Journal of the Egyptian Mathematical Society (2011) **19**, 102–105

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

Two sample Bayesian prediction intervals for order statistics based on the inverse exponential-type distributions using right censored sample

M.M. Mohie El-Din [a], Y. Abdel-Aty [a,*], A.R. Shafay [b]

[a] *Department of Mathematics, Faculty of Science, Al-Azhar University, Egypt*
[b] *Department of Mathematics, Faculty of Science, Fayoum University, Egypt*

Available online 5 January 2012

KEYWORDS

Order statistics;
Bayesian prediction intervals;
Two sample scheme;
Right censored sample;
The inverse Weibull model;
The inverse exponential model;
The inverse Rayleigh model

Abstract In this paper, two sample Bayesian prediction intervals for order statistics (OS) are obtained. This prediction is based on a certain class of the inverse exponential-type distributions using a right censored sample. A general class of prior density functions is used and the predictive cumulative function is obtained in the two samples case. The class of the inverse exponential-type distributions includes several important distributions such the inverse Weibull distribution, the inverse Burr distribution, the loglogistic distribution, the inverse Pareto distribution and the inverse paralogistic distribution. Special cases of the inverse Weibull model such as the inverse exponential model and the inverse Rayleigh model are considered.

1. Introduction

In many practical problems, one wishes to use the results of previous data to predict a future observation from the same population. One way to do this is to construct an interval which will contain the future observation with a specified probability. This interval is called a prediction interval. Prediction has been applied in medicine, engineering, business, and other areas. Bayesian prediction bounds for future observations based on certain distributions have been discussed by several authors. Bayesian prediction bounds for future observations from the exponential distribution are considered by Dunsmore [1], Lingappaiah [2], Evans and Nigm [3], Al-Hussaini and Jaheen [4], and Abdel-Aty et al. [5]. Bayesian prediction bounds for future lifetimes under the Weibull model have been derived by Evans and Nigm [6]. Bayesian prediction bounds for observables having the Burr type XII distribution were obtained by Nigm [7], Al-Hussaini and Jaheen [8], and Ali Mousa and Jaheen [9,10].

Order statistics arise naturally in many real-life applications involving data relating to life testing studies. Many authors have studied order statistics and associated inferences, see for example, David [11], Arnold et al. [12], and Balakrishnan and Cohen [13].

* Corresponding author.
E-mail address: yahia1970@yahoo.com (Y. Abdel-Aty).

Peer review under responsibility of Egyptian Mathematical Society.
doi:10.1016/j.joems.2011.10.002

Production and hosting by Elsevier

In this paper, we study two sample Bayesian prediction intervals for order statistics (OS) based on the class of the inverse exponential-type distributions using a right censored sample. A general class of prior density functions which were suggested by Al-Hussaini [14] is applied.

Throughout this paper we use the following definitions and notation.

Let X be a random variable with absolutely continuous distribution function $F(x) \equiv F(x|\theta)$ and density function $f(x) \equiv f(x|\theta)$, where the parameter $\theta \in \Theta$ may be a real vector. Corresponding to X we consider n OS $X_{1:n} \leqslant X_{2:n} \leqslant \ldots \leqslant X_{n:n}$.

The joint density function of the right type-II censored sample $X_{1:n} \leqslant X_{2:n} \leqslant \ldots \leqslant X_{r:n}$ is given by

$$f_{\underline{X}}(\underline{x}) = \frac{n!}{(n-r)!}(1-F(x_r))^{n-r}\prod_{i=1}^{r} f(x_i) = n!\sum_{i=1}^{n-r+1} c_i(n-r+1)F(x_r)^{n-r-i+1}\prod_{j=1}^{r} f(x_j), \tag{1.1}$$

where $\underline{x} = (x_1, \ldots, x_r)$, $1 \leqslant r \leqslant n$, and $c_i(n-r+1) = \frac{(-1)^{n-r-i+1}}{(i-1)!(n-r-i+1)!}$.

Let $Y_{1:m} \leqslant Y_{2:m} \leqslant \ldots \leqslant Y_{m:m}$ be the OS of the future random sample of size m from the same population. Then the marginal density function of the sth OS $Y_{s:m}$ is given by

$$f_{Y_{s:m}}(y_s) = \frac{m!}{(m-s)!(s-1)!}F(y_s)^{s-1}(1-F(y_s))^{m-s}f(y_s) = \frac{m!}{(s-1)!}\sum_{w=1}^{m-s+1} c_w(m-s+1)F(y_s)^{m-w}f(y_s), \tag{1.2}$$

where $1 \leqslant s \leqslant m$ and $c_w(m-s+1) = \frac{(-1)^{m-s-w+1}}{(w-1)!(m-s-w+1)!}$.

2. Two sample Bayesian predication intervals

The random variable X is said to be inverse exponential-type distributed if the distribution function $F(x)$ is given in the following form

$$F(x \mid \theta) = \exp[-\lambda(x)], (x > 0, \theta \in \Theta). \tag{2.1}$$

where $\lambda(x) \equiv \lambda(x;\theta)$ should be a continuous, monotone increasing, differentiable function of x such that $\lambda(x) \to \infty$ as $x \to 0^+$ and $\lambda(x) \to 0$ as $x \to \infty$. Then the probability density function is given by

$$f(x \mid \theta) = -\lambda'(x)\exp[-\lambda(x)]. \tag{2.2}$$

Specific distributions considered as particular cases of this class of distributions are inverse exponential, inverse Rayleigh, inverse Weibull, inverse Pareto, negative exponential, negative Weibull, negative Pareto, negative power, Gumbel, exponentiated-Weibull, loglogistic, Burr X, inverse Burr XII and inverse paralogistic distributions.

To obtain a Bayesian prediction interval we need a suitable prior parameter distribution. The class of prior density functions suggested by Al-Hussaini [14] given by

$$\pi(\theta;\delta) \propto C(\theta;\delta)\exp[-D(\theta;\delta)], \quad \theta \in \Theta, \tag{2.3}$$

is used, where δ is vector of prior parameters.

Theorem 2.1. *Let $X_{1:n} \leqslant X_{2:n} \leqslant \cdots \leqslant X_{r:n}$ be a right type-II censored sample that follows the distribution (2.1). Let $Y_{1:m} \leqslant Y_{2:m} \leqslant \cdots \leqslant Y_{m:m}$ be the OS of the future random sample of size m from the same population. Then 100τ % Bayesian prediction bounds for $Y_{s:m}$ based on the first sample are obtained by solving the following two equations with respect to t*

$$\frac{m!}{(s-1)!}I^{-1}\sum_{i=1}^{n-r+1}\sum_{w=1}^{m-s+1}\frac{c_i(n-r+1)c_w(m-s+1)}{m-w+1} \times \int_{\theta\in\Theta}\left(\prod_{j=1}^{r}\eta_j(\theta;\underline{x})\right)\exp[-(\zeta_i(\theta;\underline{x})+(m-w+1)\lambda(t,\theta))]d\theta = \begin{cases}(1-\tau)/2\\(1+\tau)/2,\end{cases} \tag{2.4}$$

where $1 \leqslant s \leqslant m$,

$$\eta_j(\theta;\underline{x}) = -\lambda'(x_j,\theta)[C(\theta;\delta)]^{1/r}, \tag{2.5}$$

$$\zeta_i(\theta;\underline{x}) = (n-r-i+1)\lambda(x_r,\theta) + \sum_{j=1}^{r}\lambda(x_j,\theta) + D(\theta;\delta), \tag{2.6}$$

and

$$I = \sum_{i=1}^{n-r+1} c_i(n-r+1) \times \int_{\theta\in\Theta}\left(\prod_{j=1}^{r}\eta_j(\theta;\underline{x})\right)\exp[-\zeta_i(\theta;\underline{x})]d\theta. \tag{2.7}$$

Proof

Substituting from (2.1) and (2.2) in (1.1), we obtain

$$f_{\underline{X}}(\underline{x}) = n!\sum_{i=1}^{n-r+1} c_i(n-r+1)\left(\prod_{j=1}^{r}(-\lambda'(x_j,\theta))\right) \times\exp\left[-\left((n-r-i+1)\lambda(x_r,\theta)+\sum_{j=1}^{r}\lambda(x_j,\theta)\right)\right], \tag{2.8}$$

and substituting (2.1) and (2.2) in (1.2), we obtain

$$f_{Y_{s:m}}(y_s \mid \theta) = \frac{m!}{(s-1)!}\sum_{w=1}^{m-s+1} c_w(m-s+1)(-\lambda'(y_s,\theta)) \times \exp[-(m-w+1)\lambda(y_s,\theta)]. \tag{2.9}$$

Since the posterior density function is given by

$$\pi^*(\theta \mid \underline{x}) = I^{-1}\pi(\theta;\delta)f_{\underline{X}}(\underline{x}), \tag{2.10}$$

where

$$I = \int_{\theta\in\Theta}\pi(\theta;\delta)f_{\underline{X}}(\underline{x})d\theta,$$

substituting from (2.3) and (2.8) in (2.10), we obtain the posterior density function

$$\pi^*(\theta \mid \underline{x}) = I^{-1}\sum_{i=1}^{n-r+1} c_i(n-r+1)\left(\prod_{j=1}^{r}\eta_j(\theta;\underline{x})\right)\exp[-\zeta_i(\theta;\underline{x})], \tag{2.11}$$

where

$$\eta_j(\theta;\underline{x}) = -\lambda'(x_j,\theta)[C(\theta;\delta)]^{1/r},$$
$$\zeta_i(\theta;\underline{x}) = (n-r-i+1)\lambda(x_r,\theta) + \sum_{j=1}^{r}\lambda(x_j,\theta) + D(\theta;\delta),$$

and

$$I = \sum_{i=1}^{n-r+1} c_i(n-r+1)\int_{\theta\in\Theta}\left(\prod_{j=1}^{r}\eta_j(\theta;\underline{x})\right)\exp[-\zeta_i(\theta;\underline{x})]d\theta.$$

Since the Bayesian predictive density function is given by

$$P(y_s \mid \underline{x}) = \int_{\theta\in\Theta} f_{Y_{s:m}}(y_s \mid \theta)\pi^*(\theta \mid \underline{x})d\theta, \tag{2.12}$$

combining (2.9) and (2.11) with (2.12), we get the Bayesian predictive density function

$$P(y_s \mid \underline{x}) = \frac{m!}{(s-1)!} I^{-1} \sum_{i=1}^{n-r+1} c_i(n-r+1) \int_{\theta\in\Theta} \left(\prod_{j=1}^{r} \eta_j(\theta;\underline{x})\right) \times \exp[-\zeta_i(\theta;\underline{x})] \times \sum_{w=1}^{m-s+1} c_w(m-s+1)(-\grave{\lambda}(y_s,\theta)) \times \exp[-(m-w+1)\lambda(y_s,\theta)]d\theta. \quad (2.13)$$

Since the predictive cumulative distribution function of $Y_{s:m}$ is given by

$$P(0 \leqslant Y_{s:m} \leqslant t \mid \underline{x}) = \int_0^t P(y_s \mid \underline{x}) dy_s,$$

then

$$\begin{aligned} P(0 \leqslant Y_{s:m} \leqslant t \mid \underline{x}) = {} & \frac{m!}{(s-1)!} I^{-1} \int_0^t \sum_{i=1}^{n-r+1} c_i(n-r+1) \\ & \times \int_{\theta\in\Theta} \left(\prod_{j=1}^{r} \eta_j(\theta;\underline{x})\right) \exp[-\zeta_i(\theta;\underline{x})] \\ & \times \sum_{w=1}^{m-s+1} c_w(m-s+1)(-\lambda'(y_s,\theta)) \\ & \times \exp[-(m-w+1)\lambda(y_s,\theta)]d\theta dy_s \\ = {} & \frac{m!}{(s-1)!} I^{-1} \sum_{i=1}^{n-r+1} c_i(n-r+1) \\ & \times \int_{\theta\in\Theta} \left(\prod_{j=1}^{r} \eta_j(\theta;\underline{x})\right) \exp[-\zeta_i(\theta;\underline{x})] \\ & \times \sum_{w=1}^{m-s+1} \frac{c_w(m-s+1)}{m-w+1} \\ & \times \exp[-(m-w+1)\lambda(t,\theta)]d\theta. \quad (2.14) \end{aligned}$$

Then $100\tau\%$ Bayesian prediction bounds (lower–upper) for $Y_{s:m}$ based on the first sample are obtained by solving the following two equations with respect to t

$$\frac{m!}{(s-1)!} I^{-1} \sum_{i=1}^{n-r+1} \sum_{w=1}^{m-s+1} \frac{c_i(n-r+1)c_w(m-s+1)}{m-w+1} \times \int_{\theta\in\Theta} \left(\prod_{j=1}^{r} \eta_j(\theta;\underline{x})\right) \exp[-(\zeta_i(\theta;\underline{x}) + (m-w+1)\lambda(t,\theta))]d\theta = \begin{cases} (1-\tau)/2 \\ (1+\tau)/2. \end{cases}$$

3. Example

In this section we study two sample Bayesian prediction intervals for order statistics (OS) based on the inverse Weibull model which is one of the most important models in the inverse exponential-type class of distributions. For example the inverse Weibull (IW) distribution has been used to model the degradation of mechanical components (Keller and Kanath [15]) such as the dynamic components (pistons, crankshaft, etc.) of diesel engines. Properties of IW distribution have been obtained by, for example, Calabria and Pulcini [16; 17] and Mahmoud et al. [18].

The distribution function of the inverse Weibull model is given by

$$F(x \mid \theta) = \exp[-(\alpha x)^{-\beta}], x > 0 \quad (3.1)$$

where $\theta = (\alpha,\beta)$, $\alpha > 0$ and $\beta > 0$.

Hence

$$\lambda(x) = \frac{\alpha^{-\beta}}{x^{\beta}} \quad \text{and} \quad \lambda'(x) = -\frac{\beta\alpha^{-\beta}}{x^{\beta+1}}. \quad (3.2)$$

Suppose that α is an unknown and $\beta$is known. Then we will use the prior density function which was suggested by Calabria and Pulcini [17] (when $\beta$is known) as

$$\pi(\lambda;\delta) \propto \alpha^{-c\beta-1} \exp[-d\alpha^{-\beta}], \quad (3.3)$$

where $\alpha > 0$, $\delta = (c,d)$ and $c,d > 0$.

Hence

$$C(\theta;\delta) = \alpha^{-c\beta-1} \quad \text{and} \quad D(\theta;\delta) = d\alpha^{-\beta}. \quad (3.4)$$

Using (2.14), (3.2) and (3.4) then the predictive cumulative distribution function of $Y_{s:m}$ is given by

$$\begin{aligned} & P(0 \leqslant Y_{s:m} \leqslant t \mid \underline{x}) \\ & = \frac{m!}{(s-1)!} I^{-1} \sum_{i=1}^{n-r+1} \sum_{w=1}^{m-s+1} \frac{c_i(n-r+1)c_w(m-s+1)}{m-w+1} \\ & \quad \times \int_0^\infty \left(\prod_{j=1}^{r} \frac{1}{x_j^{(\beta+1)}}\right) \beta^r \times \alpha^{-r\beta-c\beta-1} \\ & \quad \times \exp\left[-\alpha^{-\beta}\left(\sum_r^{j=1} \frac{1}{x_j^{\beta}} + \frac{(n-r-i+1)}{x_r^{\beta}} + \frac{(m-w+1)}{t^{\beta}} + d\right)\right] d\alpha \\ & = \frac{m!}{(s-1)!} I^{-1} \sum_{i=1}^{n-r+1} \sum_{w=1}^{m-s+1} \frac{c_i(n-r+1)c_w(m-s+1)}{(m-w+1)} \\ & \quad \times \left(\sum_{j=1}^{r} \frac{1}{x_j^{\beta}} + \frac{(n-r-i+1)}{x_r^{\beta}} + \frac{(m-w+1)}{t^{\beta}} + d\right)^{-(r+c)}, \end{aligned} \quad (3.5)$$

where

$$I = \sum_{i=1}^{n-r+1} \left(c_i(n-r+1) \times \left(\sum_{j=1}^{r} \frac{1}{x_j^{\beta}} + \frac{(n-r-i+1)}{x_r^{\beta}} + d\right)^{-(r+c)} \right). \quad (3.6)$$

3.1. Special cases

The inverse Weibull model contains many important special cases such as the inverse exponential model and the inverse Rayleigh model. In the following the inverse exponential model and the inverse Rayleigh model are considered.

(1) The inverse exponential model.

We can obtain the inverse exponential model as special case of the inverse Weibull model by setting $\beta = 1$. Hence the distribution function of the inverse exponential model is given by

$$F(x \mid \alpha) = \exp\left[\frac{-1}{\alpha x}\right], \quad x > 0,$$

where $\alpha > 0$,

$$\lambda(x) = \frac{1}{\alpha x} \quad \text{and} \quad \lambda'(x) = \frac{-1}{\alpha x^2}. \quad (3.7)$$

Putting $\beta = 1$ in (3.5), then the predictive cumulative distribution function of $Y_{s:m}$ is given by

$$P(0 \leqslant Y_{s:m} \leqslant t \mid \underline{x}) = \frac{m!}{(s-1)!} I^{-1} \sum_{i=1}^{n-r+1} \sum_{w=1}^{m-s+1} \frac{c_i(n-r+1)c_w(m-s+1)}{m-w+1} \times \left(\sum_{j=1}^{r} \frac{1}{x_j} + \frac{(n-r-i+1)}{x_r} + \frac{m-w+1}{t} + d\right)^{-(r+c)} \quad (3.8)$$

where

$$I=\sum_{i=1}^{n-r+1}\left\{c_i(n-r+1)\times\left(\sum j=1^r\frac{1}{x_j}+\frac{(n-r-i+1)}{x_r}+d\right)^{-(r+c)}\right\}. \quad (3.9)$$

(2) The inverse Rayleigh model.

We can obtain the inverse Rayleigh model as special case of the inverse Weibull model by setting $\beta = 1$. Hence the distribution function of the inverse Rayleigh model is given by

$$F(x \mid \theta) = \exp\left[\frac{-1}{(\alpha x)^2}\right], \quad x > 0,$$

where $\alpha > 0$,

$$\lambda(x) = \frac{1}{(\alpha x)^2} \quad \text{and} \quad \lambda'(x) = \frac{-2\alpha^{-2}}{x^3}. \quad (3.10)$$

Putting $\beta = 2$ in (3.5), then the predictive cumulative distribution function of $Y_{s:m}$ is given by

$$P(0\leqslant Y_{s:m}\leqslant t\,|\,\underline{x})=\frac{m!}{(s-1)!}I^{-1}\sum_{i=1}^{n-r+1}\sum_{w=1}^{m-s+1}\frac{c_i(n-r+1)c_w(m-s+1)}{m-w+1}$$
$$\times\left(\sum_{j=1}^{r}\frac{1}{x_j^2}+\frac{(n-r-i+1)}{x_r^2}+\frac{m-w+1}{t^2}+d\right)^{-(r+c)}, \quad (3.11)$$

where

$$I=\sum_{i=1}^{n-r+1}\left\{c_i(n-r+1)\times\left(\sum_{j=1}^{r}\frac{1}{x_j^2}+\frac{(n-r-i+1)}{x_r^2}+d\right)^{-(r+c)}\right\}.$$

Remark 3.1. We can see that (3.5) agrees with expression (19) obtained by Calabria and Pulcini [17] when β is known.

Remark 3.2. We can obtain the Bayesian prediction interval for OS from the inverse Weibull distribution when both α and β are unknown from Theorem 2.1 directly and we can see that it agree with Calabria and Pulcini [17].

Remark 3.3. We can obtain the Bayesian prediction intervals for OS from the inverse Pareto distribution and the inverse Burr distribution from Theorem 2.1 directly.

Acknowledgements

The authors would like to express their thanks to the referees and associated editor for their helpful remarks that led to improve the original manuscript.

References

[1] I.R. Dunsmore, The Bayesian predictive distribution in life testing models, Technometrics 16 (1974) 455–460.
[2] G.S. Lingappaiah, Bayesian approach to prediction and the spacings in the exponential distribution, Annals of Institute of Statistical Mathematics 31 (1979) 391–401.
[3] I.G. Evans, A.M. Nigm, Bayesian prediction for the left truncated exponential distribution, Teohnometrics 22 (2) (1980) 201–204.
[4] E.K. Al-Hussaini, Z.F. Jaheen, Parametric prediction bounds for the future median of the exponential distribution, Statistics 32 (1999) 267–275.
[5] Y. Abdel-Aty, J. Franz, M.A.W. Mahmoud, Bayesian prediction based on generalized order statistics using multiply type-II censoring, Statistics 41 (6) (2007) 495–504.
[6] I.G. Evans, A.M. Nigm, Bayesian prediction for two-parameter Weibull lifetime model, Communications in Statistics Theory and Methods 9 (1980) 649–658.
[7] A.M. Nigm, Prediction bounds for the Burr model, Communications in Statistics Theory and Methods 17 (I) (1988) 287–297.
[8] E.K. Al-Hussaini, Z.F. Jaheen, Bayesian prediction bounds for the Burr type Xll failure model, Communications in Statistics Theory and Methods 24 (7) (1995) 1829–1842.
[9] M.A.M. Ali Mousa, Z.F. Jaheen, Bayesian prediction for the Burr type XII model based on doubly censored data, Statistics 48 (1997) 337–344.
[10] M.A.M. Ali Monsa, Z.F. Jaheen, Bayesian prediction for the two-parameter Burr type XII model based on doubly censored data, Journal of Applied Statististical Science 7 (2/3) (1998) 103–111.
[11] H.A. David, Order Statistics, 2nd ed., Wiley, New York, 1981.
[12] B.C. Arnold, N. Balakrishnan, H.N. Nagaraja, A First Course in Order Statistics, Wiley, New York, 1992.
[13] N. Balakrishnan, A.C. Cohen, Order Statistics and Inference: Estimation Methods, Academic Press, San Diego, 1991.
[14] E.K. Al-Hussaini, Predicting observables from a general class of distributions, Journal of Statistical Planning and Inference 79 (1999) 79–91.
[15] A.Z. Keller, A.R.R. Kanath, 1982. Alternative reliability models for mechanical systems. Third International Conference on Reliability and Maintainability, Toulse, France.
[16] R. Calabria, G. Pulcini, On the maximum likelihood and least-squares estimation in the inverse Weibull distribution, Statististical Application 2 (1) (1990) 53–63.
[17] R. Calabria, G. Pulcini, Bayes 2-sample prediction for the inverse Weibull distribution, Communications in Statistics Theory and Methods 23 (6) (1994) 1811–1824.
[18] M.A.W. Mahmoud, K.S. Sultan, S.M. Amer, Order statistics from inverse Weibull distribution and associated inference, Computational Statistics and Data Analysis 42 (2003) 149–163.