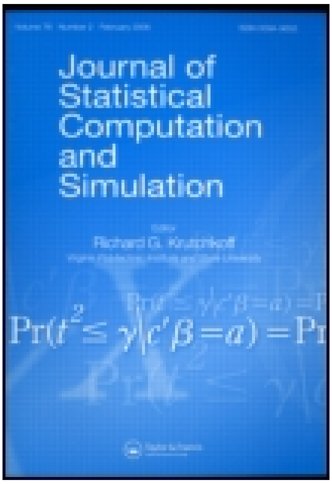

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Two-sample Bayesian prediction for sequential order statistics from exponential distribution based on multiply Type-II censored samples

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In this paper, the problem of predicting the future sequential order statistics based on observed multiply Type-II censored samples of sequential order statistics from one- and two-parameter exponential distributions is addressed. Using the Bayesian approach, the predictive and survival functions are derived and then the point and interval predictions are obtained. Finally, two numerical examples are presented for illustration.

Keywords: sequential order statistics; multiple Type-II censoring; Bayesian prediction; one-parameter exponential distribution; two-parameter exponential distribution; one-sample prediction; two-sample prediction

AMS 2000 Subject Classification: Primary: 62G30; Secondary: 62F15

1. Introduction

Sequential order statistics have been introduced as an extension of (ordinary) order statistics to model 'sequential k-out-of-n systems', where the failures of components possibly affect the surviving ones. This can be thought of as a damage caused by failures or as an increased stress put on the successively remaining components. The model of sequential order statistics is flexible in the sense that, upon each failure, the underlying distribution of the residual lifetimes of the surviving components may change. For a more detailed discussion, we refer the readers to Kamps [1] and Cramer and Kamps [2].

Let the ordered random variables $X_*^{(1)} \leq X_*^{(2)} \leq \cdots \leq X_*^{(n)}$ denote the sequential order statistics, and $x^{(1)} \leq x^{(2)} \leq \cdots \leq x^{(n)}$ be the corresponding observations of failure times. Moreover, let F_1 be the lifetime distribution at the start of the experiment, successively changing to F_j at the time of the $(j-1)$th failure, for $2 \leq j \leq n$. The ageing behaviour of the components in the system is determined by the form and shape of the underlying distributions $F_1, \ldots, F_n$. Here, these

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distribution functions are chosen as

$$F_j = 1 - (1 - F)^{\alpha_j}, \quad 1 \leq j \leq n,$$

with an absolutely continuous distribution function (cdf) F, a corresponding probability density function (pdf) f, and positive real model parameters $\alpha_1, \ldots, \alpha_n$. These assumptions imply that the failure rate of the components at work after the jth failure is given by $\alpha_{j+1} f/(1 - F)$. Therefore, the model parameter α_{j+1} reflects the influence of the jth failure on the surviving components in a natural way.

The joint density function of the sequential order statistics, $X_*^{(1)}, X_*^{(2)}, \ldots, X_*^{(n)}$, is given by

$$f_{X_*^{(1)}, X_*^{(2)}, \ldots, X_*^{(n)}}(x_1, \ldots, x_n) = n! \left(\prod_{j=1}^{n-1} \alpha_j \right) \left(\prod_{i=1}^{n-1} [1 - F(x_i)]^{m_i} f(x_i) \right) [1 - F(x_n)]^{\alpha_n - 1} f(x_n), \qquad (1)$$

where $m_i = (n - i + 1)\alpha_i - (n - i)\alpha_{i+1} - 1$ for $i = 1, \ldots, n - 1$ (see [1]).

Several well-known models of ordered random variables are included in this particular model of sequential order statistics. For instance, the choice $\alpha_1 = \cdots = \alpha_n = 1$ yields the usual order statistics, and setting

$$\alpha_j = \frac{(N - j + 1 - \sum_{i=1}^{j-1} R_i)}{(n - j + 1)}, \quad 1 \leq j \leq n,$$

with R_j and N as non-negative integers such that $N \geq n$ and $\sum_{i=1}^{n} R_i = N - n$ lead to the model of progressively Type-II censored order statistics (see [3,4]).

Statistical methods based on sequential order statistics may be found in the review article by Cramer and Kamps [3], and in the recent papers by Revathy and Chandrasekar [5], Balakrishnan *et al.* [6], Beutner [7,8], Beutner and Kamps [9], Balakrishnan *et al.* [10], Bedbur *et al.* [11], Burkschat [12], and Burkschat *et al.* [13].

In many practical problems, one would wish to use previous data to predict a future observation from the same population. One way to do this is to construct an interval that will contain the future observation with a specified probability. Such an interval is called a prediction interval. Bayesian prediction bounds for future observations from the exponential distribution have been discussed by several authors, including Dunsmore [14], Lingappaiah [15], Evans and Nigm [16], Al-Hussaini and Jaheen [17], Abdel-Aty *et al.* [18], and Mohie El-Din *et al.* [19].

Recently, Schenk *et al.* [20] considered the problems of Bayesian estimation and prediction based on observed multiply Type-II censored samples of sequential order statistics from one- and two-parameter exponential distributions. In the case of the one-parameter exponential distribution, they obtained explicit representations for the posterior distribution and the corresponding Bayes estimator under squared error loss as well as the predictor of a future failure time along with its predictive posterior distribution function. In the case of the two-parameter exponential distribution, they obtained explicit representations for the posterior distribution and the corresponding Bayes estimators under squared error loss. But, these representations unfortunately contain some errors. In this paper, the point and interval predictions for the sequential order statistics from a future sample from one-parameter exponential distribution based on observed multiply Type-II censored samples of sequential order statistics from the same distribution are first presented. In the two-parameter case, the correct representations for the posterior distribution and the corresponding Bayes estimators are given. In addition, the prediction of future sequential order statistics from the two-parameter exponential distribution based on multiply Type-II censored samples of sequential order statistics is also discussed in detail.

The rest of this paper is organized as follows. In Section 2, the description of the model of the multiply Type-II censored sample of sequential order statistics is presented. The problem of

predicting the sequential order statistics from a future sample from one-parameter exponential distribution is discussed in Section 3. In Section 4, the prediction of sequential order statistics from a future sample from the two-parameter exponential distribution is discussed. Finally, in Section 5, two numerical examples are presented for the purpose of illustrating all the inferential methods developed here.

2. The model description

In reliability analysis, experiments often get terminated before all units on test have failed due to cost and time considerations. In such cases, failure information is available only on part of the sample, and one has only partial information on all units that had not failed. Such data are said to be censored data. There are several forms of censored data. One of the most common forms of censoring is Type-II censoring. In Type-II censoring scheme, only the first $r \le n$ failure times $x^{(1)} \le x^{(2)} \le \cdots \le x^{(r)}$ would have been observed and the rest of the data will be known only to be larger than $x^{(r)}$. A generalization of Type-II censoring scheme is multiple Type-II censoring scheme. Under this scheme, we observe only the j_1th, j_2th, $\ldots, j_q$th failure times $x^{(j_1)} \le x^{(j_2)} \le \cdots \le x^{(j_q)}$, where $1 \le j_1 < j_2 < \cdots < j_q \le n$, and the rest of data are not available. Particular applications of such censoring are found in reliability theory and survival analysis. Surveys regarding censored data can be found in Nelson [21], Balakrishnan and Cohen [22], Cohen [23], Meeker and Escobar [24], Balakrishnan and Aggarwala [25], and McCool [26]. For a survey on multiple Type-II censoring, one may refer to Kong [27].

Schenk [28] derived the joint density function of multiply Type-II censored sequential order statistics $X_*^{(j_1)} \le X_*^{(j_2)} \le \cdots \le X_*^{(j_q)}$ as

$$f^{\tilde{X}_*}(\tilde{x}) = \left(\prod_{p=1}^{q} \prod_{k=j_{p-1}+1}^{j_p} \gamma_k\right) \prod_{p=1}^{q} \sum_{k=j_{p-1}+1}^{j_p} a_k^{(j_{p-1})}(j_p) \left(\frac{1-F(x^{(j_p)})}{1-F(x^{(j_{p-1})})}\right)^{\gamma_k}$$
$$\times \frac{f(x^{(j_p)})}{1-F(x^{(j_p)})}, \quad x^{(j_1)} < \cdots < x^{(j_q)}, \tag{2}$$

where F and f are the parent cdf and pdf, respectively, $\tilde{X}_* = (X_*^{(j_1)}, \ldots, X_*^{(j_q)})$ is the vector of observable variables, $\tilde{x} = (x^{(j_1)}, \ldots, x^{(j_q)})$ is a vector of realizations, and $j_0 = 0$; the constants are given by $\gamma_k = \alpha_k(n-k+1)$, $1 \le k \le n$, and

$$a_k^{(j_{p-1})}(j_p) = \prod_{\substack{i=j_{p-1}+1 \\ i \ne k}}^{j_p} \frac{1}{\gamma_i - \gamma_k}, \quad j_{p-1}+1 \le k \le j_p, \;\; 1 \le p \le q. \tag{3}$$

Notice that the above density representation is only valid for pairwise different parameters, i.e. $\gamma_i \ne \gamma_k$ for all $i, k \in \{j_{p-1}+1, \ldots, j_p\}$, $1 \le p \le q$, which is assumed throughout this paper. Clearly, this is no restriction for the usual order statistics and progressively Type-II censored order statistics.

We consider here data set consisting of s independent multiply Type-II censored samples of sequential order statistics. The ith sample contains $q_i (\ge 1)$ observations for $1 \le i \le s$, i.e., the data from the ith sample are given by

$$x_i^{(j_{i1})} \le x_i^{(j_{i2})} \le \cdots \le x_i^{(j_{iq_i})} \quad \text{with } 0 < j_{i1} < j_{i2} < \cdots < j_{iq_i} \le n_i.$$

The corresponding sequential order statistics $(X_{*i}^{(j_{ik})})_{1 \le k < q_i, 1 \le i < s}$ are assumed to be independent with respect to index i.

Let $X_{s+1}^{(1)} \leq X_{s+1}^{(2)} \leq \cdots \leq X_{s+1}^{(n_{s+1})}$ be a future independent sample of sequential order statistics from the same population. In the following sections, the Bayesian prediction of the failure time $X_{s+1}^{(j)}$ in the future $(s+1)$th sample along with its predictive distribution is developed. The marginal density function of the jth sequential order statistic $X_{s+1}^{(j)}$ is given by (see [1])

$$f^{X_{s+1}^{(j)}}(x) = \left(\prod_{k=1}^{j} \gamma_{s+1,k}\right) \sum_{k=1}^{j} a_k(j) f(x)[1 - F(x)]^{\gamma_{s+1,k}-1}, \tag{4}$$

where

$$a_k(j) = a_k^{(0)}(j) = \prod_{\substack{i=1 \\ i \neq k}}^{j} \frac{1}{\gamma_{s+1,i} - \gamma_{s+1,k}}, \quad 1 \leq k \leq j, \ 1 \leq j \leq n_{s+1}. \tag{5}$$

3. One-parameter exponential distribution

In this section, the underlying distribution is assumed to be a one-parameter exponential distribution with scale parameter $\sigma > 0$, and with pdf and cdf as

$$f(x;\sigma) = \frac{1}{\sigma} \exp\left(-\frac{x}{\sigma}\right), \quad x \geq 0, \ \sigma > 0, \tag{6}$$

and

$$F(x;\sigma) = 1 - \exp\left(-\frac{x}{\sigma}\right), \quad x \geq 0, \ \sigma > 0, \tag{7}$$

respectively. For an excellent survey on the exponential distribution, interested readers may refer to the book by Balakrishnan and Basu [29].

The parameter σ is assumed to be a realization of a random variable Σ with prior density function given by

$$\pi^{\Sigma}(\sigma) \propto \sigma^{-(b+1)} \exp\left(-\frac{a}{\sigma}\right), \ \sigma > 0, \tag{8}$$

which is an inverted gamma density, for $a > 0, b > 0$.

Schenk *et al.* [20] derived explicit representations for the posterior distribution of Σ and the corresponding Bayes estimator under squared error loss. These representations, presented in the following lemma without proof, will be used here to derive predictive and survival functions for the sequential order statistics from a future sample and the corresponding Bayesian prediction.

LEMMA 3.1 *Let us denote*

$$\begin{aligned}
\tilde{X}_{*i} &= (X_{*i}^{(j_{i1})}, \ldots, X_{*i}^{(j_{iq_i})}), \quad \tilde{x}_i = (x_i^{(j_{i1})}, \ldots, x_i^{(j_{iq_i})}), \quad i = 1, \ldots, s,\\
\tilde{X} &= (\tilde{X}_{*1}, \ldots, \tilde{X}_{*s}), \quad \tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_s),\\
\tilde{K} &= \{\tilde{k} = (k_{11}, \ldots, k_{sq_s}) \mid k_{ip} \in \{j_{i,p-1}+1, \ldots, j_{ip}\}, 1 \le p \le q_i, 1 \le i \le s\},\\
\Psi(\tilde{k}) &= \prod_{i=1}^{s}\prod_{p=1}^{q_i} a_{k_{ip}}^{(j_{i,p-1})}(j_{ip}), \quad Q = \sum_{i=1}^{s} q_i
\end{aligned}$$

and

$$V_{\tilde{k}} = \sum_{i=1}^{s}\sum_{p=1}^{q_i} \gamma_{ik_{ip}}(x_i^{(j_{ip})} - x_i^{(j_{i,p-1})}) + a.$$

Then, the posterior distribution of Σ, *given* $\tilde{X} = \tilde{x}$, *is given by*

$$\pi^{\Sigma|\tilde{X}}(\sigma|\tilde{x}) = C_1^{-1}\sigma^{-(Q+b+1)} \sum_{\tilde{k}\in\tilde{K}} \Psi(\tilde{k}) \exp\left(-\frac{V_{\tilde{k}}}{\sigma}\right), \tag{9}$$

with the normalizing constant

$$C_1 = \Gamma(Q+b) \sum_{\tilde{k}\in\tilde{K}} \Psi(\tilde{k})(V_{\tilde{k}})^{-(Q+b)}. \tag{10}$$

Therefore, the Bayes estimator of σ *under the squared error loss is given by*

$$\hat{\sigma} = \frac{1}{Q+b-1}\frac{\sum_{\tilde{k}\in\tilde{K}} \Psi(\tilde{k})(V_{\tilde{k}})^{-(Q+b-1)}}{\sum_{\tilde{k}\in\tilde{K}} \Psi(\tilde{k})(V_{\tilde{k}})^{-(Q+b)}}, \tag{11}$$

and the posterior second moment of σ *is given by*

$$\widehat{\sigma^2} = \frac{1}{(Q+b-1)(Q+b-2)}\frac{\sum_{\tilde{k}\in\tilde{K}} \Psi(\tilde{k})(V_{\tilde{k}})^{-(Q+b-2)}}{\sum_{\tilde{k}\in\tilde{K}} \Psi(\tilde{k})(V_{\tilde{k}})^{-(Q+b)}}. \tag{12}$$

THEOREM 3.2 *Using the notation of Lemma 3.1, the predictive density function of* $X_{s+1}^{(j)}$, *given* $\tilde{X} = \tilde{x}$, *is given by*

$$f^{X_{s+1}^{(j)}|\tilde{X}}(x|\tilde{x}) = C_1^{-1}\Gamma(Q+b+1)\left(\prod_{k=1}^{j}\gamma_{s+1,k}\right)\sum_{k=1}^{j} a_k(j) \sum_{\tilde{k}\in\tilde{K}} \Psi(\tilde{k})(V_{s+1,\tilde{k}}^{k}(x))^{-(Q+b+1)}, \tag{13}$$

where $a_k(j)$ *is as in Equation (5) and* $V_{s+1,\tilde{k}}^{k}(x) = V_{\tilde{k}} + \gamma_{s+1,k}x$. *Therefore, the predictive survival function of* $X_{s+1}^{(j)}$, *given* $\tilde{X} = \tilde{x}$, *is given by*

$$\bar{F}^{X_{s+1}^{(j)}|\tilde{X}}(t|\tilde{x}) = C_1^{-1}\Gamma(Q+b)\left(\prod_{k=1}^{j}\gamma_{s+1,k}\right)\sum_{k=1}^{j} \frac{a_k(j)}{\gamma_{s+1,k}} \sum_{\tilde{k}\in\tilde{K}} \Psi(\tilde{k})(V_{s+1,\tilde{k}}^{k}(t))^{-(Q+b)}.$$

Proof The conditional independence of $\tilde{X}_{*1},\ldots,\tilde{X}_{*s}$ and $\tilde{X}_{s+1}=(X_{s+1}^{(1)},X_{s+1}^{(2)},\ldots,X_{s+1}^{(q_{s+1})})$ readily yields

$$f^{X_{s+1}^{(j)}|\tilde{X}}(x|\tilde{x})=f^{X_{s+1}^{(j)}}(x)$$
$$=\left(\prod_{k=1}^{j}\gamma_{s+1,k}\right)\sum_{k=1}^{j}a_k(j)\sigma^{-1}\exp\left(-\frac{\gamma_{s+1,k}}{\sigma}x\right),\quad x\geq 0.$$

Thus, the predictive density function of $X_{s+1}^{(j)}$, given $\tilde{X}=\tilde{x}$, is given by

$$f^{X_{s+1}^{(j)}|\tilde{X}}(x|\tilde{x})=\int_0^\infty f^{X_{s+1}^{(j)}|\tilde{X}}(x|\tilde{x})\pi^{\Sigma|\tilde{X}}(\sigma|\tilde{x})\,\mathrm{d}\sigma$$
$$=C_1^{-1}\left(\prod_{k=1}^{j}\gamma_{s+1,k}\right)\sum_{k=1}^{j}a_k(j)\sum_{\tilde{k}\in\tilde{K}}\Psi(\tilde{k})\int_0^\infty\sigma^{-(Q+b+2)}\exp\left(-\frac{V_{s+1,\tilde{k}}^k(x)}{\sigma}\right)\mathrm{d}\sigma$$
$$=C_1^{-1}\Gamma(Q+b+1)\left(\prod_{k=1}^{j}\gamma_{s+1,k}\right)\sum_{k=1}^{j}a_k(j)\sum_{\tilde{k}\in\tilde{K}}\Psi(\tilde{k})(V_{s+1,\tilde{k}}^k(x))^{-(Q+b+1)}.\quad(14)$$

From Equation (14), the predictive survival function of $X_{s+1}^{(j)}$, given $\tilde{X}=\tilde{x}$, is simply obtained as

$$\bar{F}^{X_{s+1}^{(j)}|\tilde{X}}(t|\tilde{x})=\int_t^\infty f^{X_{s+1}^{(j)}|\tilde{X}}(x|\tilde{x})\,\mathrm{d}x$$
$$=C_1^{-1}\Gamma(Q+b)\left(\prod_{k=1}^{j}\gamma_{s+1,k}\right)\sum_{k=1}^{j}\frac{a_k(j)}{\gamma_{s+1,k}}\sum_{\tilde{k}\in\tilde{K}}\Psi(\tilde{k})(V_{s+1,\tilde{k}}^k(t))^{-(Q+b)}.$$

This completes the proof of the theorem. ■

LEMMA 3.3 *Let $r\in N$, $\gamma_l>0$, $1\leq l\leq r$, and $\gamma_l\neq\gamma_j$ for $l,j\in\{1,\ldots,r\}$, $l\neq j$, and define*

$$a_j(r)=a_j^{(0)}(r)=\prod_{\substack{l=1\\l\neq j}}^{r}\frac{1}{\gamma_l-\gamma_j},\quad r\geq 2\ (\textit{with } a_1(1)=1).$$

Moreover, let $l_0\in N_0$ and $l_r=0$. Then,

$$\left(\prod_{j=1}^{r}\gamma_j\right)\sum_{j=1}^{r}\frac{1}{\gamma_j^{l_0+1}}a_j(r)=\sum_{l_0\geq l_1\geq\cdots\geq l_{r-1}\geq 0}\prod_{j=1}^{r}\gamma_j^{l_{r-j+1}-l_{r-j}}.\quad(15)$$

In particular, for $l_0=0$,

$$\left(\prod_{j=1}^{r}\gamma_j\right)\sum_{j=1}^{r}\frac{1}{\gamma_j}a_j(r)=1;\quad(16)$$

for $l_0=1$,

$$\left(\prod_{j=1}^{r}\gamma_j\right)\sum_{j=1}^{r}\frac{1}{\gamma_j^2}a_j(r)=\sum_{j=1}^{r}\frac{1}{\gamma_j};\quad(17)$$

and for $l_0 = 2$,

$$\left(\prod_{j=1}^{r}\gamma_j\right)\sum_{j=1}^{r}\frac{1}{\gamma_j^3}a_j(r) = \sum_{j=1}^{r}\frac{1}{\gamma_j^2} + \sum_{\substack{l,k=1\\ l<k}}^{r}\frac{1}{\gamma_l\gamma_k}. \tag{18}$$

For a proof of this lemma, one may refer to Schenk *et al.* [20].

THEOREM 3.4 *The predictive failure of the jth sequential order statistic in the* $(s+1)$*th sample, given* $\tilde{X} = \tilde{x}$, *is given by*

$$\hat{X}_{s+1}^{(j)} = \hat{\sigma}\sum_{k=1}^{j}\frac{1}{\gamma_{s+1,k}}, \tag{19}$$

and the posterior variance of $\hat{X}_{s+1}^{(j)}$ *is given by*

$$Var(\hat{X}_{s+1}^{(j)}|\tilde{X}) = (\widehat{\sigma^2} - \hat{\sigma}^2)\left(\sum_{k=1}^{j}\frac{1}{\gamma_{s+1,k}}\right)^2 + \widehat{\sigma^2}\sum_{k=1}^{j}\frac{1}{\gamma_{s+1,k}^2}. \tag{20}$$

Proof The predictor of $X_{s+1}^{(j)}$ is given by the mean of the predictive density function of $X_{s+1}^{(j)}$. Thus, we have

$$\begin{aligned}\hat{X}_{s+1}^{(j)} &= E(X_{s+1}^{(j)}|\tilde{X}) = \int_0^\infty t f^{X_{s+1}^{(j)}|\tilde{X}}(t|\tilde{x})\,\mathrm{d}t\\ &= C_1^{-1}\Gamma(Q+b+1)\left(\prod_{k=1}^{j}\gamma_{s+1,k}\right)\sum_{k=1}^{j}a_k(j)\sum_{\tilde{k}\in\tilde{K}}\Psi(\tilde{k})\int_0^\infty t(V_{s+1,\tilde{k}}^{k}(t))^{-(Q+b+1)}\mathrm{d}t.\end{aligned}$$

Upon using integration by parts, $\hat{X}_{s+1}^{(j)}$ becomes

$$\hat{X}_{s+1}^{(j)} = C_1^{-1}\Gamma(Q+b-1)\left(\prod_{k=1}^{j}\gamma_{s+1,k}\right)\sum_{k=1}^{j}\frac{a_k(j)}{(\gamma_{s+1,k})^2}\sum_{\tilde{k}\in\tilde{K}}\Psi(\tilde{k})(V_{\tilde{k}})^{-(Q+b-1)}. \tag{21}$$

Upon using Equations (16) and (17), the representation in Equation (21) simplifies to

$$\hat{X}_{s+1}^{(j)} = \hat{\sigma}\sum_{k=1}^{j}\frac{1}{\gamma_{s+1,k}}.$$

Similarly, the posterior second moment of $X_{s+1}^{(j)}$ is given by

$$\begin{aligned}E({X_{s+1}^{(j)}}^2|\tilde{X}) &= 2C_1^{-1}\Gamma(Q+b-2)\left(\prod_{k=1}^{j}\gamma_{s+1,k}\right)\sum_{k=1}^{j}\frac{a_k(j)}{(\gamma_{s+1,k})^3}\sum_{\tilde{k}\in\tilde{K}}\Psi(\tilde{k})(V_{\tilde{k}})^{-(Q+b-2)}\\ &= 2\widehat{\sigma^2}\left(\sum_{k=1}^{j}\frac{1}{\gamma_{s+1,k}^2} + \sum_{\substack{l,k=1\\ l<k}}^{j}\frac{1}{\gamma_{s+1,l}\gamma_{s+1,k}}\right).\end{aligned}$$

Thus, the posterior variance of $\hat{X}_{s+1}^{(j)}$ is given by

$$\mathrm{Var}(\hat{X}_{s+1}^{(j)}|\tilde{X}) = (\widehat{\sigma^2} - \hat{\sigma}^2)\left(\sum_{k=1}^{j}\frac{1}{\gamma_{s+1,k}}\right)^2 + \widehat{\sigma^2}\sum_{k=1}^{j}\frac{1}{\gamma_{s+1,k}^2},$$

which completes the proof of the theorem. ■

Remark 3.5 The Bayesian predictive bounds of a two-sided equi-tailed $100(1-\tau)\%$ interval for the jth sequential order statistic in the $(s+1)$th sample, given $\tilde{X} = \tilde{x}$, can be obtained by solving the following two equations:

$$\bar{F}^{X_{s+1}^{(j)}|\tilde{X}}(L|\tilde{x}) = 1 - \frac{\tau}{2} \quad \text{and} \quad \bar{F}^{X_{s+1}^{(j)}|\tilde{X}}(U|\tilde{x}) = \frac{\tau}{2},$$

where $\bar{F}^{X_{s+1}^{(j)}|\tilde{X}}(t|\tilde{x})$ is the predictive survival function of $X_{s+1}^{(j)}$ given in Theorem 3.2, and L and U denote the lower and upper bounds, respectively.

For the highest posterior density (HPD) method, the following two equations need to be solved:

$$\bar{F}^{X_{s+1}^{(j)}|\tilde{X}}(L_{X_{s+1}^{(j)}}|\tilde{x}) - \bar{F}^{X_{s+1}^{(j)}|\tilde{X}}(U_{X_{s+1}^{(j)}}|\tilde{x}) = 1 - \tau$$

and

$$f^{X_{s+1}^{(j)}|\tilde{X}}(L_{X_{s+1}^{(j)}}|\tilde{x}) = f^{X_{s+1}^{(j)}|\tilde{X}}(U_{X_{s+1}^{(j)}}|\tilde{x}),$$

where $f^{X_{s+1}^{(j)}|\tilde{X}}(x|\tilde{x})$ is the predictive density function of $X_{s+1}^{(j)}$ given in Theorem 3.2, and $L_{X_{s+1}^{(j)}}$ and $U_{X_{s+1}^{(j)}}$ denote the HPD lower and upper bounds, respectively.

4. Two-parameter exponential distribution

In this section, the underlying distribution is assumed to be a two-parameter exponential distribution with location parameter $\mu \in \mathbb{R}$ and scale parameter $\sigma > 0$, and with pdf and cdf as

$$f(x;\sigma,\mu) = \frac{1}{\sigma}\exp\left(-\frac{x-\mu}{\sigma}\right), \quad x \geq \mu,\ \sigma > 0,\ \mu \in \mathbb{R}, \tag{22}$$

and

$$F(x;\sigma,\mu) = 1 - \exp\left(-\frac{x-\mu}{\sigma}\right), \quad x \geq \mu,\ \sigma > 0,\ \mu \in \mathbb{R}, \tag{23}$$

respectively. The parameters σ and μ are assumed to be realizations of two random variables Σ and Δ with joint prior density function given by

$$\pi^{(\Sigma,\Delta)}(\sigma,\mu) \propto \sigma^{-(b+1)}\exp\left(-\frac{a-c\mu}{\sigma}\right)1_{[N,M]}(\mu), \quad \sigma > 0,\ \mu \in \mathbb{R}, \tag{24}$$

which is a proper density for $a > cM$, $b > 1$, $c > 0$ and $M > N$ with $N, M \in \mathbb{R}$ (see [16,30] for related results with a similar prior). Notice that the choice $a = 0$, $b \in R$, $c = 0$ and $M \to \infty, N \to -\infty$ leads to the improper prior $\pi^{(\Sigma,\Delta)}(\sigma,\mu) \propto \sigma^{-(b+1)}$, $\sigma > 0$, $\mu \in \mathbb{R}$.

Schenk *et al.* [20] derived explicit representations for the joint posterior distribution of Σ and Δ and the corresponding Bayes estimators under the squared error loss. But, these representations

unfortunately contain some errors. Here, the correct representations are presented in the following two lemmas without proof for the sake of brevity.

LEMMA 4.1 *Under the notation in Lemma 3.1, let $Q+b-1>0$ and*

$$z=\min\{x_1^{(j_{11})},\ldots,x_s^{(j_{s1})}\},\quad M_0=\min\{z,M\},$$

$$H_{\tilde{k}}(t)=V_{\tilde{k}}-t\left(c+\sum_{i=1}^{s}\gamma_{ik_{i1}}\right)$$

with $N<z$ and $H_{\tilde{k}}(M_0)>0$. Then, the joint posterior distribution of Σ and Δ, given $\tilde{X}=\tilde{x}$, is given by

$$\pi^{(\Sigma,\Delta)|\tilde{X}}(\sigma,\mu|\tilde{x})=C_2^{-1}\sigma^{-(Q+b+1)}\sum_{\tilde{k}\in\tilde{K}}\Psi(\tilde{k})\exp\left(-\frac{H_{\tilde{k}}(\mu)}{\sigma}\right)1_{[N,M_0]}(\mu), \tag{25}$$

with the normalizing constant

$$C_2=\Gamma(Q+b-1)\sum_{\tilde{k}\in\tilde{K}}\frac{\Psi(\tilde{k})}{c+\sum_{i=1}^{s}\gamma_{ik_{i1}}}([H_{\tilde{k}}(M_0)]^{-(Q+b-1)}-[H_{\tilde{k}}(N)]^{-(Q+b-1)}). \tag{26}$$

LEMMA 4.2 *Under the notation in Lemma 4.1, if $Q+b-3>0$, then the Bayes estimators of $\sigma,\sigma^2,\mu,\mu^2$ and $\sigma\mu$ under the squared error loss are given, respectively, by*

$$\hat{\sigma}=\frac{1}{Q+b-2}\frac{\sum_{\tilde{k}\in\tilde{K}}(\Psi(\tilde{k})/(c+\sum_{i=1}^{s}\gamma_{ik_{i1}}))([H_{\tilde{k}}(M_0)]^{-(Q+b-2)}-[H_{\tilde{k}}(N)]^{-(Q+b-2)})}{\sum_{\tilde{k}\in\tilde{K}}(\Psi(\tilde{k})/(c+\sum_{i=1}^{s}\gamma_{ik_{i1}}))([H_{\tilde{k}}(M_0)]^{-(Q+b-1)}-[H_{\tilde{k}}(N)]^{-(Q+b-1)})}, \tag{27}$$

$$\widehat{\sigma^2}=\frac{1}{(Q+b-2)(Q+b-3)}\times\frac{\sum_{\tilde{k}\in\tilde{K}}(\Psi(\tilde{k})/(c+\sum_{i=1}^{s}\gamma_{ik_{i1}}))([H_{\tilde{k}}(M_0)]^{-(Q+b-3)}-[H_{\tilde{k}}(N)]^{-(Q+b-3)})}{\sum_{\tilde{k}\in\tilde{K}}(\Psi(\tilde{k})/(c+\sum_{i=1}^{s}\gamma_{ik_{i1}}))([H_{\tilde{k}}(M_0)]^{-(Q+b-1)}-[H_{\tilde{k}}(N)]^{-(Q+b-1)})}, \tag{28}$$

$$\begin{aligned}\hat{\mu}=&\frac{\sum_{\tilde{k}\in\tilde{K}}(\Psi(\tilde{k})/(c+\sum_{i=1}^{s}\gamma_{ik_{i1}}))(M_0[H_{\tilde{k}}(M_0)]^{-(Q+b-1)}-N[H_{\tilde{k}}(N)]^{-(Q+b-1)})}{\sum_{\tilde{k}\in\tilde{K}}(\Psi(\tilde{k})/(c+\sum_{i=1}^{s}\gamma_{ik_{i1}}))([H_{\tilde{k}}(M_0)]^{-(Q+b-1)}-[H_{\tilde{k}}(N)]^{-(Q+b-1)})}\\&-\frac{1}{Q+b-2}\frac{\sum_{\tilde{k}\in\tilde{K}}(\Psi(\tilde{k})/(c+\sum_{i=1}^{s}\gamma_{ik_{i1}})^2)([H_{\tilde{k}}(M_0)]^{-(Q+b-2)}-[H_{\tilde{k}}(N)]^{-(Q+b-2)})}{\sum_{\tilde{k}\in\tilde{K}}(\Psi(\tilde{k})/(c+\sum_{i=1}^{s}\gamma_{ik_{i1}}))([H_{\tilde{k}}(M_0)]^{-(Q+b-1)}-[H_{\tilde{k}}(N)]^{-(Q+b-1)})},\end{aligned} \tag{29}$$

$$\begin{aligned}\widehat{\mu^2}=&\frac{\sum_{\tilde{k}\in\tilde{K}}(\Psi(\tilde{k})/(c+\sum_{i=1}^{s}\gamma_{ik_{i1}}))(M_0^2[H_{\tilde{k}}(M_0)]^{-(Q+b-1)}-N^2[H_{\tilde{k}}(N)]^{-(Q+b-1)})}{\sum_{\tilde{k}\in\tilde{K}}(\Psi(\tilde{k})/(c+\sum_{i=1}^{s}\gamma_{ik_{i1}}))([H_{\tilde{k}}(M_0)]^{-(Q+b-1)}-[H_{\tilde{k}}(N)]^{-(Q+b-1)})}\\&-\frac{2}{Q+b-2}\\&\times\frac{\sum_{\tilde{k}\in\tilde{K}}(\Psi(\tilde{k})/(c+\sum_{i=1}^{s}\gamma_{ik_{i1}})^2)(M_0[H_{\tilde{k}}(M_0)]^{-(Q+b-2)}-N[H_{\tilde{k}}(N)]^{-(Q+b-2)})}{\sum_{\tilde{k}\in\tilde{K}}(\Psi(\tilde{k})/(c+\sum_{i=1}^{s}\gamma_{ik_{i1}}))([H_{\tilde{k}}(M_0)]^{-(Q+b-1)}-[H_{\tilde{k}}(N)]^{-(Q+b-1)})}\\&+\frac{2}{(Q+b-2)(Q+b-3)}\\&\times\frac{\sum_{\tilde{k}\in\tilde{K}}(\Psi(\tilde{k})/(c+\sum_{i=1}^{s}\gamma_{ik_{i1}})^3)([H_{\tilde{k}}(M_0)]^{-(Q+b-3)}-[H_{\tilde{k}}(N)]^{-(Q+b-3)})}{\sum_{\tilde{k}\in\tilde{K}}(\Psi(\tilde{k})/(c+\sum_{i=1}^{s}\gamma_{ik_{i1}}))([H_{\tilde{k}}(M_0)]^{-(Q+b-1)}-[H_{\tilde{k}}(N)]^{-(Q+b-1)})},\end{aligned} \tag{30}$$

and

$$\widehat{\sigma\mu} = \frac{1}{Q+b-2}\frac{\sum_{\tilde{k}\in\tilde{K}}(\Psi(\tilde{k})/(c+\sum_{i=1}^{s}\gamma_{ik_{i1}}))(M_0[H_{\tilde{k}}(M_0)]^{-(Q+b-2)} - N[H_{\tilde{k}}(N)]^{-(Q+b-2)})}{\sum_{\tilde{k}\in\tilde{K}}(\Psi(\tilde{k})/(c+\sum_{i=1}^{s}\gamma_{ik_{i1}}))([H_{\tilde{k}}(M_0)]^{-(Q+b-1)} - [H_{\tilde{k}}(N)]^{-(Q+b-1)})}$$
$$- \frac{1}{(Q+b-2)(Q+b-3)}$$
$$\times \frac{\sum_{\tilde{k}\in\tilde{K}}(\Psi(\tilde{k})/(c+\sum_{i=1}^{s}\gamma_{ik_{i1}})^2)([H_{\tilde{k}}(M_0)]^{-(Q+b-3)} - [H_{\tilde{k}}(N)]^{-(Q+b-3)})}{\sum_{\tilde{k}\in\tilde{K}}(\Psi(\tilde{k})/(c+\sum_{i=1}^{s}\gamma_{ik_{i1}}))([H_{\tilde{k}}(M_0)]^{-(Q+b-1)} - [H_{\tilde{k}}(N)]^{-(Q+b-1)})}. \tag{31}$$

THEOREM 4.3 *Under the notation in Lemma 4.1, the predictive density function of $X_{s+1}^{(j)}$, given $\tilde{X} = \tilde{x}$, is given by*

$$f^{X_{s+1}^{(j)}|\tilde{X}}(x|\tilde{x}) = \begin{cases} f_1^{X_{s+1}^{(j)}|\tilde{X}}(x|\tilde{x}), & N < x < M_0, \\ f_2^{X_{s+1}^{(j)}|\tilde{X}}(x|\tilde{x}), & x \geq M_0, \end{cases} \tag{32}$$

where

$$f_1^{X_{s+1}^{(j)}|\tilde{X}}(x|\tilde{x}) = C_2^{-1}\Gamma(Q+b)\left(\prod_{k=1}^{j}\gamma_{s+1,k}\right)\sum_{k=1}^{j}a_k(j)\sum_{\tilde{k}\in\tilde{K}}\frac{\Psi(\tilde{k})}{c+\gamma_{s+1,k}+\sum_{i=1}^{s}\gamma_{ik_{i1}}}$$
$$\times ([H_{\tilde{k}}(x)]^{-(Q+b)} - [G_{s+1,\tilde{k}}^{k}(N,x)]^{-(Q+b)})$$

and

$$f_2^{X_{s+1}^{(j)}|\tilde{X}}(x|\tilde{x}) = C_2^{-1}\Gamma(Q+b)\left(\prod_{k=1}^{j}\gamma_{s+1,k}\right)\sum_{k=1}^{j}a_k(j)\sum_{\tilde{k}\in\tilde{K}}\frac{\Psi(\tilde{k})}{c+\gamma_{s+1,k}+\sum_{i=1}^{s}\gamma_{ik_{i1}}}$$
$$\times ([G_{s+1,\tilde{k}}^{k}(M_0,x)]^{-(Q+b)} - [G_{s+1,\tilde{k}}^{k}(N,x)]^{-(Q+b)}),$$

with $G_{s+1,\tilde{k}}^{k}(t,x) = H_{\tilde{k}}(t) + \gamma_{s+1,k}(x-t)$. Therefore, the predictive survival function of $X_{s+1}^{(j)}$, given $\tilde{X} = \tilde{x}$, is given by

$$\bar{F}^{X_{s+1}^{(j)}|\tilde{X}}(t|\tilde{x}) = \begin{cases} \bar{F}_1^{X_{s+1}^{(j)}|\tilde{X}}(t|\tilde{x}), & N < t < M_0, \\ \bar{F}_2^{X_{s+1}^{(j)}|\tilde{X}}(t|\tilde{x}), & t \geq M_0, \end{cases} \tag{33}$$

where

$$\bar{F}_1^{X_{s+1}^{(j)}|\tilde{X}}(t|\tilde{x}) = C_2^{-1}\Gamma(Q+b-1)\left(\prod_{k=1}^{j}\gamma_{s+1,k}\right)\sum_{k=1}^{j}a_k(j)\sum_{\tilde{k}\in\tilde{K}}\frac{\Psi(\tilde{k})}{c+\gamma_{s+1,k}+\sum_{i=1}^{s}\gamma_{k_{i1}}}$$
$$\times \left[\frac{1}{c+\sum_{i=1}^{s}\gamma_{k_{i1}}}([H_{\tilde{k}}(M_0)]^{-(Q+b-1)} - [H_{\tilde{k}}(t)]^{-(Q+b-1)})\right.$$
$$\left.+\frac{1}{\gamma_{s+1,k}}([H_{\tilde{k}}(M_0)]^{-(Q+b-1)} - [G_{s+1,\tilde{k}}^{k}(N,t)]^{-(Q+b-1)})\right]$$

and

$$\bar{F}_2^{X_{s+1}^{(j)}|\tilde{X}}(t|\tilde{x}) = C_2^{-1}\Gamma(Q+b-1)\left(\prod_{k=1}^{j}\gamma_{s+1,k}\right)\sum_{k=1}^{j}\frac{a_k(j)}{\gamma_{s+1,k}}\sum_{\tilde{k}\in\tilde{K}}\frac{\Psi(\tilde{k})}{c+\gamma_{s+1,k}+\sum_{i=1}^{s}\gamma_{k_{i1}}}$$
$$\times ([G_{s+1,\tilde{k}}^{k}(M_0,t)]^{-(Q+b-1)} - [G_{s+1,\tilde{k}}^{k}(N,t)]^{-(Q+b-1)}).$$

Proof The predictive density function of $X_{s+1}^{(j)}$, given $\tilde{X} = \tilde{x}$, is given by

$$f^{X_{s+1}^{(j)}|\tilde{X}}(x|\tilde{x}) = \begin{cases} f_1^{X_{s+1}^{(j)}|\tilde{X}}(x|\tilde{x}), & N < x < M_0, \\ f_2^{X_{s+1}^{(j)}|\tilde{X}}(x|\tilde{x}), & x \geq M_0, \end{cases} \tag{34}$$

where

$$\begin{aligned} f_1^{X_{s+1}^{(j)}|\tilde{X}}(x|\tilde{x}) &= \int_N^x \int_0^\infty f^{X_{s+1}^{(j)}|\tilde{X}}(x|\tilde{x}) \pi^{(\Sigma,\Delta)|\tilde{X}}(\sigma, \mu|\tilde{x}) \, \mathrm{d}\sigma \, \mathrm{d}\mu \\ &= C_2^{-1} \left(\prod_{k=1}^{j} \gamma_{s+1,k} \right) \sum_{k=1}^{j} a_k(j) \sum_{\tilde{k} \in \tilde{K}} \Psi(\tilde{k}) \\ &\quad \times \int_N^x \int_0^\infty \sigma^{-(Q+b+2)} \exp\left(-\frac{G_{s+1,\tilde{k}}^k(\mu, x)}{\sigma} \right) \mathrm{d}\sigma \, \mathrm{d}\mu \\ &= C_2^{-1} \Gamma(Q+b) \left(\prod_{k=1}^{j} \gamma_{s+1,k} \right) \sum_{k=1}^{j} a_k(j) \sum_{\tilde{k} \in \tilde{K}} \frac{\Psi(\tilde{k})}{c + \gamma_{s+1,k} + \sum_{i=1}^{s} \gamma_{ik_{i1}}} \\ &\quad \times \left([H_{\tilde{k}}(x)]^{-(Q+b)} - [G_{s+1,\tilde{k}}^k(N, x)]^{-(Q+b)} \right) \end{aligned}$$

and

$$\begin{aligned} f_2^{X_{s+1}^{(j)}|\tilde{X}}(x|\tilde{x}) &= \int_N^{M_0} \int_0^\infty f^{X_{s+1}^{(j)}|\tilde{X}}(x|\tilde{x}) \pi^{(\Sigma,\Delta)|\tilde{X}}(\sigma, \mu|\tilde{x}) \, \mathrm{d}\sigma \, \mathrm{d}\mu \\ &= C_2^{-1} \left(\prod_{k=1}^{j} \gamma_{s+1,k} \right) \sum_{k=1}^{j} a_k(j) \sum_{\tilde{k} \in \tilde{K}} \Psi(\tilde{k}) \\ &\quad \times \int_N^{M_0} \int_0^\infty \sigma^{-(Q+b+2)} \exp\left(-\frac{G_{s+1,\tilde{k}}^k(\mu, x)}{\sigma} \right) \mathrm{d}\sigma \, \mathrm{d}\mu \\ &= C_2^{-1} \Gamma(Q+b) \left(\prod_{k=1}^{j} \gamma_{s+1,k} \right) \sum_{k=1}^{j} a_k(j) \sum_{\tilde{k} \in \tilde{K}} \frac{\Psi(\tilde{k})}{c + \gamma_{s+1,k} + \sum_{i=1}^{s} \gamma_{ik_{i1}}} \\ &\quad \times \left([G_{s+1,\tilde{k}}^k(M_0, x)]^{-(Q+b)} - [G_{s+1,\tilde{k}}^k(N, x)]^{-(Q+b)} \right). \end{aligned}$$

From Equation (34), the predictive survival function of $X_{s+1}^{(j)}$, given $\tilde{X} = \tilde{x}$, is simply obtained as

$$\bar{F}^{X_{s+1}^{(j)}|\tilde{X}}(t|\tilde{x}) = \begin{cases} \bar{F}_1^{X_{s+1}^{(j)}|\tilde{X}}(t|\tilde{x}), & N < t < M_0, \\ \bar{F}_2^{X_{s+1}^{(j)}|\tilde{X}}(t|\tilde{x}), & t \geq M_0, \end{cases}$$

where

$$
\begin{aligned}
\bar{F}_1^{X_{s+1}^{(j)}|\tilde{X}}(t|\tilde{x}) &= \int_t^{M_0} f_1^{X_{s+1}^{(j)}|\tilde{X}}(x|\tilde{x})\,\mathrm{d}x + \int_{M_0}^{\infty} f_2^{X_{s+1}^{(j)}|\tilde{X}}(x|\tilde{x})\,\mathrm{d}x \\
&= C_2^{-1}\Gamma(Q+b-1)\left(\prod_{k=1}^{j}\gamma_{s+1,k}\right)\sum_{k=1}^{j} a_k(j)\sum_{\tilde{k}\in\tilde{K}} \frac{\Psi(\tilde{k})}{c+\gamma_{s+1,k}+\sum_{i=1}^{s}\gamma_{k_{i1}}} \\
&\quad\times\left[\frac{1}{c+\sum_{i=1}^{s}\gamma_{k_{i1}}}([H_{\tilde{k}}(M_0)]^{-(Q+b-1)}-[H_{\tilde{k}}(t)]^{-(Q+b-1)})\right. \\
&\quad\left.+\frac{1}{\gamma_{s+1,k}}([H_{\tilde{k}}(M_0)]^{-(Q+b-1)}-[G_{s+1,\tilde{k}}^{k}(N,t)]^{-(Q+b-1)})\right]
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{F}_2^{X_{s+1}^{(j)}|\tilde{X}}(t|\tilde{x}) &= \int_t^{\infty} f_2^{X_{s+1}^{(j)}|\tilde{X}}(x|\tilde{x})\,\mathrm{d}x \\
&= C_2^{-1}\Gamma(Q+b-1)\left(\prod_{k=1}^{j}\gamma_{s+1,k}\right)\sum_{k=1}^{j}\frac{a_k(j)}{\gamma_{s+1,k}}\sum_{\tilde{k}\in\tilde{K}}\frac{\Psi(\tilde{k})}{c+\gamma_{s+1,k}+\sum_{i=1}^{s}\gamma_{k_{i1}}} \\
&\quad\times([G_{s+1,\tilde{k}}^{k}(M_0,t)]^{-(Q+b-1)}-[G_{s+1,\tilde{k}}^{k}(N,t)]^{-(Q+b-1)}).
\end{aligned}
$$

This completes the proof of the theorem. ■

THEOREM 4.4 *The predictive failure of the jth sequential order statistic in the* $(s+1)$*th sample, given* $\tilde{X}=\tilde{x}$*, is given by*

$$
\hat{X}_{s+1}^{(j)} = \hat{\mu} + \hat{\sigma}\sum_{k=1}^{j}\frac{1}{\gamma_{s+1,k}}, \tag{35}
$$

where $\hat{\sigma}$ *and* $\hat{\mu}$ *are the Bayes estimators of* σ *and* μ *as presented in Equations (27) and (29), respectively. The posterior variance of* $\hat{X}_{s+1}^{(j)}$ *is given by*

$$
\begin{aligned}
Var(\hat{X}_{s+1}^{(j)}|\tilde{X}) &= (\widehat{\mu^2}-\hat{\mu}^2)+2(\widehat{\sigma\mu}-\hat{\sigma}\hat{\mu})\sum_{k=1}^{j}\frac{1}{\gamma_{s+1,k}}+(\widehat{\sigma^2}-\hat{\sigma}^2)\left(\sum_{k=1}^{j}\frac{1}{\gamma_{s+1,k}}\right)^2 \\
&\quad+\widehat{\sigma^2}\sum_{k=1}^{j}\frac{1}{\gamma_{s+1,k}^2}m,
\end{aligned} \tag{36}
$$

where $\widehat{\sigma^2}$, $\widehat{\mu^2}$ *and* $\widehat{\sigma\mu}$ *are as presented in Equations (28), (30), and (31), respectively.*

Proof The predictor of $X_{s+1}^{(j)}$ is given by the mean of $X_{s+1}^{(j)}$, given $\tilde{X} = \tilde{x}$. Thus, we have

$$\begin{aligned}
\hat{X}_{s+1}^{(j)} &= E(X_{s+1}^{(j)}|\tilde{X}) = \int_N^{\infty} tf^{X_{s+1}^{(j)}|\tilde{X}}(t|\tilde{x})\,\mathrm{d}t \\
&= \int_N^{M_0} tf_1^{X_{s+1}^{(j)}|\tilde{X}}(t|\tilde{x})\,\mathrm{d}t + \int_{M_0}^{\infty} tf_2^{X_{s+1}^{(j)}|\tilde{X}}(t|\tilde{x})\,\mathrm{d}t \\
&= C^{-1}\Gamma(Q+b)\left(\prod_{k=1}^{j}\gamma_{s+1,k}\right)\sum_{k=1}^{j}a_k(j)\sum_{\tilde{k}\in\tilde{K}}\frac{\Psi(\tilde{k})}{c+\gamma_{s+1,k}+\sum_{i=1}^{s}\gamma_{ik_{i1}}} \\
&\quad\times\left\{\int_N^{M_0} t([H_{\tilde{k}}(t)]^{-(Q+b)} - [G_{s+1,\tilde{k}}^{k}(N,t)]^{-(Q+b)})\,\mathrm{d}t\right. \\
&\quad\left.+\int_{M_0}^{\infty} t([G_{s+1,\tilde{k}}^{k}(M_0,t)]^{-(Q+b)} - [G_{s+1,\tilde{k}}^{k}(N,t)]^{-(Q+b)})\,\mathrm{d}t\right\}.
\end{aligned}$$

Upon using integration by parts, $\hat{X}_{s+1}^{(j)}$ is obtained as

$$\begin{aligned}
\hat{X}_{s+1}^{(j)} &= C^{-1}\Gamma(Q+b-1)\left(\prod_{k=1}^{j}\gamma_{s+1,k}\right)\sum_{k=1}^{j}a_k(j)\sum_{\tilde{k}\in\tilde{K}}\frac{\Psi(\tilde{k})}{c+\gamma_{s+1,k}+\sum_{i=1}^{s}\gamma_{ik_{i1}}} \\
&\quad\times\left\{\frac{c+\gamma_{s+1,k}+\sum_{i=1}^{s}\gamma_{ik_{i1}}}{\gamma_{s+1,k}(c+\sum_{i=1}^{s}\gamma_{ik_{i1}})}(M_0[H_{\tilde{k}}(M_0)]^{-(Q+b-1)} - N[H_{\tilde{k}}(N)]^{-(Q+b-1)})\right. \\
&\quad\left.-\frac{1}{Q+b-2}\frac{\gamma_{s+1,k}^2-(c+\sum_{i=1}^{s}\gamma_{ik_{i1}})^2}{\gamma_{s+1,k}^2(c+\sum_{i=1}^{s}\gamma_{ik_{i1}})^2}([H_{\tilde{k}}(M_0)]^{-(Q+b-2)} - [H_{\tilde{k}}(N)]^{-(Q+b-2)})\right\} \\
&= C^{-1}\Gamma(Q+b-1)\left(\prod_{k=1}^{j}\gamma_{s+1,k}\right)\sum_{k=1}^{j}a_k(j)\sum_{\tilde{k}\in\tilde{K}}\Psi(\tilde{k}) \\
&\quad\times\left\{\frac{1}{\gamma_{s+1,k}(c+\sum_{i=1}^{s}\gamma_{ik_{i1}})}(M_0[H_{\tilde{k}}(M_0)]^{-(Q+b-1)} - N[H_{\tilde{k}}(N)]^{-(Q+b-1)})\right. \\
&\quad-\frac{1}{Q+b-2}\frac{1}{\gamma_{s+1,k}(c+\sum_{i=1}^{s}\gamma_{ik_{i1}})^2}([H_{\tilde{k}}(M_0)]^{-(Q+b-2)} - [H_{\tilde{k}}(N)]^{-(Q+b-2)}) \\
&\quad\left.+\frac{1}{Q+b-2}\frac{1}{\gamma_{s+1,k}^2(c+\sum_{i=1}^{s}\gamma_{ik_{i1}})}([H_{\tilde{k}}(M_0)]^{-(Q+b-2)} - [H_{\tilde{k}}(N)]^{-(Q+b-2)})\right\}. \qquad (37)
\end{aligned}$$

Now by using Equations (16) and (17), the expression in Equation (37) simplifies to

$$\hat{X}_{s+1}^{(j)} = \hat{\mu} + \hat{\sigma}\sum_{k=1}^{j}\frac{1}{\gamma_{s+1,k}}.$$

Similarly, the posterior second moment of $X_{s+1}^{(j)}$ is given by

$$E({X_{s+1}^{(j)}}^2|\tilde{X}) = \widehat{\mu^2} + 2\widehat{\sigma\mu}\sum_{k=1}^{j}\frac{1}{\gamma_{s+1,k}} + 2\widehat{\sigma^2}\left(\sum_{k=1}^{j}\frac{1}{\gamma_{s+1,k}^2} + \sum_{\substack{l,k=1\\ l<k}}^{j}\frac{1}{\gamma_{s+1,l}\gamma_{s+1,k}}\right).$$

Thus, the posterior variance of $\hat{X}_{s+1}^{(j)}$ is given by

$$\operatorname{Var}(\hat{X}_{s+1}^{(j)}|\tilde{X}) = (\widehat{\mu^2} - \hat{\mu}^2) + 2(\widehat{\sigma\mu} - \hat{\sigma}\hat{\mu})\sum_{k=1}^{j}\frac{1}{\gamma_{s+1,k}} + (\widehat{\sigma^2} - \hat{\sigma}^2)\left(\sum_{k=1}^{j}\frac{1}{\gamma_{s+1,k}}\right)^2 + \widehat{\sigma^2}\sum_{k=1}^{j}\frac{1}{\gamma_{s+1,k}^2},$$

which completes the proof of the theorem. ■

Remark 4.5 The Bayesian predictive bounds of a two-sided equi-tailed $100(1-\tau)\%$ interval for the jth sequential order statistic in the $(s+1)$th sample, given $\tilde{X} = \tilde{x}$, can be obtained by solving the following two equations:

$$\bar{F}^{X_{s+1}^{(j)}|\tilde{X}}(L|\tilde{x}) = 1 - \frac{\tau}{2} \quad \text{and} \quad \bar{F}^{X_{s+1}^{(j)}|\tilde{X}}(U|\tilde{x}) = \frac{\tau}{2},$$

where $\bar{F}^{X_{s+1}^{(j)}|\tilde{X}}(t|\tilde{x})$ is the predictive survival function of $X_{s+1}^{(j)}$ given in Theorem 4.3, and L and U denote the lower and upper bounds, respectively.

For the HPD method, the following two equations need to be solved:

$$\bar{F}^{X_{s+1}^{(j)}|\tilde{X}}(L_{X_{s+1}^{(j)}}|\tilde{x}) - \bar{F}^{X_{s+1}^{(j)}|\tilde{X}}(U_{X_{s+1}^{(j)}}|\tilde{x}) = 1 - \tau$$

and

$$f^{X_{s+1}^{(j)}|\tilde{X}}(L_{X_{s+1}^{(j)}}|\tilde{x}) = f^{X_{s+1}^{(j)}|\tilde{X}}(U_{X_{s+1}^{(j)}}|\tilde{x}),$$

where $f^{X_{s+1}^{(j)}|\tilde{X}}(x|\tilde{x})$ is the predictive density function of $X_{s+1}^{(j)}$ given in Theorem 4.3, and $L_{X_{s+1}^{(j)}}$ and $U_{X_{s+1}^{(j)}}$ denote the HPD lower and upper bounds, respectively.

Remark 4.6 By choosing $\alpha_{ij} = 1$ for $i = 1, \dots, s+1$ and $j = 1, \dots, n_i$, the Bayesian predictor $\hat{X}_{s+1}^{(j)}$ and its predictive posterior distribution under the model of the usual order statistics from one- and two-parameter exponential distributions are readily obtained. Similarly, the Bayesian predictor $\hat{X}_{*i}^{(r_i)}$ and its predictive posterior distribution under the model of progressively Type-II censored order statistics from one- and two-parameter exponential distributions are readily obtained by setting $\alpha_{ij} = (N_i - j + 1 - \sum_{j-1}^{q=1} R_{iq})/(n_i - j + 1)$ for $i = 1, \dots, s+1$ and $j = 1, \dots, n_i$.

5. Numerical examples

5.1. *Numerical example 1*

To illustrate the inferential procedures developed in Section 3, we consider $s = 4$ simulated samples from a sequential 2-out-of-5 system based on components following the one-parameter

Table 1. Bayes estimates and the posterior standard errors for the scale parameter of the one-parameter exponential distribution.

Prior	$\hat{\sigma}$	$\mathrm{SE}(\hat{\sigma})$
Jeffreys' prior	26.966	7.479
Improper prior 1	29.040	8.383
Improper prior 2	25.168	6.727
Informative prior 1	20.807	2.135
Informative prior 2	20.519	1.687

exponential distribution with $\sigma = 20$:

$$x_1^{(1)} = 1.940, \quad x_1^{(2)} = --, \quad x_1^{(3)} = 6.216, \quad x_1^{(4)} = 12.060, \quad x_1^{(5)} = --,$$

$$x_2^{(1)} = 3.815, \quad x_2^{(2)} = 5.125, \quad x_2^{(3)} = 6.280, \quad x_2^{(4)} = 17.345, \quad x_2^{(5)} = --,$$

$$x_3^{(1)} = 5.566, \quad x_3^{(2)} = 5.692, \quad x_3^{(3)} = 20.247, \quad x_3^{(4)} = --, \quad x_3^{(5)} = --,$$

$$x_4^{(1)} = --, \quad x_4^{(2)} = 3.354, \quad x_4^{(3)} = 19.134, \quad x_4^{(4)} = 47.88, \quad x_4^{(5)} = --\,.$$

The influence of failures on the remaining components in the system is described by the increasing sequence of parameters $\alpha_1 = 1, \alpha_2 = 1.2, \alpha_3 = 1.4, \alpha_4 = 1.6, \alpha_5 = 1.8$. Notice that in the first and fourth samples, some failure times are missing. Moreover, the failure of the fifth unit in the second sample is not observed, while in the third sample we only observed the first three failures.

These samples are now assumed to have come from the one-parameter exponential distribution, with σ being unknown. Based on the above multiply Type-II censored samples, the results in Section 3 have been used to calculate the Bayes estimates $\hat{\sigma}$ and the posterior standard errors $\mathrm{SE}(\hat{\sigma})$. In addition, suppose we want to predict the failure times from a future sample. In this case, the predictors $\hat{X}_{*5}^{(j)}$, for $j = 1, \ldots, 5$, along with the posterior standard errors $\mathrm{SE}(\hat{X}_{*5}^{(j)})$, and equi-tailed intervals and HPD intervals for $X_{*5}^{(j)}$ are obtained for five different choices of the hyperparameters a and b, namely,

(1) Jeffreys' prior: $\pi^{(\Sigma)}(\sigma) \propto \sigma^{-1}$ $(a = 0, b = 0)$.
(2) Improper prior 1: $\pi^{(\Sigma)}(\sigma) \propto 1$ $(a = 0, b = -1)$.
(3) Improper prior 2: $\pi^{(\Sigma)}(\sigma) \propto \sigma^{-2}$ $(a = 0, b = 1)$.
(4) Informative prior 1: $E(\Sigma) = 20$ and $\mathrm{Var}(\Sigma) = 5$ $(a = 1620, b = 82)$.
(5) Informative prior 2: $E(\Sigma) = 20$ and $\mathrm{Var}(\Sigma) = 3$ $(a = 8060/3, b = 406/3)$.

The results so obtained are presented in Tables 1 and 2, and from the results in these tables, the following points can be observed:

(1) A comparison of the results for the informative priors with the corresponding ones for the non-informative priors reveals that the former produces more precise results, as we would expect.
(2) The results obtained based on Improper prior 2 are the best among those obtained based on non-informative priors.
(3) The results obtained based on Informative prior 2 (with $\mathrm{Var}(\Sigma) = 3$) are more precise than the corresponding ones based on Informative prior 1 (with $\mathrm{Var}(\Sigma) = 5$), as we would expect.
(4) Table 2 shows that the HPD prediction interval is more precise than the corresponding equi-tailed interval for all cases considered.
(5) The predictive density functions of $X_{*5}^{(j)}$ are highly right skewed and consequently the upper bounds of equi-tailed prediction intervals become quite large as compared to those of HPD intervals; thus, equi-tailed prediction intervals become less precise.

Table 2. 95% Bayesian prediction bounds for $X_{*5}^{(j)}$ $(j = 1, \dots, 5)$ from the one-parameter exponential distribution.

				Equi-tailed interval			HPD interval		
j	Prior	$\hat{X}_{*5}^{(j)}$	$\mathrm{SE}(\hat{X}_{*5}^{(j)})$	$L_{X_{*5}^{(j)}}$	$U_{X_{*5}^{(j)}}$	Width	$L_{X_{*5}^{(j)}}$	$U_{X_{*5}^{(j)}}$	Width
1	Jeffreys' prior	5.393	5.793	0.128	21.051	20.923	0.000	16.691	16.691
	Improper prior 1	5.808	6.273	0.137	22.762	22.625	0.000	18.015	18.015
	Improper prior 2	5.034	5.381	0.120	19.578	19.458	0.000	15.547	15.547
	Informative prior 1	4.161	4.205	0.104	15.486	15.382	0.000	13.354	13.354
	Informative prior 2	4.104	4.131	0.103	15.224	15.121	0.000	12.335	12.335
2	Jeffreys' prior	11.011	8.639	1.214	33.404	32.190	0.155	27.722	27.567
	Improper prior 1	11.858	9.376	1.299	36.191	34.892	0.162	29.969	29.807
	Improper prior 2	10.277	8.009	1.140	31.013	29.873	0.145	25.786	25.641
	Informative prior 1	8.496	6.103	1.014	23.989	22.975	0.169	20.421	20.252
	Informative prior 2	8.379	5.985	1.005	23.546	22.541	0.170	20.077	19.907
3	Jeffreys' prior	17.431	11.536	3.204	46.725	43.521	1.260	39.874	38.614
	Improper prior 1	18.772	12.544	3.423	50.690	47.267	1.326	43.145	41.819
	Improper prior 2	16.269	10.676	3.012	43.331	40.319	1.200	37.061	35.861
	Informative prior 1	13.450	7.950	2.719	33.007	30.288	1.283	29.053	27.770
	Informative prior 2	13.264	7.782	2.698	32.360	29.662	1.288	28.542	27.254
4	Jeffreys' prior	25.858	15.415	6.127	64.612	58.485	3.282	56.005	52.723
	Improper prior 1	27.847	16.788	6.534	70.154	63.620	3.460	60.637	57.177
	Improper prior 2	24.134	14.246	5.760	59.872	54.112	3.122	52.025	48.903
	Informative prior 1	19.953	10.402	5.262	45.133	39.871	3.281	40.475	37.194
	Informative prior 2	19.676	10.168	5.226	44.213	38.987	3.291	39.740	36.449
5	Jeffreys' prior	40.839	23.583	10.866	100.355	89.489	6.445	87.027	80.582
	Improper prior 1	43.981	25.698	11.589	108.971	97.382	6.802	94.240	87.438
	Improper prior 2	38.116	21.785	10.228	92.987	82.759	6.124	80.832	74.708
	Informative prior 1	31.513	15.796	9.425	70.077	60.652	6.346	62.773	56.427
	Informative prior 2	31.076	15.431	9.367	68.651	59.284	6.359	61.624	55.265

5.2. *Numerical example 2*

To illustrate the inferential procedures developed in Section 4, we consider $s = 5$ simulated samples from a sequential 2-out-of-5 system based on components following the two-parameter exponential distribution with $\sigma = 20$ and $\mu = 10$:

$$x_1^{(1)} = --, \quad x_1^{(2)} = 16.601, \quad x_1^{(3)} = 20.426, \quad x_1^{(4)} = 25.408, \quad x_1^{(5)} = --,$$
$$x_2^{(1)} = 10.980, \quad x_2^{(2)} = 11.150, \quad x_2^{(3)} = 22.861, \quad x_2^{(4)} = 23.902, \quad x_2^{(5)} = --,$$
$$x_3^{(1)} = --, \quad x_3^{(2)} = 12.378, \quad x_3^{(3)} = --, \quad x_3^{(4)} = 18.140, \quad x_3^{(5)} = --,$$
$$x_4^{(1)} = 10.527, \quad x_4^{(2)} = 20.866, \quad x_4^{(3)} = 35.856, \quad x_4^{(4)} = 37.913, \quad x_4^{(5)} = --,$$
$$x_5^{(1)} = 10.714, \quad x_5^{(2)} = 13.822, \quad x_5^{(3)} = 20.945, \quad x_5^{(4)} = --, \quad x_5^{(5)} = --.$$

The influence of failures on the remaining components in the system is described by the increasing sequence of parameters $\alpha_1 = 1, \alpha_2 = 1.2, \alpha_3 = 1.4, \alpha_4 = 1.6, \alpha_5 = 1.8$. Notice that in the first and third samples, some failure times are missing. Moreover, the failure of the fifth unit in the second and fourth samples is not observed, while in the last sample, we only observed the first three failures.

These samples are now assumed to have come from the two-parameter exponential distribution, with both parameters σ and μ being unknown. Based on the above multiply Type-II censored samples, the Bayes estimates and the posterior standard errors for the unknown parameters σ and μ are determined. In addition, suppose we want to predict the failure times from a future

Table 3. Bayes estimates and the posterior standard errors for the parameters of the two-parameter exponential distribution.

Prior	$\hat{\sigma}$	SE($\hat{\sigma}$)	$\hat{\mu}$	SE($\hat{\mu}$)
Jeffreys' prior	18.910	4.866	9.404	1.146
Improper prior 1	20.162	5.378	9.308	1.245
Improper prior 2	17.806	4.433	9.487	1.060
Informative prior 1	18.627	3.078	9.986	0.543
Informative prior 2	18.604	3.140	9.957	0.572

Table 4. 95% Bayesian prediction bounds for $X_{*6}^{(j)}$ ($j = 1, \ldots, 5$) from the two-parameter exponential distribution.

				Equi-tailed interval			HPD interval		
j	Prior	$\hat{X}_{*6}^{(j)}$	SE($\hat{X}_{*6}^{(j)}$)	$L_{X_{*6}^{(j)}}$	$U_{X_{*6}^{(j)}}$	Width	$L_{X_{*6}^{(j)}}$	$U_{X_{*6}^{(j)}}$	Width
1	Jeffreys' prior	13.186	4.097	8.163	23.987	15.824	7.207	22.052	14.845
	Improper prior 1	13.341	4.384	7.956	24.891	16.935	6.931	22.825	15.894
	Improper prior 2	13.048	3.846	8.341	23.192	14.851	7.446	21.372	13.926
	Informative prior 1	13.712	3.847	9.674	24.025	14.351	8.917	21.797	12.880
	Informative prior 2	13.678	3.849	9.603	23.990	14.387	8.831	21.785	12.954
2	Jeffreys' prior	17.126	5.965	9.860	32.405	22.545	8.683	29.517	20.834
	Improper prior 1	17.541	6.391	9.783	33.927	24.144	9.771	33.866	24.095
	Improper prior 2	16.757	5.593	9.926	31.073	21.147	9.062	28.929	19.867
	Informative prior 1	17.592	5.597	10.797	31.864	21.067	10.195	29.374	19.179
	Informative prior 2	17.553	5.600	10.756	31.834	21.078	10.172	29.436	19.264
3	Jeffreys' prior	21.628	7.892	11.568	41.502	29.934	10.144	37.520	27.376
	Improper prior 1	22.342	8.469	11.608	43.702	32.094	10.061	39.370	29.309
	Improper prior 2	20.997	7.390	11.530	39.577	28.047	9.581	35.331	25.750
	Informative prior 1	22.027	7.332	12.324	40.227	27.903	11.024	36.528	25.504
	Informative prior 2	21.983	7.338	12.280	40.207	27.927	10.985	36.525	25.540
4	Jeffreys' prior	27.537	10.501	13.772	53.760	39.988	11.818	48.369	36.551
	Improper prior 1	28.643	11.284	13.948	56.880	42.932	11.825	51.210	39.385
	Improper prior 2	26.562	9.820	13.618	51.038	37.420	11.788	46.044	34.256
	Informative prior 1	27.848	9.649	14.548	51.479	36.931	12.717	46.923	34.206
	Informative prior 2	27.797	9.660	14.505	51.471	36.966	12.669	46.884	34.215
5	Jeffreys' prior	38.042	16.077	17.255	78.429	61.174	14.203	69.718	55.515
	Improper prior 1	39.843	17.291	17.633	83.357	65.724	14.363	73.876	59.513
	Improper prior 2	36.454	15.024	16.917	74.131	57.214	14.057	66.073	52.016
	Informative prior 1	38.197	14.689	18.169	74.433	56.264	15.312	67.225	51.913
	Informative prior 2	38.132	14.708	18.110	74.440	56.330	15.255	67.214	51.959

sample. In this case, the predictors $\hat{X}_{*6}^{(j)}$ along with posterior standard errors SE($\hat{X}_{*6}^{(j)}$), and equi-tailed intervals and the HPD intervals for $X_{*6}^{(j)}$ are all obtained for five different choices of the hyperparameters a, b, c, M and N, namely,

(1) Jeffreys' prior: $\pi^{(\Sigma,\Delta)}(\sigma, \mu) \propto \sigma^{-1}$ ($a = 0, b = 0, c = 0, M \to \infty, N \to -\infty$).
(2) Improper prior 1: $\pi^{(\Sigma,\Delta)}(\sigma, \mu) \propto 1$ ($a = 0, b = -1, c = 0, M \to \infty, N \to -\infty$).
(3) Improper prior 2: $\pi^{(\Sigma,\Delta)}(\sigma, \mu) \propto \sigma^{-2}$ ($a = 0, b = 1, c = 0, M \to \infty, N \to -\infty$).
(4) Informative prior 1: $E(\Sigma) = 20$, $\mathrm{Var}(\Sigma) = 22$, $E(\Delta) = 10$ and $\mathrm{Var}(\Delta) = 1.2$ ($a = 595.990, b = 21.182, c = 19.235, M = 11.040, N \to -\infty$).
(5) Informative prior 2: $E(\Sigma) = 20$, $\mathrm{Var}(\Sigma) = 24$, $E(\Delta) = 10$ and $\mathrm{Var}(\Delta) = 1.5$ ($a = 546.153, b = 19.667, c = 17.282, M = 11.157, N \to -\infty$).

The results so obtained are presented in Tables 3 and 4, and from the results in these tables, the following points can be observed:

(1) A comparison of the results for the informative priors with the corresponding ones for the non-informative priors reveals that the former produces more precise results, as we would expect.
(2) The results obtained based on improper prior 2 are the best among those obtained based on non-informative priors.
(3) The results obtained based on the informative prior 1 (with $\mathrm{Var}(\Sigma) = 22$ and $\mathrm{Var}(\Delta) = 1.2$) are more precise than the corresponding ones based on the informative prior 2 (with $\mathrm{Var}(\Sigma) = 22$ and $\mathrm{Var}(\Delta) = 1.5$), as we would expect.
(4) Table 4 shows that the HPD prediction interval is more precise than the corresponding equi-tailed interval for all cases considered.
(5) Here again, the predictive density functions of $X_{*6}^{(j)}$ are highly right skewed and consequently the upper bounds of equi-tailed prediction intervals become quite large as compared to those of HPD intervals; thus, equi-tailed prediction intervals become less precise.

6. Conclusions and discussion

In this paper, Bayesian prediction of future sequential order statistics has been discussed based on multiply Type-II censored samples of sequential order statistics observed from one- and two-parameter exponential distributions. Both point and interval predictions have been developed, and two examples have been presented to illustrate the results. The computational results show that predictions based on an informative prior with less variability are more precise than those based on an informative prior with more variability. Moreover, the HPD prediction intervals are more precise than the equi-tailed prediction intervals.

The results developed here may be extended to other forms of censored samples of sequential order statistics from one- and two-parameter exponential distributions. It may also be of interest to develop corresponding results when the lifetimes are from a Pareto distribution. We are currently working on these problems and hope to report these findings in a future paper.

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