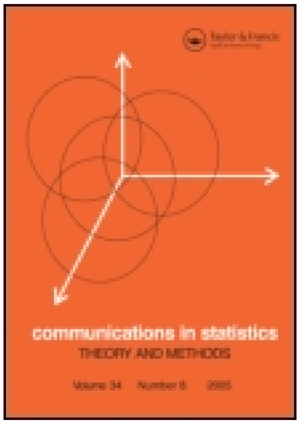

Communications in Statistics - Theory and Methods
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/lsta20

Two-Sample Bayesian Prediction Intervals of Generalized Order Statistics Based on Multiply Type II Censored Data

M. M. Mohie El-Din [a], Y. Abdel-Aty [a] & A. R. Shafay [b]

[a] Department of Mathematics, Al-Azhar University, Cairo, Egypt

[b] Department of Mathematics, Fayoum University, Fayoum, Egypt

Available online: 14 Dec 2011

Communications in Statistics—Theory and Methods, 41: 381–392, 2012

ISSN: 0361-0926 print/1532-415X online
DOI: 10.1080/03610926.2010.505694

Two-Sample Bayesian Prediction Intervals of Generalized Order Statistics Based on Multiply Type II Censored Data

M. M. MOHIE EL-DIN[1], Y. ABDEL-ATY[1], AND A. R. SHAFAY[2]

[1]Department of Mathematics, Al-Azhar University, Cairo, Egypt
[2]Department of Mathematics, Fayoum University, Fayoum, Egypt

In this article, two-sample Bayesian prediction intervals of generalized order statistics (GOS) based on multiply Type II censored data are derived. To illustrate these results, the Pareto, Weibull, and Burr-Type XII distributions are used as examples. Finally, a numerical illustration of the sequential order statistics from the Pareto distribution is presented.

Keywords Burr-Type XII distribution; Generalized order statistics; Multiply Type II censoring; Pareto distribution; Two-sample Bayesian prediction intervals; Weibull distribution.

Mathematics Subject Classification Primary 62G30; Secondary 62F15.

1. Introduction

Kamps (1995) introduced the idea of the generalized order statistics (GOS) to unify several important concepts such as ordinary order statistics (OOS) (Arnold and Balakrishnan, 1989; Arnold et al., 1992; Balakrishnan and Cohen, 1991; David, 1981); sequential order statistics (SOS) (Cramer and Kamps, 1996, 2001); ordering via truncated distributions, censoring schemes, record values (Ahsanullah, 1995; Nevzorov, 1987); and progressive Type II censored sample (Balakrishnan and Aggarwala, 2000; Balakrishnan, 2007). Therefore, all of these models are contained in the model of GOS.

In reliability analysis, experiments must often terminate before all units on test have failed. In such cases, one has failure information only on part of the sample. On all units which have not failed, one has only partial information. Such data are said to be censored data. There are several forms of censored data. One of the most common form of censoring is Type II censoring. In Type II censoring scheme, a total of n units are placed on test, but instead of continuing until all n units have

Received April 13, 2009; Accepted June 25, 2010
Address correspondence to M. M. Mohie El-Din, Faculty of Science, Department of Mathematics, Al-Azhar University, Cairo, Egypt; E-mail: mmmmoheeldin@yahoo.com

failed, the test is terminated at the rth ($1 \leq r \leq n$) failure. A generalization of Type II censoring scheme is multiply Type II censoring scheme. Under this scheme, we observe only j_1th, j_2th, $\ldots$, j_rth failure times $X_{j_1} \leq X_{j_2} \leq \cdots \leq X_{j_r}$ ($1 \leq r \leq n$) and the rest of data is not available.

In many practical problems, one would wish to use previous data to predict a future observation from the same population. One way to do this is to construct an interval which will contain the future observation with a specified probability. This interval is called a prediction interval. Bayesian prediction bounds for future observations have been discussed by several authors, including Dunsmore (1974), Aitchison and Dunsmore (1975), Nigm and Hamdy (1987), Nigm (1988, 1989), AL-Hussaini and Jaheen (1995), AL-Hussaini (1999), and AL-Hussaini and Ahmed (2003a). The Bayesian prediction problem of GOS has been studied by AL-Hussaini and Ahmed (2003b). They got the one-sample Bayesian prediction bounds for the future sth GOS based on a right Type II censored data. Abdel-Aty et al. (2007) obtained one-sample Bayesian prediction bounds for the future sth GOS based on a multiply Type II censored data. AL-Hussaini and Al-Awadhi (2010) obtained two-sample Bayesian prediction bounds for the future sth GOS based on a right Type II censored data.

In this article, we use the general exponential form of the underlying distribution and the general conjugate prior of the prior distribution to derive general procedure for determining the two-sample Bayesian prediction intervals of the future sth GOS when the informative sample is a multiply Type II censored sample. In Sec. 2, we derive the Bayesian predictive survival function and the Bayesian prediction bounds for the future sth GOS. In Sec. 3, we present the results for Pareto, Weibull, and Burr-Type XII distributions as illustrative examples. Finally, in Sec. 4, we present numerical computations for prediction intervals of the future sth sequential order statistics.

Motivated by the fact that the survival function (SF) $\overline{F}(x \mid \theta) = 1 - F(x \mid \theta)$ corresponding to any cumulative distribution function (cdf) $F(x \mid \theta)$, $\theta \in \Theta$ can be written in the form

$$\overline{F}(x \mid \theta) = \exp[-\lambda(x; \theta)], \tag{1.1}$$

where $\lambda(x; \theta) = -\ln \overline{F}(x \mid \theta)$, we shall consider the underling population SF to be given by (1.1). Conditions on $\lambda(x; \theta)$ should be imposed so that $\overline{F}(x \mid \theta)$ is an SF. That is, $\lambda(x; \theta)$ should be a continuous, monotone increasing, differentiable function, $\lambda(x) \to 0$ as $x \to -\infty$ and $\lambda(x) \to \infty$ as $x \to \infty$. The probability density function (pdf) corresponding to (1.1) is

$$f(x \mid \theta) = \lambda'(x; \theta) \exp[-\lambda(x; \theta)]. \tag{1.2}$$

With the appropriate choice of $\lambda(x; \theta)$ (notice that the derivative of $\lambda(x; \theta)$ with respect to x is the hazard rate function), several distributions that are important in life testing (and other areas as well) can be obtained. For example, if $\lambda(x) = -\alpha \ln(\beta/x)$ we obtain the Pareto(α, β) distribution. If $\lambda(x; \theta) = \alpha x^{\beta}$, we obtain the Weibull($\alpha, \beta$) distribution; the Burr-Type XII(α, β) distribution is obtained by setting $\lambda(x) = \alpha \ln(1 + x^{\beta})$; and so on. Appropriate conditions should be imposed on $\lambda(x; \theta)$ to suit the domain on which $\overline{F}(x \mid \theta)$ is defined. For example, if $\overline{F}(x \mid \theta)$ is defined only on the positive half of the real line (as for the Weibull and

Burr-Type XII distributions), then $\lambda(x) \to 0$ as $x \to 0^+$ and $\lambda(x) \to \infty$ as $x \to \infty$. If $\overline{F}(x \mid \theta)$ is defined on (β, ∞) (as in the Pareto distribution), then $\lambda(x) \to 0$ as $x \to \beta^+$ and $\lambda(x) \to \infty$ as $x \to \infty$, and so on. The use of the exponential form of SF (1.1) is more flexible than the direct use of $\overline{F}(x \mid \theta)$ in drawing general conclusions.

Throughout this article we use the following definitions and notation.

Let X be a random variable with survival function $\overline{F}(x \mid \theta)$ and density function $f(x \mid \theta)$, where the parameter $\theta \in \Theta$ may be a real-vector. Corresponding to X we consider n GOS $X_1, X_2, \ldots, X_n$ defined in the sense of Kamps (1995). The variable X_i depend on parameters: $X_i \equiv X_i(i, n, \underline{\mathbf{m}}, k)$, $i = 1, \ldots, n$ where $n \in \mathbb{N}$, $n \geq 2$, $k > 0$, $\underline{\mathbf{m}} = (m_1, m_2, \ldots, m_{n-1}) \in \mathbb{R}^{n-1}$ such that $\gamma_r = k + n - r + M_r \geq 1$, $\forall r \in \{1, 2, \ldots, n-1\}$ with $M_r = \sum_{j=r}^{n-1} m_j$, $\gamma_r - \gamma_{r+1} - 1 = m_r$ and $\gamma_n = k$.

Moreover, we notice:

(1) The constant c_{r-1} are defined by

$$c_{r-1} = \prod_{j=1}^{r} \gamma_j, \quad r = 1, \ldots, n.$$

(2) The constant $c_i^{(r)}(s)$ are defined by

$$c_i^{(r)}(s) = \prod_{\substack{j=r+1 \\ j \neq i}}^{s} \frac{1}{\gamma_j - \gamma_i},$$

where $r + 1 \leq i \leq s \leq n$, $r = 0, 1, \ldots, n$, $\gamma_i \neq \gamma_j \forall i \neq j$, and $c_i(s) = c_i^{(0)}(s)$.

(3)

$$c = \prod_{i=1}^{r} \prod_{k=j_{i-1}+1}^{j_i} \gamma_k, \quad r = 1, \ldots, n.$$

The joint density function of GOS $X_1, X_2, \ldots, X_n$ is given by

$$f_{\underline{\mathbf{X}}}(\underline{\mathbf{x}}) = k \left(\prod_{j=1}^{n-1} \gamma_j \right) \left(\prod_{i=1}^{n-1} \overline{F}(x_i)^{m_i} f(x_i) \right) \overline{F}(x_n)^{k-1} f(x_n), \tag{1.3}$$

in the cone $F^{-1}(0) \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq F^{-1}(1)$ and $\underline{\mathbf{x}} = (x_1, \ldots, x_n)$.

The joint density function of the first GOS $X_1, X_2, \ldots, X_r$ is given by

$$\begin{aligned} f_{\underline{\mathbf{X}}}(\underline{\mathbf{x}}) &= c_{r-1} \left(\prod_{i=1}^{r-1} \overline{F}(x_i)^{m_i} f(x_i) \right) \overline{F}(x_r)^{\gamma_r - 1} f(x_r) \\ &= c_{r-1} \prod_{i=1}^{r} \left(\frac{\overline{F}(x_i)}{\overline{F}(x_{i-1})} \right)^{\gamma_i} \frac{f(x_i)}{\overline{F}(x_i)}, \end{aligned} \tag{1.4}$$

where $\underline{\mathbf{x}} = (x_1, \ldots, x_r)$, $r \leq n$ and $\overline{F}(x_0) = \overline{F}(-\infty) = 1$.

The joint density function of multiply Type II censored GOS $X_{j_1}, X_{j_2}, \ldots, X_{j_r}$ is given by

$$f_{\underline{\mathbf{X}}}(\underline{\mathbf{x}}) = c \prod_{i=1}^{r} \sum_{k_i=j_{i-1}+1}^{j_i} c_{k_i}^{(j_i-1)}(j_i) \left(\frac{\overline{F}(x_{j_i})}{\overline{F}(x_{j_{i-1}})} \right)^{\gamma_{k_i}} \frac{f(x_{j_i})}{\overline{F}(x_{j_i})}, \tag{1.5}$$

where $\underline{\mathbf{x}} = (x_{j_1}, \ldots, x_{j_r})$, $1 \leq j_1 < \cdots < j_r \leq n$, and $j_0 = 0$.

Upon using (1.1) and (1.2) in (1.5), we obtain the likelihood function of $X_{j_1}, X_{j_2}, \ldots, X_{j_r}$ as follows:

$$L(\theta; \underline{\mathbf{x}}) = c \prod_{i=1}^{r} \sum_{k_i=j_{i-1}+1}^{j_i} c_{k_i}^{(j_i-1)}(j_i) \lambda'(x_{j_i}) \exp[-\gamma_{k_i}\{\lambda(x_{j_i}) - \lambda(x_{j_{i-1}})\}]. \tag{1.6}$$

Let $Y_1, Y_2, \ldots, Y_N$ be future independent sample of GOS with size N from the same population. Then the marginal density function of the sth GOS Y_s is given by

$$f_{Y_s}(y_s) = c_{s-1} f(y_s) \sum_{w=1}^{s} c_w(s) \overline{F}(y_s)^{\tilde{\gamma}_w - 1}, \tag{1.7}$$

where $1 \leq s \leq N$, $\tilde{\gamma}_w = k + N - w + M_w \geq 1, \forall w \in \{1, 2, \ldots, N-1\}$ with $M_w = \sum_{j=w}^{N-1} m_j$, $\tilde{\gamma}_w - \tilde{\gamma}_{w+1} - 1 = m_w$ and $\tilde{\gamma}_N = k$.

Upon substituting (1.1) and (1.2) in (1.7), we obtain

$$f_{Y_s}(y_s \mid \theta) = c_{s-1} \sum_{w=1}^{s} c_w(s) \lambda'(y_s) \exp[-\widetilde{\gamma}_w \lambda(y_s)]. \tag{1.8}$$

2. Two-Sample Bayesian Prediction Intervals

For the Bayesian prediction setup we need a suitable prior parameter distribution. We consider here a general conjugate prior, suggested by AL-Hussaini (1999), that includes most priors used in literature given by

$$\pi(\theta; \delta) \propto C(\theta; \delta) \exp[-D(\theta; \delta)], \tag{2.1}$$

where $\theta \in \Theta$ is the vector of parameters of the distribution under consideration and δ is the vector of prior parameters.

Lemma 2.1. *From* (1.6) *and* (2.1), *we obtain the posterior density function as follows*:

$$\pi^*(\theta \mid \underline{\mathbf{x}}) = I^{-1} \prod_{i=1}^{r} \sum_{k_i=j_{i-1}+1}^{j_i} c_{k_i}^{(j_i-1)}(j_i) \eta_i(\theta; \underline{\mathbf{x}}) \exp[-\zeta_{k_i}(\theta; \underline{\mathbf{x}})], \tag{2.2}$$

where

$$\eta_i(\theta; \underline{\mathbf{x}}) = \lambda'(x_{j_i})[C(\theta; \delta)]^{1/r},$$

$$\zeta_{k_i}(\theta; \underline{\mathbf{x}}) = \gamma_{k_i}\{\lambda(x_{j_i}) - \lambda(x_{j_{i-1}})\} + \frac{D(\theta; \delta)}{r},$$

and

$$I = \int_{\theta\in\Theta} \prod_{i=1}^{r} \sum_{k_i=j_{i-1}+1}^{j_i} c_{k_i}^{(j_{i-1})}(j_i)\eta_i(\theta; \underline{\mathbf{x}}) \exp[-\zeta_{k_i}(\theta; \underline{\mathbf{x}})]d\theta.$$

Theorem 2.1. *Suppose* $\underline{\mathbf{X}} = (X_{j_1}, X_{j_2}, \ldots, X_{j_r})$ *is the informative multiply Type II censored sample from GOS. Let* $Y_1, Y_2, \ldots, Y_N$ *be a future independent sample of GOS with size N from the same population. Then the* $100\tau\%$ *Bayesian prediction bounds for* Y_s *are obtained by solving the following equations with respect to* t:

$$c_{s-1}I^{-1} \int_{\theta\in\Theta} \left\{ \prod_{i=1}^{r} \sum_{k_i=j_{i-1}+1}^{j_i} c_{k_i}^{(j_{i-1})}(j_i)\eta_i(\theta; \underline{\mathbf{x}}) \exp[-\zeta_{k_i}(\theta; \underline{\mathbf{x}})] \right\} \times \sum_{w=1}^{s} \frac{c_w(s)}{\tilde{\gamma}_w} \exp[-\tilde{\gamma}_w \lambda(t)]d\theta = \begin{cases} (1+\tau)/2 \\ (1-\tau)/2, \end{cases} \tag{2.3}$$

where $1 \le s \le N$.

Proof. Since the Bayesian predictive density function is given by

$$h(y_s \mid \underline{\mathbf{x}}) = \int_{\theta\in\Theta} f_{Y_s}(y_s \mid \theta)\pi^*(\theta \mid \underline{\mathbf{x}})d\theta, \tag{2.4}$$

upon using (1.8) and (2.2) in (2.4), we obtain the predictive density function as

$$h(y_s \mid \underline{\mathbf{x}}) = c_{s-1}I^{-1} \int_{\theta\in\Theta} \left\{ \prod_{i=1}^{r} \sum_{k_i=j_{i-1}+1}^{j_i} c_{k_i}^{(j_{i-1})}(j_i)\eta_i(\theta; \underline{\mathbf{x}}) \exp[-\zeta_{k_i}(\theta; \underline{\mathbf{x}})] \right\} \times \sum_{w=1}^{s} c_w(s)\lambda'(y_s) \exp[-\tilde{\gamma}_w \lambda(y_s)]d\theta. \tag{2.5}$$

Since the predictive survival function of Y_s is given by

$$P(Y_s > t \mid \underline{\mathbf{x}}) = \int_t^{\infty} h(y_s \mid \underline{\mathbf{x}})dy_s,$$

we obtain

$$\begin{aligned} P(Y_s > t \mid \underline{\mathbf{x}}) &= c_{s-1}I^{-1} \int_t^{\infty} \int_{\theta\in\Theta} \left\{ \prod_{i=1}^{r} \sum_{k_i=j_{i-1}+1}^{j_i} c_{k_i}^{(j_{i-1})}(j_i)\eta_i(\theta; \underline{\mathbf{x}}) \exp[-\zeta_{k_i}(\theta; \underline{\mathbf{x}})] \right\} \\ &\quad \times \sum_{w=1}^{s} c_w(s)\lambda'(y_s) \exp[-\tilde{\gamma}_w \lambda(y_s)]d\theta dy_s \\ &= c_{s-1}I^{-1} \int_{\theta\in\Theta} \left\{ \prod_{i=1}^{r} \sum_{k_i=j_{i-1}+1}^{j_i} c_{k_i}^{(j_{i-1})}(j_i)\eta_i(\theta; \underline{\mathbf{x}}) \exp[-\zeta_{k_i}(\theta; \underline{\mathbf{x}})] \right\} \\ &\quad \times \sum_{w=1}^{s} \frac{c_w(s)}{\tilde{\gamma}_w} \exp[-\tilde{\gamma}_w \lambda(t)]d\theta. \end{aligned} \tag{2.6}$$

Consequently, the $100\tau\%$ Bayesian prediction bounds for Y_s based on the informative sample are obtained by solving the following equations with respect to t:

$$P(Y_s > t \mid \underline{\mathbf{x}}) = \frac{1+\tau}{2}, \quad P(Y_s > t \mid \underline{\mathbf{x}}) = \frac{1-\tau}{2}.$$

Remark 2.1. If $\gamma_i = (n-i+1)\rho$, $\rho \in R^+$, and $i = 1, \ldots, n$, then Theorem 2.1 gives the predictive bounds for the future sample based on the case of sequential order statistics (SOS). If $\rho = 1$, we will obtain the results based on ordinary order statistics (OOS).

Remark 2.2. The Bayesian predictive survival function of Y_s may be written in the following form:

$$P(Y_s > t \mid \underline{\mathbf{x}}) = c_{s-1} I^{-1} \sum_{k_1=j_0+1}^{j_1} \cdots \sum_{k_r=j_{r-1}+1}^{j_r} \sum_{w=1}^{s} \frac{c_w(s) \prod_{i=1}^{r} c_{k_i}^{(j_{i-1})}(j_i)}{\tilde{\gamma}_w} \times \int_{\theta} \eta(\theta; \underline{\mathbf{x}}) \exp[-\tilde{\gamma}_w \lambda(t) - \zeta(\theta; \underline{\mathbf{x}})] d\theta, \tag{2.7}$$

where $1 \le s \le N$,

$$\eta(\theta; \underline{\mathbf{x}}) = C(\theta; \delta) \prod_{i=1}^{r} \lambda'(x_{j_i}),$$

$$\zeta(\theta; \underline{\mathbf{x}}) = \sum_{i=1}^{r} \gamma_{k_i} \{\lambda(x_{j_i}) - \lambda(x_{j_{i-1}})\} + D(\theta; \delta),$$

and

$$I = \sum_{k_1=j_0+1}^{j_1} \cdots \sum_{k_r=j_{r-1}+1}^{j_r} \left(\prod_{i=1}^{r} c_{k_i}^{(j_{i-1})}(j_i) \right) \int_{\theta \in \Theta} \eta(\theta; \underline{\mathbf{x}}) \exp[-\zeta(\theta; \underline{\mathbf{x}})] d\theta.$$

Corollary 2.1. *In the case when the observed sample is right Type II censored (i.e., $j_i = i$, $1 \le i \le r$), then the predictive survival function of Y_s becomes*

$$P(Y_s > t \mid \underline{\mathbf{x}}) = c_{s-1} I^{-1} \sum_{w=1}^{s} \frac{c_w(s)}{\tilde{\gamma}_w} \int_{\theta \in \Theta} \eta_1(\theta; \underline{\mathbf{x}})) \exp[-\tilde{\gamma}_w \lambda(t) - \zeta_1(\theta; \underline{\mathbf{x}})] d\theta, \tag{2.8}$$

where $1 \le s \le N$,

$$\eta_1(\theta; \underline{\mathbf{x}}) = C(\theta; \delta) \prod_{i=1}^{r} \lambda'(x_i),$$

$$\zeta_1(\theta; \underline{\mathbf{x}}) = \sum_{i=1}^{r} \gamma_i \{\lambda(x_i) - \lambda(x_{i-1})\} + D(\theta; \delta),$$

and

$$I = \int_{\theta \in \Theta} \eta_1(\theta; \underline{\mathbf{x}})) \exp[-\zeta_1(\theta; \underline{\mathbf{x}})] d\theta.$$

Corollary 2.2. *In the case when the observed sample is right Type II censored, if* $m_1 = \cdots = m_{n-1} = m \neq -1$, *then the predictive survival function of* Y_s *reduces to expression* (16) *of AL-Hussaini and Al-Awadhi* (2010).

Corollary 2.3. *In the case when the observed sample is right Type II censored, if* $k = 1$ *and* $m_1 = \cdots = m_{n-1} = 0$ (*ordinary order statistics*), *then the predictive survival function of* Y_s *reduces to expression* (26) *of AL-Hussaini* (1999).

3. Examples

In this section, we discuss the Bayesian prediction of observations from Pareto(α, β), Weibull(α, β), and Burr-Type XII(α, β) as illustrative examples. We obtain these Bayesian prediction intervals when α and β both are unknown.

3.1. *Pareto(α, β) Model*

The distribution function of this model is given by

$$F(x \mid \theta) = 1 - \exp[-\alpha \ln(x/\beta)], \quad x > \beta, \tag{3.1}$$

where $\theta = (\alpha, \beta)$, $\alpha > 0$, and $\beta > 0$. Hence,

$$\lambda(x) = \alpha \ln(x/\beta) \quad \text{and} \quad \lambda'(x) = \frac{\alpha}{x}. \tag{3.2}$$

Suppose both α and β are unknown. Then, we will use the prior density function suggested by Lwin (1972) and generalized by Arnold and Press (1989). They proposed the parameter α to be distributed as gamma(c,d) and the conditional distribution of β, given α, follows the power function (αa, b) distribution. Then, the joint prior density function of α and β is given by

$$\pi(\alpha, \beta) = \pi_1(\alpha)\pi_2(\beta \mid \alpha),$$

where

$$\pi_1(\alpha) = \frac{d^c}{\Gamma(c)} \alpha^{c-1} \exp[-d\alpha], \tag{3.3}$$

and

$$\pi_2(\beta \mid \alpha) = \alpha a \beta^{\alpha a - 1} b^{-\alpha a}. \tag{3.4}$$

Then, we find

$$C(\theta; \delta) = \alpha^c \beta^{-1} \quad \text{and} \quad D(\theta; \delta) = \alpha\left[d + a \ln \frac{b}{\beta}\right], \tag{3.5}$$

where $\delta = (a, b, c, d)$, $a, b, c, d > 0$, $\theta = (\alpha, \beta)$, $\alpha > 0$, and $0 < \beta < b$.

The predictive survival function of Y_s in this case is given by

$$P(Y_s > t \mid \underline{\mathbf{x}}) = c_{s-1} I^{-1} \sum_{k_1=j_0+1}^{j_1} \cdots \sum_{k_r=j_{r-1}+1}^{j_r} \sum_{w=1}^{s} c_{k_1}^{(j_0)}(j_1) \cdots c_{k_r}^{(j_{r-1})}(j_r) \frac{c_w(s)}{\tilde{\gamma}_w}$$
$$\times \int_0^L \int_0^\infty \alpha^{(r+c)} \beta^{-1} \left(\prod_{i=1}^{r} \frac{1}{x_{j_i}} \right) \exp\Bigg[-\alpha \Bigg\{ \sum_{i=2}^{r} \gamma_{k_i} \ln\left(\frac{x_{j_i}}{x_{j_{i-1}}} \right)$$
$$+ \gamma_{k_1} \ln\left(\frac{x_{j_1}}{\beta} \right) + a \ln \frac{b}{\beta} + d + \tilde{\gamma}_w \ln\left(\frac{t}{\beta} \right) \Bigg\} \Bigg] d\alpha \, d\beta$$
$$= c_{s-1} I^{*-1} \sum_{k_1=j_0+1}^{j_1} \cdots \sum_{k_r=j_{r-1}+1}^{j_r} \sum_{w=1}^{s} c_{k_1}^{(j_0)}(j_1) \cdots c_{k_r}^{(j_{r-1})}(j_r) \frac{c_w(s)}{\tilde{\gamma}_w (a + \gamma_{k_1} + \tilde{\gamma}_w)}$$
$$\times \left(\sum_{i=2}^{r} \gamma_{k_i} \ln\left(\frac{x_{j_i}}{x_{j_{i-1}}} \right) + \gamma_{k_1} \ln\left(\frac{x_{j_1}}{L} \right) + a \ln\left(\frac{b}{L} \right) \right.$$
$$\left. + \tilde{\gamma}_w \ln\left(\frac{t}{L} \right) + d \right)^{-(r+c)}, \tag{3.6}$$

where

$$I^* = \sum_{k_1=j_0+1}^{j_1} \cdots \sum_{k_r=j_{r-1}+1}^{j_r} \frac{c_{k_1}^{(j_0)}(j_1) \cdots c_{k_r}^{(j_{r-1})}(j_r)}{(a + \gamma_{k_1})}$$
$$\times \left(\sum_{i=2}^{r} \gamma_{k_i} \ln\left(\frac{x_{j_i}}{x_{j_{i-1}}} \right) + \gamma_{k_1} \ln\left(\frac{x_{j_1}}{L} \right) + a \ln\left(\frac{b}{L} \right) + d \right)^{-(r+c)},$$

and $L = \min(x_{j_1}, b)$.

3.2. *Weibull(α, β) Model*

The distribution function of this model is given by

$$F(x \mid \theta) = 1 - \exp[-\alpha x^{\beta})], \quad x > 0, \tag{3.7}$$

where $\theta = (\alpha, \beta)$, $\alpha > 0$, and $\beta > 0$. So,

$$\lambda(x) = \alpha x^{\beta} \quad \text{and} \quad \lambda'(x) = \alpha \beta x^{\beta - 1}. \tag{3.8}$$

Suppose both α and β are unknown. In this case, we will use the prior density function suggested by Nigm (1989), given by

$$\pi(\alpha, \beta) = \pi_1(\alpha) \pi_2(\beta \mid \alpha),$$

where both π_1 and π_2 are gamma distributions. Then, the joint prior density function of α and β is

$$\pi(\alpha, \beta) \propto \beta^{2a} \alpha^{a + \frac{c}{\phi(\beta)}} \exp[-(b\beta + d\alpha\psi(\beta))],$$

where $a > -1$, $b, c, d > 0$, $\alpha, \beta > 0$, $\phi(\beta)$, and $\psi(\beta)$ are increasing functions of β. So,

$$C(\theta; \delta) = \beta^{2a}\alpha^{a+\frac{c}{\phi(\beta)}} \quad \text{and} \quad D(\theta; \delta) = b\beta + d\alpha\psi(\beta) \tag{3.9}$$

where $\delta = (a, b, c, d)$ and $\theta = (\alpha, \beta)$.

The predictive survival function of Y_s in this case is given by

$$\begin{aligned} P(Y_s > t \mid \underline{\mathbf{x}}) &= c_{s-1} I^{-1} \sum_{k_1=j_0+1}^{j_1} \cdots \sum_{k_r=j_{r-1}+1}^{j_r} \sum_{w=1}^{s} c_{k_1}^{(j_0)}(j_1) \cdots c_{k_r}^{(j_{r-1})}(j_r) \frac{c_w(s)}{\tilde{\gamma}_w} \\ &\quad \times \int_0^\infty \int_0^\infty \alpha^{(r+a+\frac{c}{\phi(\beta)})} \beta^{r+2a} \left(\prod_{i=1}^{r} x_{j_i}^{\beta-1} \right) \exp[-b\beta] \\ &\quad \times \exp\left[-\alpha \left\{ \sum_{i=1}^{r} \gamma_{k_i} (x_{j_i}^{\beta} - x_{j_{i-1}}^{\beta}) + d\psi(\beta) + \tilde{\gamma}_w t^{\beta} \right\} \right] d\alpha\, d\beta \\ &= c_{s-1} I^{-1} \sum_{k_1=j_0+1}^{j_1} \cdots \sum_{k_r=j_{r-1}+1}^{j_r} \sum_{w=1}^{s} c_{k_1}^{(j_0)}(j_1) \cdots c_{k_r}^{(j_{r-1})}(j_r) \frac{c_w(s)}{\tilde{\gamma}_w} \\ &\quad \times \int_0^\infty \Gamma(\delta) \beta^{r+2a} \left(\prod_{i=1}^{r} x_{j_i}^{\beta-1} \right) \exp[-b\beta] \\ &\quad \times \left(\sum_{i=1}^{r} \gamma_{k_i} (x_{j_i}^{\beta} - x_{j_{i-1}}^{\beta}) + d\psi(\beta) + \tilde{\gamma}_w t^{\beta} \right)^{-\delta} d\beta, \end{aligned} \tag{3.10}$$

where $\delta = r + a + \frac{c}{\phi(\beta)} + 1$,

$$\begin{aligned} I &= \sum_{k_1=j_0+1}^{j_1} \cdots \sum_{k_r=j_{r-1}+1}^{j_r} c_{k_1}^{(j_0)}(j_1) \cdots c_{k_r}^{(j_{r-1})}(j_r) \int_0^\infty \Gamma(\delta) \beta^{r+2a} \left(\prod_{i=1}^{r} x_{j_i}^{\beta-1} \right) \exp[-b\beta] \\ &\quad \times \left(\sum_{i=1}^{r} \gamma_{k_i} \left(x_{j_i}^{\beta} - x_{j_{i-1}}^{\beta}\right) + d\psi(\beta) \right)^{-\delta} d\beta. \end{aligned}$$

3.3. *Burr-Type XII*(α, β) *Model*

The distribution function in this case is given by

$$F(x \mid \theta) = 1 - [1 + x^{\beta}]^{-\alpha} = 1 - \exp[-\alpha \ln(1 + x^{\beta})], \tag{3.11}$$

where $\theta = (\alpha, \beta)$, $\alpha > 0$, $\beta > 0$, and $x > 0$. So,

$$\lambda(x) = \alpha \ln(1 + x^{\beta}) \quad \text{and} \quad \lambda'(x) = \frac{\alpha\beta x^{\beta-1}}{1 + x^{\beta}}. \tag{3.12}$$

Suppose both α and β are unknown. We will use in this case the bivariate prior density function suggested by AL-Hussaini and Jaheen (1992), given by

$$\pi(\alpha, \beta) \propto \beta^{a+d} \alpha^{a} \exp\left[-\frac{\beta}{b} \right] \exp\left[-\alpha \left(\frac{\beta}{d} \right) \right],$$

where $a > -1,\ b, c, d > 0$, and $\alpha, \beta > 0$. So,

$$C(\theta;\delta) = \beta^{a+d}\alpha^{a}\exp\left[-\frac{\beta}{b}\right] \quad\text{and}\quad D(\theta;\delta) = \alpha\left(\frac{\beta}{d}\right), \tag{3.13}$$

where $\delta = (a, b, c, d)$ and $\theta = (\alpha, \beta)$.

The predictive survival function of Y_s in this case is given by

$$\begin{aligned}
P(Y_s > t \mid \underline{\mathbf{x}}) &= c_{s-1} I^{-1} \sum_{k_1=j_0+1}^{j_1} \cdots \sum_{k_r=j_{r-1}+1}^{j_r} \sum_{w=1}^{s} c_{k_1}^{(j_0)}(j_1)\cdots c_{k_r}^{(j_{r-1})}(j_r)\frac{c_w(s)}{\tilde{\gamma}_w} \\
&\quad\times \int_0^\infty\int_0^\infty \alpha^{(r+a)}\beta^{r+a+d}\left(\prod_{i=1}^{r}\frac{x_{j_i}^{\beta-1}}{1+x_{j_i}^{\beta}}\right)\exp\left[-\frac{\beta}{b}\right] \\
&\quad\times \exp\left[-\alpha\left\{\sum_{i=1}^{r}\gamma_{k_i}\ln\left(\frac{1+x_{j_i}^{\beta}}{1+x_{j_{i-1}}^{\beta}}\right) + \frac{\beta}{d} + \tilde{\gamma}_w\ln(1+t^{\beta})\right\}\right] d\alpha\, d\beta \\
&= c_{s-1} I^{*-1} \sum_{k_1=j_0+1}^{j_1} \cdots \sum_{k_r=j_{r-1}+1}^{j_r} c_{k_1}^{(j_0)}(j_1)\cdots c_{k_r}^{(j_{r-1})}(j_r)\frac{c_w(s)}{\tilde{\gamma}_w} \\
&\quad\times \int_0^\infty \beta^{r+a+d}\left(\prod_{i=1}^{r}\frac{x_{j_i}^{\beta-1}}{1+x_{j_i}^{\beta}}\right)\exp\left[-\frac{\beta}{b}\right] \\
&\quad\times \left(\sum_{i=1}^{r}\gamma_{k_i}\ln\left(\frac{1+x_{j_i}^{\beta}}{1+x_{j_{i-1}}^{\beta}}\right) + \frac{\beta}{d} + \tilde{\gamma}_w\ln(1+t^{\beta})\right)^{-(r+a+1)} d\beta,
\end{aligned} \tag{3.14}$$

where

$$\begin{aligned}
I^* &= \sum_{k_1=j_0+1}^{j_1} \cdots \sum_{k_r=j_{r-1}+1}^{j_r} c_{k_1}^{(j_0)}(j_1)\cdots c_{k_r}^{(j_{r-1})}(j_r)\int_0^\infty \beta^{r+a+d}\left(\prod_{i=1}^{r}\frac{x_{j_i}^{\beta-1}}{1+x_{j_i}^{\beta}}\right)\exp\left[-\frac{\beta}{b}\right] \\
&\quad\times \left(\sum_{i=1}^{r}\gamma_{k_i}\ln\left(\frac{1+x_{j_i}^{\beta}}{1+x_{j_{i-1}}^{\beta}}\right) + \frac{\beta}{d}\right)^{-(r+a+1)} d\beta.
\end{aligned}$$

4. Numerical Illustration

In this section, we present the numerical study for Pareto (α, β) distribution when α and β both are unknown. In this study, we used the case of sequential order statistics (i.e., $\gamma_i = (n-i+1)\rho,\ \rho \in R^+$, and $i = 1, \ldots, n$). We generated a sequential order statistics sample of size $n = 20$ from the Pareto distribution whose pdf is given by (3.1). We selected the prior parameter values (a, b, c, d), these values of the prior parameters are chosen such that the prior variances are quite small with moderate values of the prior means.

The generated sample is listed as the following:

6.009514962	6.116853114	6.133726002	6.156733740	6.230574864
6.296355954	6.565306266	6.938945634	7.000456296	7.063521180
7.108905156	7.110793272	7.292923152	7.412751966	7.576398852
7.585510812	7.630819110	7.721700492	8.580489822	9.190879674.

Table 1
The 95% Bayesian prediction bounds for Y_s where $s = 1, 5, 10, 15, 20$

ρ	0.5		1		5		10	
s	L_{Y_s}	U_{Y_s}	L_{Y_s}	U_{Y_s}	L_{Y_s}	U_{Y_s}	L_{Y_s}	U_{Y_s}
1	5.94671	6.38720	5.96519	6.26192	5.98099	6.16484	5.98303	6.15290
5	6.03924	7.65134	6.022259	7.07945	6.01326	6.65527	6.01247	6.60422
10	6.50084	10.25275	6.33645	8.63890	6.20957	7.53612	6.19408	7.40878
15	7.28653	16.83238	6.84742	12.10554	6.51851	9.30402	6.47803	9.00317
20	9.98527	131.84065	8.48730	49.14520	7.45430	22.32723	7.33447	20.23096

Table 2
The 95% Bayesian prediction bounds for Y_s where $s = 1, 5, 10, 15, 20$

ρ	0.5		1		5		10	
s	L_{Y_s}	U_{Y_s}	L_{Y_s}	U_{Y_s}	L_{Y_s}	U_{Y_s}	L_{Y_s}	U_{Y_s}
1	5.95902	6.28934	5.96941	6.22127	5.97826	6.16798	5.97940	6.16139
5	6.03204	7.13812	6.02094	6.85466	6.01357	6.63696	6.01283	6.61030
10	6.41474	8.71874	6.31533	7.99041	6.23678	7.45300	6.22761	7.38850
15	7.06842	12.23084	6.80413	10.35821	6.59972	9.07013	6.57411	8.92092
20	9.21561	52.5237	8.34505	31.67258	7.70221	21.13205	7.62548	20.08976

Let the size of the future sample be $N = 20$, the credibility level be $\tau = 0.95$, $\gamma_i = (n - i + 1)\rho$, and $\tilde{\gamma}_w = (N - w + 1)\rho$ where ρ takes the values 0.5, 1, 5, and 10 ($\rho = 1$ this means the ordinary order statistics (OOS)).

If $r = 5$ and $(j_1, j_2, j_3, j_4, j_5) = (1, 3, 4, 6, 7)$. Then Table 1 presents the Bayesian prediction bounds for Y_s where $s = 1, 5, 10, 15, 20$.

If $r = 10$ and $(j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8, j_9, j_{10}) = (1, 3, 4, 6, 7, 9, 11, 13, 14, 15)$, then Table 2 presents the Bayesian prediction bounds for Y_s where $s = 1, 5, 10, 15, 20$.

Remarks.

1. From the Tables 1 and 2, we notice that the lengths of the Bayesian prediction intervals are short when there are a large number of observed values and they are decreasing in ρ.
2. If the vector of prior parameters δ is unknown, the empirical Bayesian approach could be used in estimating such prior parameters based on past samples (see, for example, Maritz and Lwin, 1989). Otherwise, one could use the hierarchical Bayesian method in which some suitable prior for δ is to be proposed (see, for example, Geisser, 1990; Bernardo and Smith, 1994).

Acknowledgment

The authors would like to express their thanks to the referees and professor N. Balakrishnan (Editor-in-Chief, *Communications in Statistics – Theory and Methods*) for their helpful remarks that led to improve the original manuscript.

References

Abdel-Aty, Y., Franz, J., Mahmoud, M. A. W. (2007). Bayesian prediction based on generalized order statistics using multiply type II censored. *Statistics* 41:495–504.

Ahsanullah, M. (1995). *Record Statistics*. Commack, NY: Nova Science Publisher, Inc.

Aitchison, J., Dunsmore, I. (1975). *Statistical Prediction Analysis*. Cambridge, UK: Cambridge University Press.

AL-Hussaini, E. K. (1999). Predicting observables from a general class of distributions. *J. Statist. Plann. Infer.* 79:79–91.

AL-Hussaini, E. K., Ahmed, A. A. (2003a). On Bayesian interval prediction for future records. *Test* 12:79–99.

AL-Hussaini, E. K., Ahmed, A. A. (2003b). On Bayesian predictive distributions of generalized order statistics. *Metrika* 57:165–176.

AL-Hussaini, E. K., Al-Awadhi, F. (2010). Bayes two-sample prediction of generalized order statistics with fixed and random sample size. *J. Statist. Computat. Simul.* 80:13–28.

AL-Hussaini, E. K., Jaheen, Z. F. (1992). Bayesian estimation of the parameters, reliability and failure rate functions of the Burr type XII failure model. *J. Statist. Computat. Simul.* 44:31–40.

AL-Hussaini, E. K., Jaheen, Z. F. (1995). Bayesian prediction bounds for the Burr type XII model. *Comm. Statist. Theor. Meth.* 24:1829–1842.

Arnold, B. C., Balakrishnan, N. (1989). *Relations, Bounds and Approximations for Order Statistics*. Berlin: Springer.

Arnold, B. C., Press, S. J. (1989). Bayesian estimation and prediction for Pareto data. *J. Amer. Statist. Assoc.* 84:1079–1084.

Arnold, B. C., Balakrishnan, N., Nagaraja, H. N. (1992). *A First Course in Order Statistics*. New York: Wiley.

Balakrishnan, N. (2007). Progressive censoring methodology: An appraisal (with discussions). *Test* 16:211–296.

Balakrishnan, N., Aggarwala, R. (2000). *Progressive Censoring: Theory, Methods and Applications*. Boston: Birkhä user.

Balakrishnan, N., Cohen, A. C. (1991). *Order Statistics and Inference: Estimation Methods*. Boston: Academic Press.

Bernardo, J. M., Smith, A. F. M. (1994). *Bayesian Theory*. New York: Wiley.

Cramer, E., Kamps, U. (1996). Sequential order statistics and k-out-of-n systems with sequentially adjusted failure rates. *Ann. Instit. Statist. Math.* 48:535–549.

Cramer, E., Kamps, U. (2001). Estimation with sequential order statistics from exponential distribution. *Ann. Instit. Statist. Math.* 53:307–324.

David, H. A. (1981). *Order Statistics*. 2nd ed. New York: Wiley.

Dunsmore, I. R. (1974). The Bayesian predictive distribution in life testing models. *Technometrics* 16:455–460.

Geisser, S. (1990). On hierarchical Bayes procedures for predicting simple exponential survival. *Biometrics* 46:225–230.

Kamps, U. (1995). *A Concept of Generalized Order Statistics*. Stuttgart, Germany: Teubner.

Lwin, T. (1972). Estimating the tail of the Paretian law. *Skandinavian Aktuarietidskr* 55: 170–178.

Maritz, J. S., Lwin, T. (1989). *Empirical Bayes Methods*. 2nd ed. London: Chapman and Hall.

Nevzorov, V. B. (1987). Records. *Theor. Probab. Appl.* 32:201–228.

Nigm, A. M. (1988). Prediction bounds for the Burr model. *Commun. Statist. Theor. Meth.* 17:289–297.

Nigm, A. M. (1989). An informative Bayesian prediction for the Weibull lifetime distribution. *Comm. Statist. Theor. Meth.* 18:897–911.

Nigm, A. M., Hamdy, H. I. (1987). Bayesian prediction bounds for the Pareto lifetime model. *Commun. Statist. Theor. Meth.* 16:1761–1772.