{1}{(a+j\omega)^2}$$

$$\frac{1}{(a+j\omega)(b+j\omega)} = \frac{A}{a+j\omega} + \frac{B}{b+j\omega}$$

$$= \left(\frac{1}{b-a}\right)\cdot\left(\frac{1}{a+j\omega}\right) + \left(-\frac{1}{b-a}\right)\left(\frac{1}{b+j\omega}\right)$$

$$= \frac{1}{b-a}\left[\frac{1}{a+j\omega} - \frac{1}{b+j\omega}\right]$$

$b = a$ WOULD NOT ALLOW US TO SOLVE THIS USING PARTIAL FRACTIONS

IF $b \neq a$

$$Y(j\omega) = \frac{1}{b-a}\left[\frac{1}{a+j\omega} - \frac{1}{b+j\omega}\right]$$

WE KNOW THAT

$$\int_{-\infty}^{\infty} e^{-at} u(t)\, e^{-j\omega t}\, dt = \frac{1}{a+j\omega}$$

THIS GIVES US

$$e^{-at} u(t) \overset{F}{\longleftrightarrow} \frac{1}{a+j\omega}$$

$$e^{-bt} u(t) \overset{F}{\longleftrightarrow} \frac{1}{b+j\omega}$$

$$y(t) = \frac{1}{b-a}\left(e^{-at} - e^{-bt}\right)u(t), \quad b \neq a$$

FOR $b = a$

$$Y(j\omega) = \frac{1}{(a+j\omega)(b+j\omega)}$$

$$= \frac{1}{(a+j\omega)^2}$$

WE KNOW THAT

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{u}{v} = \frac{1}{a+jw}$$

$$\frac{d}{dw}\left(\frac{1}{a+jw}\right) = \frac{(a+jw)0 - 1(0+j)}{(a+jw)^2}$$

$$= \cancel{\frac{-j}{a^2}}$$

$$= \frac{-j}{(a+jw)^2}$$

$$\Rightarrow j\frac{d}{dw}\left(\frac{1}{a+jw}\right) = \frac{j(-j)}{(a+jw)^2}$$

$$= \frac{1}{(a+jw)^2}$$

$$\therefore \; Y(jw) = j\frac{d}{dw}\left(\frac{1}{a+jw}\right)$$

FROM TABLE 4.1
PROPERTY 4.3.6
PAGE (328)

DIFFERENTATION IN FREQUENCY

$$t\,x(t) \overset{F}{\longleftrightarrow} j\frac{d}{d\omega}X(j\omega)$$

AND $e^{-at}u(t) \overset{F}{\longleftrightarrow} \dfrac{1}{a+j\omega}$

$$\therefore\ t\left(e^{-at}u(t)\right) \overset{F}{\longleftrightarrow} j\frac{d}{d\omega}\left(\frac{1}{a+j\omega}\right)$$

ALSO IN TABLE 4.2 (PAGE 329)

$$\therefore\ y(t) = t\,e^{-at}u(t)\ , \qquad a=b$$

COMBINING BOTH SOLUTIONS

$$y(t) = \begin{cases} t\,e^{-at}u(t) & \text{if } a=b \\ \dfrac{1}{b-a}\left(e^{-at} - e^{-bt}\right)u(t) & \text{o.w} \end{cases}$$