Welcome to Engineering Mathematics 2

We will cover 4 topics today

1. **The Dot Product**
2. **The Cross Product**
3. **The Scalar Triple Product**
4. **The Vector Moment**

The Dot Product

The multiplication of 2 vectors is defined in 2 different ways.

The first way is known as the **dot** (or scalar) product. Unsurprisingly the calculated result is a scalar.

The second way is known as the **cross** (or vector) product. The calculated result is a vector.

The dot product is written as

$$\mathbf{a} \bullet \mathbf{b}$$

The cross product is written as

$$\mathbf{a} \times \mathbf{b}$$

The Dot Product

Suppose that two vectors are given in the following component form

$$\mathbf{a} = (a_1, a_2, a_3)$$

$$\mathbf{b} = (b_1, b_2, b_3)$$

The dot product is calculated as

$$\mathbf{a} \bullet \mathbf{b} = a_1.b_1 + a_2.b_2 + a_3.b_3$$

<u>Question</u>

Find $(\mathbf{a} - \mathbf{b}) \bullet (\mathbf{a} + \mathbf{b})$

When

$$\mathbf{a} = (-1, 0, 1)$$

$$\mathbf{b} = (2, 3, 2)$$

$$\mathbf{a} + \mathbf{b} = (1, 3, 3)$$

$$\mathbf{a} - \mathbf{b} = (-3, -3, -1)$$

Therefore

$$(\mathbf{a} - \mathbf{b}) \bullet (\mathbf{a} + \mathbf{b})$$

$$= (-3.1) + (-3.3) + (-1.3) = -15$$

The Dot Product

Properties

a) Commutative Property

$$\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$$

b) Distributive Property

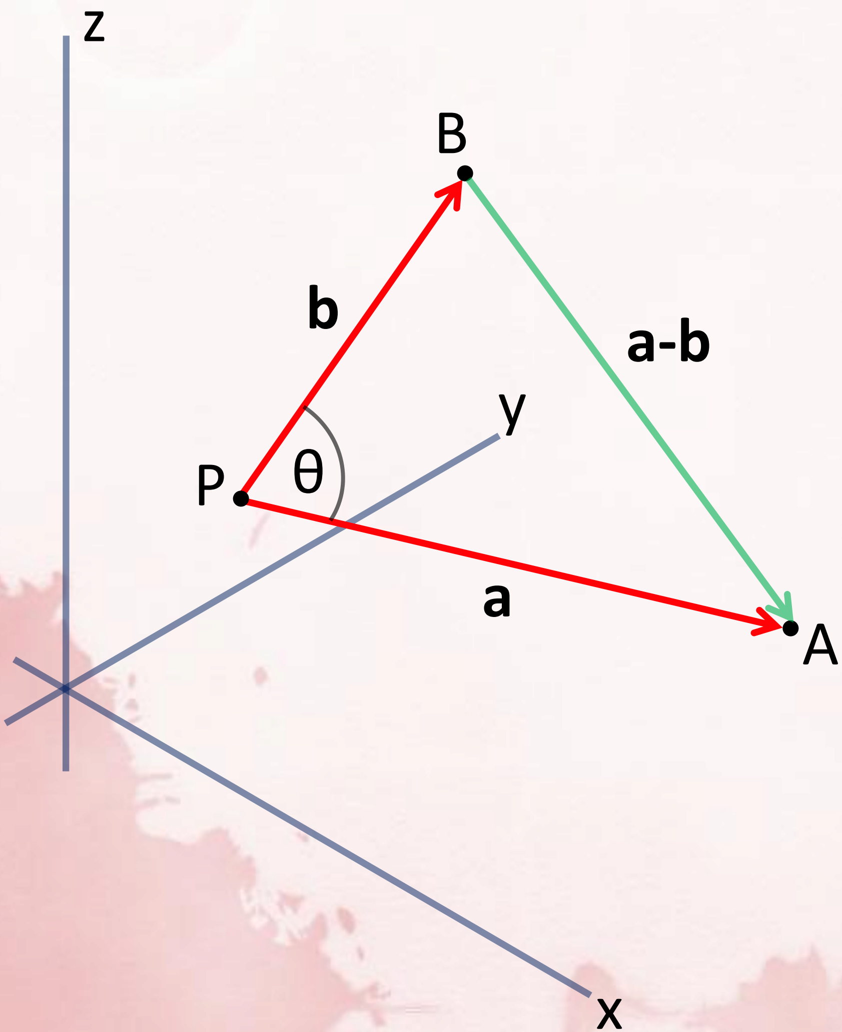
$$\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$$

c) Connection with the magnitude

$$\mathbf{a} \bullet \mathbf{a} = a_1^2 + a_2^2 + a_3^2 = |\mathbf{a}|^2$$

(For two dimensions, omit the third component).

The Dot Product



Consider the points P, A & B.

$$\overrightarrow{PA} = \mathbf{a} \qquad PA = \left|\overrightarrow{PA}\right| = |\mathbf{a}|$$

$$\overrightarrow{PB} = \mathbf{b} \qquad PB = \left|\overrightarrow{PB}\right| = |\mathbf{b}|$$

$$\overrightarrow{BA} = \mathbf{a} - \mathbf{b} \qquad BA = \left|\overrightarrow{BA}\right| = |\mathbf{a} - \mathbf{b}|$$

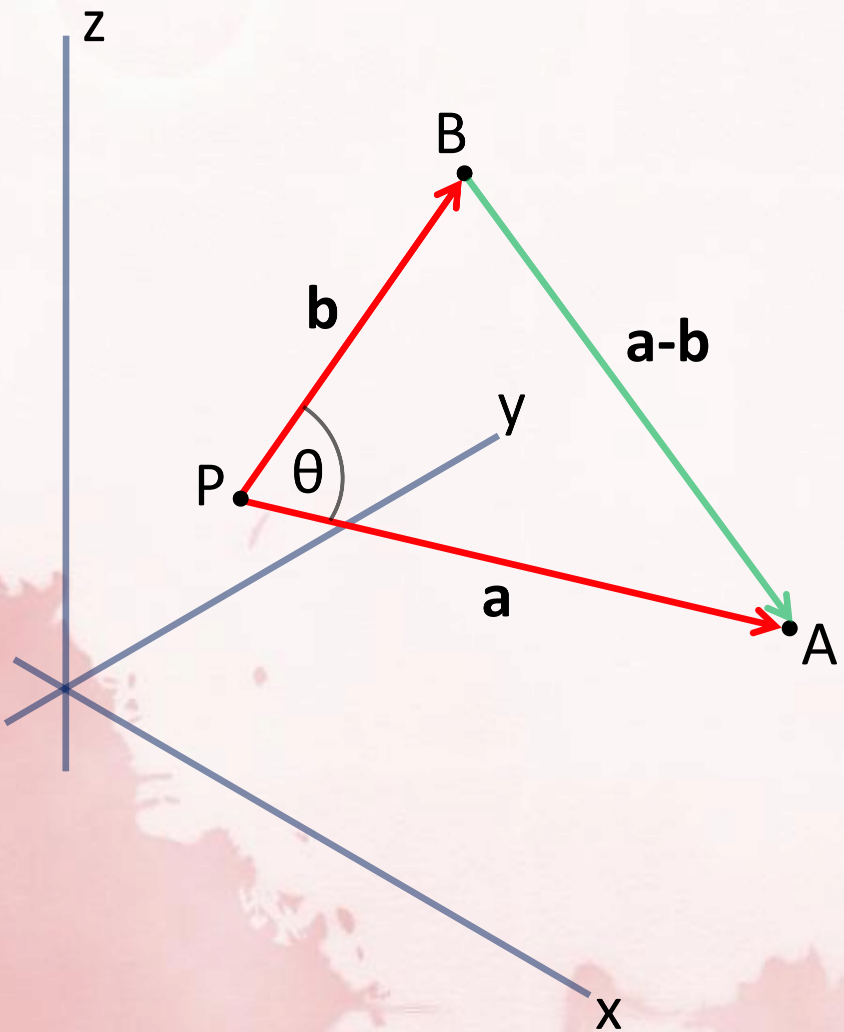
The cosine rule says that

$$BA^2 = PA^2 + PB^2 - 2.PA.PB.Cos(\theta)$$

or

$$|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2.|\mathbf{a}|.|\mathbf{b}|.Cos(\theta)$$

The Dot Product



Also,

$$|\mathbf{a}-\mathbf{b}|^2 = (\mathbf{a}-\mathbf{b})\bullet(\mathbf{a}-\mathbf{b})$$
$$= \mathbf{a}\bullet\mathbf{a}+\mathbf{b}\bullet\mathbf{b}-2\mathbf{a}\bullet\mathbf{b}$$

$$|\mathbf{a}-\mathbf{b}|^2 = |\mathbf{a}|^2+|\mathbf{b}|^2-2\mathbf{a}\bullet\mathbf{b}$$

Comparing with the previous result of

$$|\mathbf{a}-\mathbf{b}|^2 = |\mathbf{a}|^2+|\mathbf{b}|^2-2.|\mathbf{a}|.|\mathbf{b}|.\mathrm{Cos}(\theta)$$

We can easily see that

$$\mathbf{a}\bullet\mathbf{b} = |\mathbf{a}|.|\mathbf{b}|.\mathrm{Cos}(\theta)$$

Where 0 ≤ θ ≥ 180. I.e. θ is the internal angle

The Dot Product

Question

Given three points A : (1, 1, 1); B : (3, 2, 3) and C : (0, -1, 1). Find the angle between $\overrightarrow{CA}$ and $\overrightarrow{CB}$.

Let $\overrightarrow{CA} = \mathbf{a} = (1,1,1) - (0,-1,1) = (1,2,0)$ $\quad \overrightarrow{CB} = \mathbf{b} = (3,2,3) - (0,-1,1) = (3,3,2)$

$$|\mathbf{a}| = \sqrt{1^2 + 2^2 + 0^2} = \sqrt{5} \qquad |\mathbf{b}| = \sqrt{3^2 + 3^2 + 2^2} = \sqrt{22}$$

$$\mathbf{a} \bullet \mathbf{b} = 1.3 + 2.3 + 0.2 = 9$$

$$\mathrm{Cos}(\theta) = \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}|.|\mathbf{b}|} = \frac{9}{\sqrt{110}} = 0.858$$

$$\theta = 30.9^{o}$$

The Dot Product

In the previous lecture we defined a unit vector.

Three important unit vectors are

$$\mathbf{i} = \hat{\mathbf{x}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{j} = \hat{\mathbf{y}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{k} = \hat{\mathbf{z}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

In terms of **i**, **j** and **k** we can write down three dot products that are equal to 1.

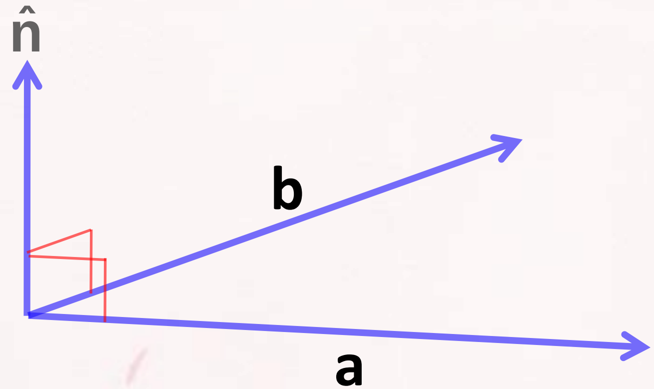
$$\mathbf{i} \bullet \mathbf{i} = \mathbf{j} \bullet \mathbf{j} = \mathbf{k} \bullet \mathbf{k} = 1$$

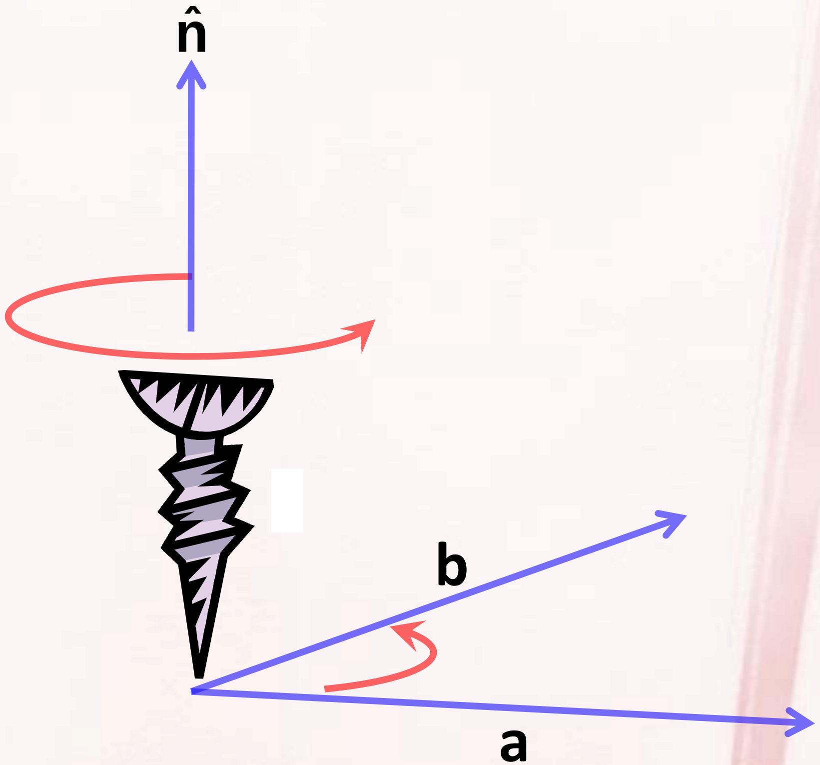
In terms of **i**, **j** and **k** we can write down three dot products that are equal to 0.

$$\mathbf{i} \bullet \mathbf{j} = \mathbf{j} \bullet \mathbf{k} = \mathbf{i} \bullet \mathbf{k} = 0$$

The Cross Product

Consider the following vectors.





The vector a and b are both in the same plane whereas **n̂** is perpendicular to both of them. i.e.

$$\hat{\mathbf{n}} \bullet \mathbf{a} = \hat{\mathbf{n}} \bullet \mathbf{b} = 0$$

If a right handed screw is turned from **a** to **b** then the screw will move in the direction of the unit normal. Hence, we can tell the direction of the normal using the "right hand rule".

The Cross Product

The cross product has applications in problems that involve rotation (i.e. moments and angular velocities).

The cross product is denoted by a bold multiplication sign or by a caret sign. I.e.

$$\mathbf{a}\times\mathbf{b}$$

Or

$$\mathbf{a}\wedge\mathbf{b}$$

The cross product is defined in the following way

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Where; **i**, **j** & **k** are the unit vectors in the x, y & z directions respectively. a_i & b_i are the components of the **a** and **b** vectors in the 1, 2 & 3 directions (x, y, & z respectively).

The Cross Product

<u>Question</u>

Find the cross products **a** x **b** and **b** x **a**

a = 2**i** – **j** + 3**k**; **b** = -**i** + 2**j** + 4**k**

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ -1 & 2 & 4 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -1 & 3 \\ 2 & 4 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & 3 \\ -1 & 4 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$\mathbf{a}\times\mathbf{b} = \mathbf{i}.\left((-1).4 - 3.2\right) - \mathbf{j}.\left(2.4 - 3.(-1)\right) + \mathbf{k}.\left(2.2 - (-1).(-1)\right)$$

$$\mathbf{a}\times\mathbf{b} = -10.\mathbf{i} - 11.\mathbf{j} + 3.\mathbf{k}$$

The Cross Product

<u>Question</u>

Find the cross products **a** x **b** and **b** x **a**

a = 2**i** – **j** + 3**k**; **b** = -**i** + 2**j** + 4**k**

$$\mathbf{b}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 4 \\ 2 & -1 & 3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 2 & 4 \\ -1 & 3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -1 & 4 \\ 2 & 3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix}$$

$$\mathbf{b}\times\mathbf{a} = \mathbf{i}.\left(3.2-(-1).4\right) - \mathbf{j}.\left(3.(-1)-2.4\right) + \mathbf{k}.\left((-1).(-1)-2.2\right)$$

$$\mathbf{b}\times\mathbf{a} = 10.\mathbf{i} + 11.\mathbf{j} - 3.\mathbf{k}$$

The Cross Product

Properties

a) Commutative Property (the cross product does not commute)

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

b) Distributive Property

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

c) Scalar Multiplication

$$\mathbf{a} \times (\lambda \mathbf{b}) = \lambda \mathbf{a} \times \mathbf{b}$$

d) If **a** and **b** are Parallel then

$$\mathbf{a} \times \mathbf{b} = 0$$

The Cross Product

Consider the following cross product

$$\mathbf{i}\times\mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= \mathbf{i}\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \mathbf{k}$$

We can also show that

$$\mathbf{j}\times\mathbf{k} = \mathbf{i}$$

$$\mathbf{k}\times\mathbf{i} = \mathbf{j}$$

$$\mathbf{j}\times\mathbf{i} = -\mathbf{k}$$

$$\mathbf{i}\times\mathbf{k} = -\mathbf{j}$$

$$\mathbf{k}\times\mathbf{j} = -\mathbf{i}$$

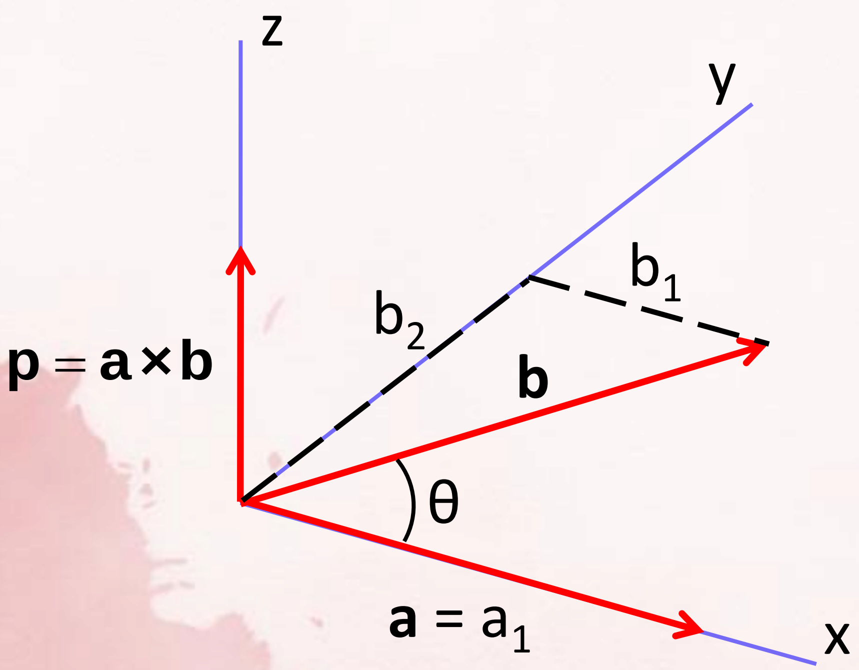
$$\mathbf{i}\times\mathbf{i} = \mathbf{0}$$

$$\mathbf{j}\times\mathbf{j} = \mathbf{0}$$

$$\mathbf{k}\times\mathbf{k} = \mathbf{0}$$

The Cross Product

Consider the following diagram



We can deduce that

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & 0 & 0 \\ b_1 & b_2 & 0 \end{vmatrix} = \mathbf{k}.a_1.b_2$$

$$b_2 = |\mathbf{b}|.\mathrm{Sin}(\theta)$$

$$a_1 = |\mathbf{a}|$$

Therefore

$$\mathbf{a} \times \mathbf{b} = \hat{\mathbf{n}}.|\mathbf{a}|.|\mathbf{b}|.\mathrm{Sin}(\theta)$$

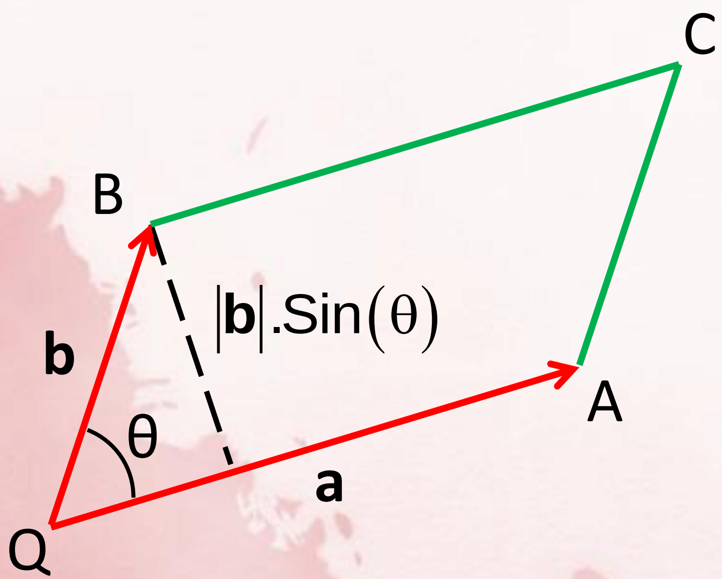
Where $\hat{\mathbf{n}}$ is the unit normal to **a** and **b**. Hence, we can also write

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|.|\mathbf{b}|.\mathrm{Sin}(\theta)$$

The Cross Product

Question

Let $\mathbf{a} = \overrightarrow{QA}$ and $\mathbf{b} = \overrightarrow{QB}$ be two vectors (from Q) representing the sides of a parallelogram. Show that the area of a parallelogram is equal to $|\mathbf{a} \times \mathbf{b}|$



Area = base x perpendicular height

Area = $|QA|.|\mathbf{b}|.\text{Sin}(\theta) = |\mathbf{a}|.|\mathbf{b}|.\text{Sin}(\theta)$

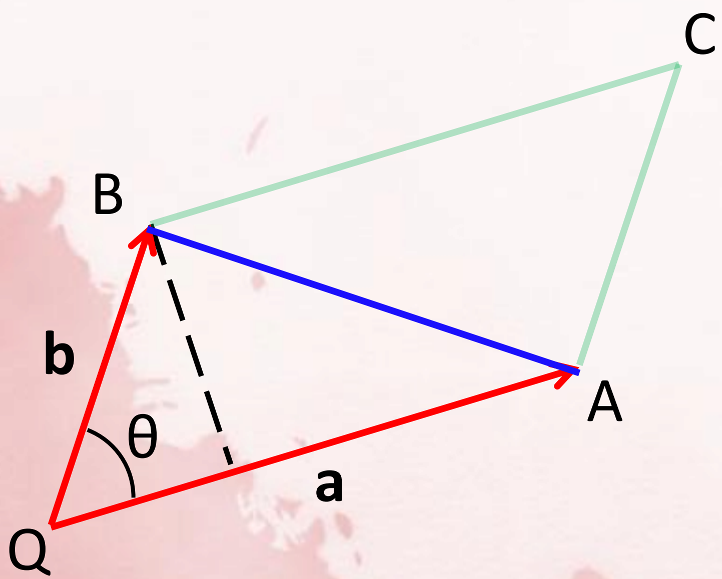
$= |\mathbf{a} \times \mathbf{b}|$

Therefore, what is the area of a triangle in terms of **a** and **b**?

The Cross Product

Question

Let $\mathbf{a} = \overrightarrow{QA}$ and $\mathbf{b} = \overrightarrow{QB}$ be two vectors (from Q) representing the sides of a parallelogram. Show that the area of a parallelogram is equal to $|\mathbf{a} \times \mathbf{b}|$



Area = base x perpendicular height

Area = $|QA|.|\mathbf{b}|.\text{Sin}(\theta) = |\mathbf{a}|.|\mathbf{b}|.\text{Sin}(\theta)$

$= |\mathbf{a} \times \mathbf{b}|$

Therefore, what is the area of a triangle in terms of **a** and **b**?

$$\text{Area} = \frac{|\mathbf{a}|.|\mathbf{b}|.\text{Sin}(\theta)}{2} = \frac{|\mathbf{a} \times \mathbf{b}|}{2}$$

The Scalar Triple Product

The scalar quantity **a**.(**b** x **c**) is called a **Scalar Triple Product**.

The triple product is calculated as

$$\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Commutative Property

$$\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\bullet(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\bullet(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\bullet(\mathbf{c}\times\mathbf{b}) = \mathbf{b}\bullet(\mathbf{a}\times\mathbf{c}) = \mathbf{c}\bullet(\mathbf{b}\times\mathbf{a})$$

$$= -\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c})$$

If any two vectors are equal or parallel then

$$\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c}) = 0$$

If the lengths of the sides of a parallelepiped are given by the vectors **a**, **b** & **c**; then,

$$\text{Volume} = |\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c})|$$

If three vectors are coplanar when drawn from the same point, then

$$\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c}) = 0$$

The Scalar Triple Product

<u>Question</u>

Show that the points A : (1, 2, 2); B : (3, 4, 5); C : (-1, 0 , -1) lie on a plane that passes through the origin.

If the origin is designated by O; then, $\mathbf{a} = \overrightarrow{OA} \quad \mathbf{b} = \overrightarrow{OB} \quad \mathbf{c} = \overrightarrow{OC}$

To show that **a**, **b** & **c** are coplanar we evaluate

$$\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \bullet \left[\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} \right]$$

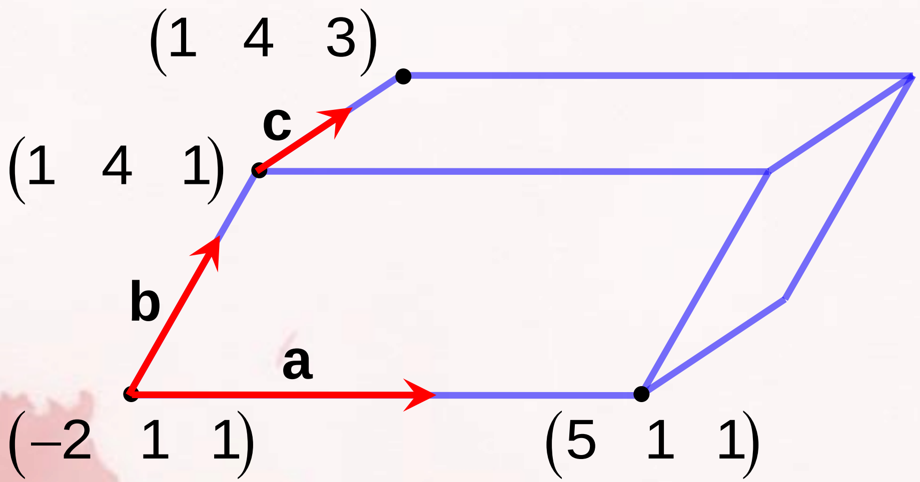
$$= \begin{vmatrix} 1 & 2 & 2 \\ 3 & 4 & 5 \\ -1 & 0 & -1 \end{vmatrix} = -1 \begin{vmatrix} 2 & 2 \\ 4 & 5 \end{vmatrix} - 1 \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -1.(10-8) - 1.(4-6) = 0$$

Therefore **a**, **b** & **c** are coplanar *and* pass through the origin.

The Scalar Triple Product

<u>Question</u>

What is the volume of the following parallelepiped?



The parallelepiped is comprised of the following three vectors

$$\mathbf{a} = \begin{pmatrix}7 & 0 & 0\end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix}3 & 3 & 0\end{pmatrix}$$

$$\mathbf{c} = \begin{pmatrix}0 & 0 & 2\end{pmatrix}$$

The volume is calculated using the triple product, i.e.

$$\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix}7 & 0 & 0\\ 3 & 3 & 0\\ 0 & 0 & 2\end{vmatrix} = 7\begin{vmatrix}3 & 0\\ 0 & 2\end{vmatrix} = 42$$

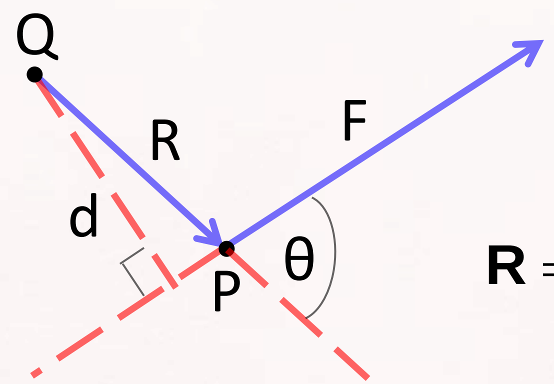
Applications

<u>The 'Vector Moment'</u>

Suppose that (in three dimensions) a force (**F**) is acting at a point (P) in a body. The moment or torque about an arbitrary point (Q) is defined as

$$M = |\mathbf{F}|.d$$

Where 'd' is the perpendicular distance between P and Q. This is shown in the following diagram.



$$\mathbf{R} = \overrightarrow{QP}$$

It can be seen that

$$d = |\mathbf{R}|.\mathrm{Sin}(\theta) \qquad 0 \le \theta \le 180$$

Thus, $M = |\mathbf{F}||\mathbf{R}|.\mathrm{Sin}(\theta)$

Hence, we can deduce that

$$M = |\mathbf{R} \times \mathbf{F}|$$

or

$$\mathbf{M} = \mathbf{R} \times \mathbf{F}$$

Notice that R comes first and **M** is bold in the second equation.

Applications

Question

A force F = **i** - **j** + 2**k** acts at (1, 2, 1). Find its vector moment about the point (2, 1, 1).

$$R = \begin{pmatrix}1 & 2 & 1\end{pmatrix} - \begin{pmatrix}2 & 1 & 1\end{pmatrix}$$
$$= \begin{pmatrix}-1 & 1 & 0\end{pmatrix}$$

The moment is given by

$$\mathbf{M} = \mathbf{R} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ 1 & -1 & 2 \end{vmatrix}$$

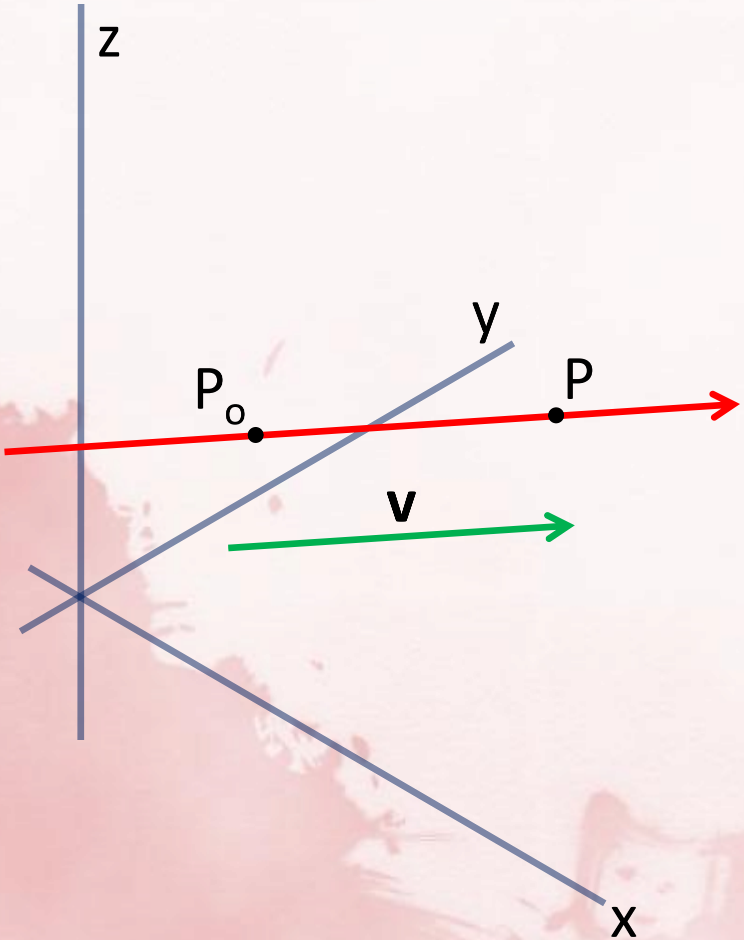
$$= \mathbf{i}\begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -1 & 0 \\ 1 & 2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= 2.\mathbf{i} + 2.\mathbf{j}$$

$$M = \sqrt{8}$$

Lines

A line can be specified in two ways. Either using two points or a point and a direction.



We can define

$$P_o = P_o(x_o, y_o, z_o) \quad P = P(x, y, z)$$

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

Let t be an arbitrary scalar then,

$$(x - x_o)\mathbf{i} + (y - y_o)\mathbf{j} + (z - z_o)\mathbf{k}$$
$$= t(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})$$

Which can be rewritten as

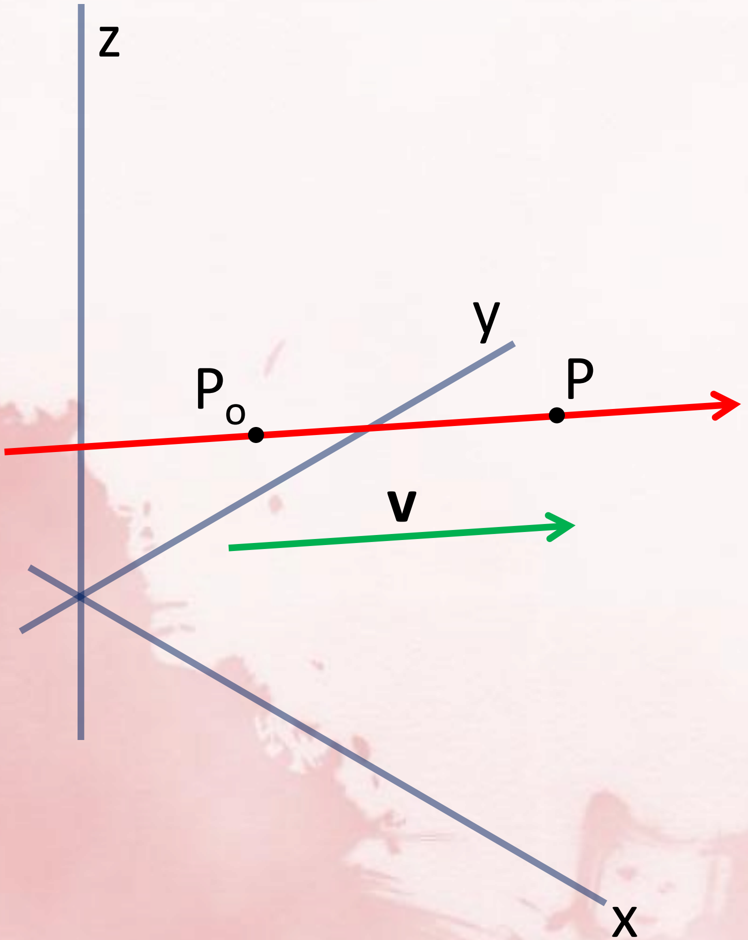
$$x.\mathbf{i} + y.\mathbf{j} + z.\mathbf{k}$$
$$= x_o.\mathbf{i} + y_o.\mathbf{j} + z_o.\mathbf{k} + t(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})$$

Or more simply if $\mathbf{r}_o$ is the position vector of P_o

$$\mathbf{r}(t) = \mathbf{r}_o + t.\mathbf{v}$$

Lines

A line can be specified in two ways. Either using two points or a point and a direction.



Hence we can derive the parametric equations for a line

$$x = x_o + t.v_1$$

$$y = y_o + t.v_2$$

$$z = z_o + t.v_3$$

$$-\infty < t < \infty$$

Lines

Question

Find the parametric equations for the line through (-2, 0, 4) parallel to **v** = 2.**i** + 4.**j** – 2.**k**

$$\mathbf{r}(t) = \mathbf{r}_o + t.\mathbf{v}$$
$$= \left(-2.\mathbf{i} + 0.\mathbf{j} + 4.\mathbf{k}\right) + t.(2.\mathbf{i} + 4.\mathbf{j} - 2.\mathbf{k})$$

$$x = -2 + 2t$$
$$y = 4t$$
$$z = 4 - 2t$$

Find the parametric equations for the line through P(-3, 2, -3) and Q(1, -1, 4) using the vector $\overline{PQ}$.

$$\overrightarrow{PQ} = \left(1 - (-3)\right).\mathbf{i} + (-1 - 2).\mathbf{j}$$
$$+ \left(4 - (-3)\right).\mathbf{k} = 4.\mathbf{i} - 3.\mathbf{j} + 7.\mathbf{k}$$

Thus

$$x = -3 + 4t$$
$$y = 2 - 3t$$
$$z = -3 + 7t$$

Find the parametric equations using the vector $\overline{QP}$.

$$x = 1 - 4t$$
$$y = -1 + 3t$$
$$z = 4 - 7t$$

Lines

We can generalise the previous result in the following way.

Suppose we have two position vectors (**a** and **b**). We can write the equation of a straight line as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + k.\begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 + k.(b_1 - a_1) \\ a_2 + k.(b_2 - a_2) \\ a_3 + k.(b_3 - a_3) \end{pmatrix}$$

Hence we can deduce that,

$$x = a_1 + k.(b_1 - a_1)$$
$$y = a_2 + k.(b_2 - a_2)$$
$$z = a_3 + k.(b_3 - a_3)$$

Or, in other words

$$k = \frac{x - a_1}{b_1 - a_1} \quad k = \frac{y - a_2}{b_2 - a_2} \quad k = \frac{z - a_3}{b_3 - a_3}$$

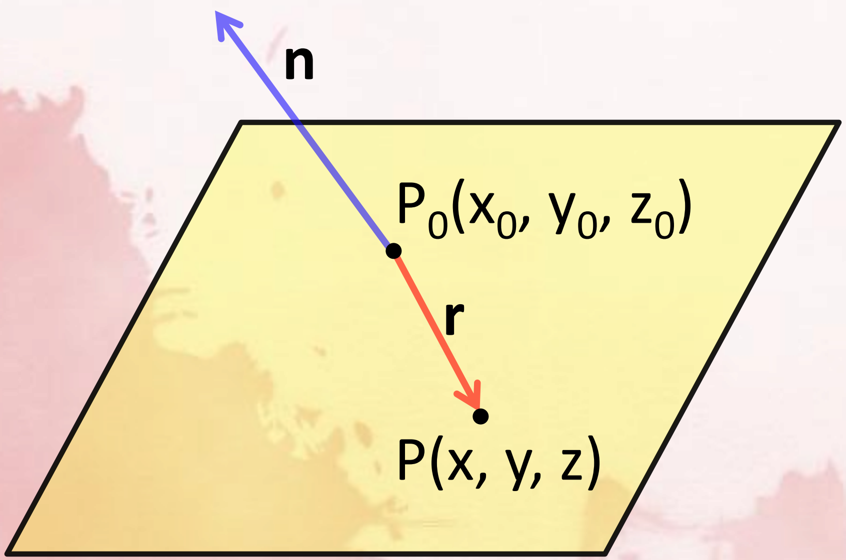
This gives us the **Cartesian form** of a straight line in three dimensions.

$$\frac{x - a_1}{b_1 - a_1} = \frac{y - a_2}{b_2 - a_2} = \frac{z - a_3}{b_3 - a_3}$$

Planes

A plane can be specified in three ways. I.e. using one point and a normal direction, one point and two vectors or three points.

A normal direction defines the 'tilt' or orientation of a plane.



Suppose that a plane passes through a point $P_0(x_0,y_0,z_0)$ and is normal to the non-zero vector **n**. We can deduce that

$$\mathbf{n}\bullet\mathbf{r} = \mathbf{n}\bullet\overrightarrow{P_0P} = 0$$

Let

$$\mathbf{n} = A.\mathbf{i} + B.\mathbf{j} + C.\mathbf{k}$$

$$\mathbf{r} = \overrightarrow{P_0P}$$

$$= (x - x_0).\mathbf{i} + (y - y_0).\mathbf{j} + (z - z_0).\mathbf{k}$$

Then

$$\mathbf{n}\bullet\mathbf{r} = 0$$

$$= A.(x - x_0) + B.(y - y_0) + C.(z - z_0)$$

Or (in Cartesian form)

$$A.x + B.y + C.z = D$$

$$D = A.x_0 + B.y_0 + C.z_0$$

Planes

Question

Find an equation for the plane through $P_0(-3, 0, 7)$ and perpendicular to **n** = 5.**i** + 2.**j** - **k**

Using

$$A.(x - x_0) + B.(y - y_0) + C.(z - z_0) = 0$$

The component equation is

$$5.(x + 3) + 2(y - 0) + (-1).(z - 7) = 0$$

Simplifying we obtain

$$5x + 2y - z = -22$$

Question

Find an equation for the plane through A(0, 0, 1); B(2, 0, 0) and C(0, 3, 0).

First we find the normal to the plane

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = 3.\mathbf{i} + 2.\mathbf{j} + 6.\mathbf{k}$$

Now we can use one the points and the normal to find the equation of the plane. Let's choose point A.

$$3.(x - 0) + 2.(y - 0) + 6.(z - 1) = 0$$

$$3.x + 2.y + 6.z = 6$$

Planes

The solution to the previous question gives the following general result. If we have a point and two vectors then we can find the Cartesian equation of the plane using

$$\overrightarrow{P_0P}.(\mathbf{v} \times \mathbf{u}) = 0$$

Where **v** and **u** are the two coplanar vectors.

In the solution to the previous question we derived our two in plane vectors using points. Hence, if we are given three points we can generalise the solution to

$$\overrightarrow{P_0P}.\left(\left(A - P_0\right) \times \left(B - P_0\right)\right) = 0$$

Where A and B are positional vectors.

Notice that if

$$\mathbf{n} = A.\mathbf{i} + B.\mathbf{j} + C.\mathbf{k}$$

Then

$$A.x + B.y + C.z = D$$

<u>This will be useful later</u>

Planes

The angle between two planes is given by the angle between their normals.

<u>Question</u>

Find the angle between the planes **3.x - 6.y - 2.z = 15** & **2.x + y – 2.z = 5**

$$\mathbf{n}_1 = \begin{pmatrix} 3 & -6 & -2 \end{pmatrix} \qquad \mathbf{n}_2 = \begin{pmatrix} 2 & 1 & -2 \end{pmatrix}$$

We know that $\mathbf{n}_1 \bullet \mathbf{n}_2 = |\mathbf{n}_1|.|\mathbf{n}_2|.\mathrm{Cos}(\theta)$ Thus $\theta = \mathrm{Cos}^{-1}\left(\dfrac{\mathbf{n}_1 \bullet \mathbf{n}_2}{|\mathbf{n}_1|.|\mathbf{n}_2|}\right)$

Hence $\theta = \mathrm{Cos}^{-1}\left(\dfrac{3.2 + (-6).1 + (-2).(-2)}{\sqrt{49.9}}\right) = \mathrm{Cos}^{-1}\left(\dfrac{4}{21}\right) = 1.38\mathrm{rad}$

Questions

Sometimes we want to know where a line and a plane intersect. This is particularly useful in computer graphics.

Find the point where the line

$$x = \frac{8}{3} + 2.t \qquad y = -2.t \qquad z = 1 + t$$

Intersects the plane **3x + 2y +6z = 6**

$$3.\left(\frac{8}{3} + 2.t\right) + 2.(-2.t) + 6.(1 + t) = 6$$

$$8 + 6.t - 4.t + 6 + 6.t = 6$$

$$t = -1$$

Hence the point of intersection is $\begin{pmatrix} \frac{2}{3} & 2 & 0 \end{pmatrix}$

Conclusion

MyMathLab: HG1M12 Example Sheet 1b

Essential reading for next week

HELM Workbook 9.5 Lines and Planes

We have covered 4 topics today

1. **The Dot Product**
2. **The Cross Product**
3. **The Scalar Triple Product**
4. **The Vector Moment**