ENTC 3331
RF FUNDAMENTALS

Fall 2006

Chapter 3

Vector Analysis

Vector Analysis

Scalar Quantity
(mass, speed, voltage, current and power)

1- Real number (one variable)

2- Complex number (two variables)

Vector Algebra
(velocity, electric field and magnetic field)

Magnitude & direction

Specified by one of the following coordinates best applied to application:

1- Cartesian (rectangular)
2- Cylindrical
3- Spherical

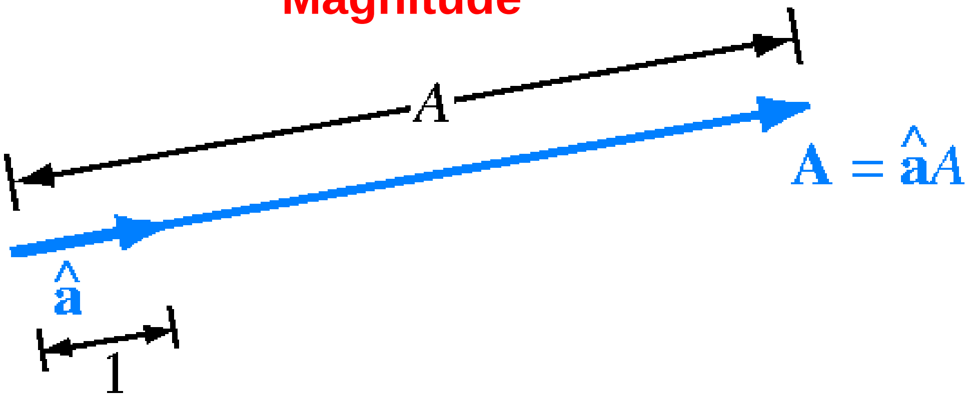
Conventions

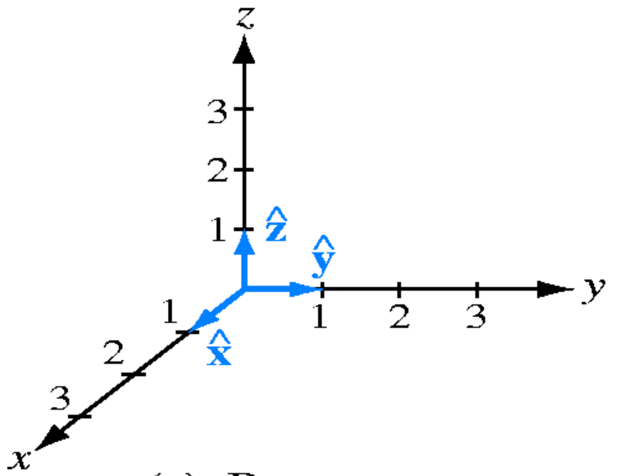
- Vector quantities denoted as $\vec{v}$ or ***v***
- We will use column format vectors:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \neq \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \quad \left(= \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T \right)$$

- Each vector is defined with respect to a set of basis vectors (which define a co-ordinate system).

Magnitude





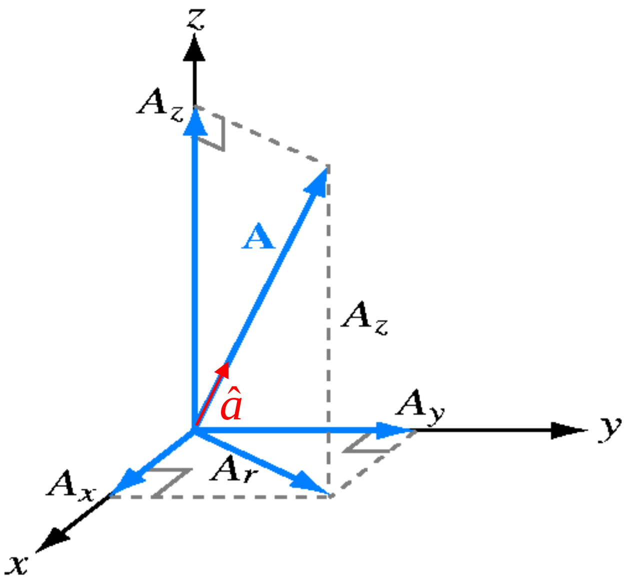
(a) Base vectors

Unit vector

$$\vec{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z$$

$$A = \left|\vec{A}\right| = {}^{+}\sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$\hat{a} = \frac{\vec{A}}{A} = \frac{\hat{x}A_x + \hat{y}A_y + \hat{z}A_z}{{}^{+}\sqrt{A_x^2 + A_y^2 + A_z^2}}$$



(b) Components of **A**

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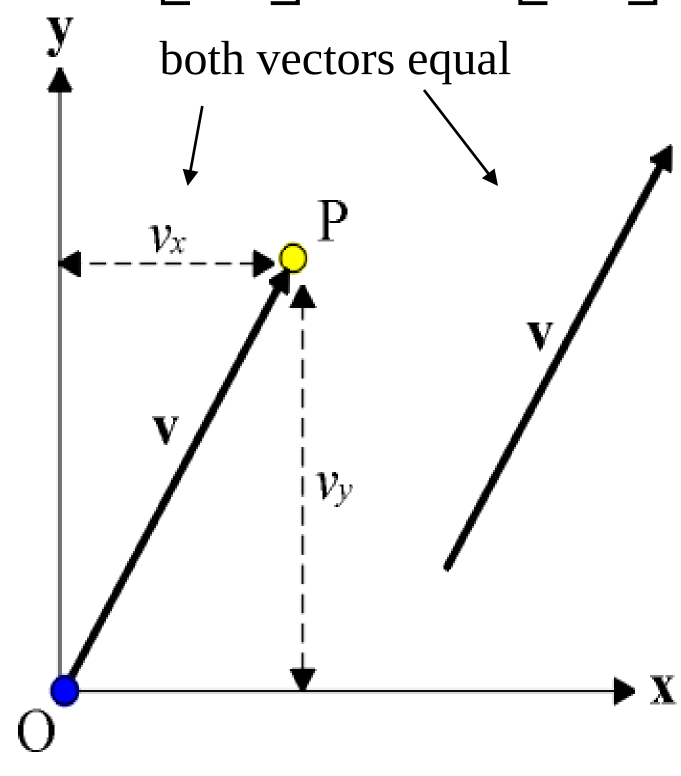
Vectors

- Vectors represent directions
  - points represent positions
- Both are meaningless without reference to a *coordinate system*
  - vectors require a set of *basis vectors*
  - points require an *origin* and a *vector space*

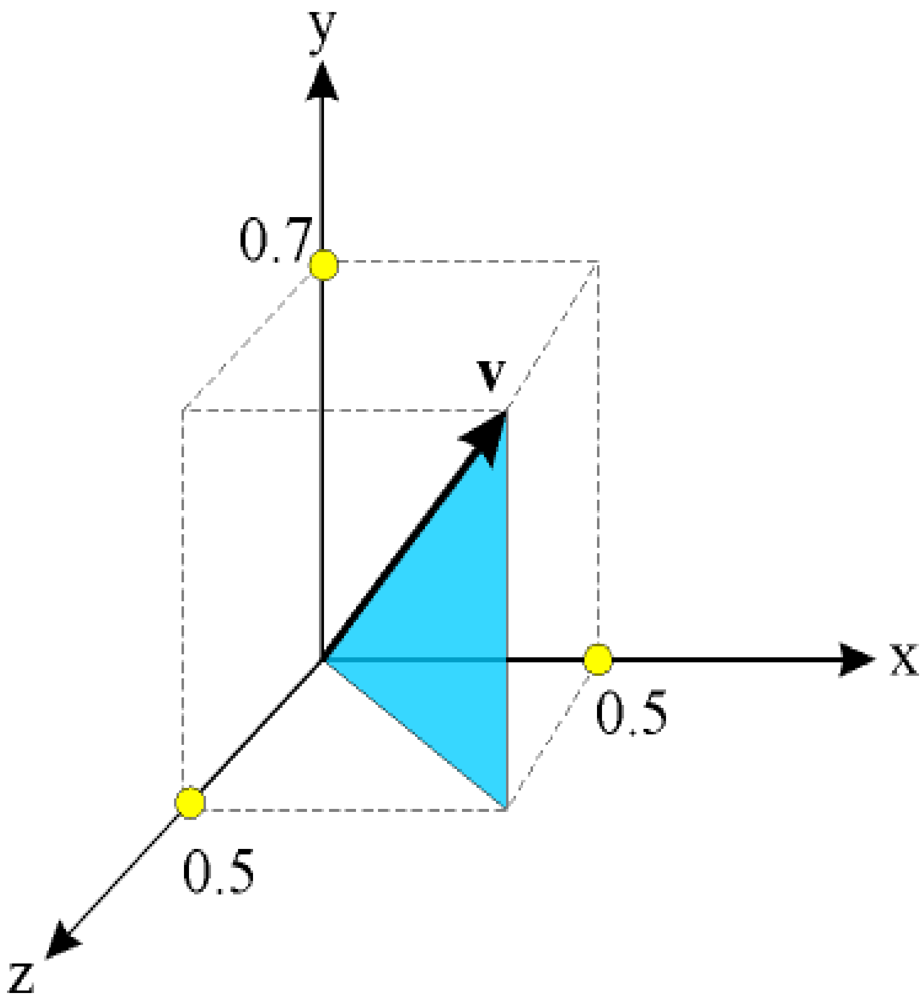
$$\mathbf{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix} \qquad \mathrm{P} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$



Co-ordinate Systems

- Until now you have probably used a *Cartesian basis*:
  - basis vectors are *mutually orthogonal* and *unit length*
  - basis vectors named **x**, **y** and **z**
- We need to define the relationship between the 3 vectors.

Cartesian Coordinate System



$$\mathbf{v} = \begin{bmatrix} 0.5 \\ 0.7 \\ 0.5 \end{bmatrix}$$

Vector Magnitude

- The *magnitude* or *norm* of a vector of dimension *n* is given by the standard *Euclidean distance metric*:

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

- For example:

$$\left\| \begin{matrix} 1 \\ 3 \\ 1 \end{matrix} \right\| = \sqrt{1^2 + 3^2 + 1^2} = \sqrt{11}$$

- Vectors of length 1(unit) are *normal vectors*.

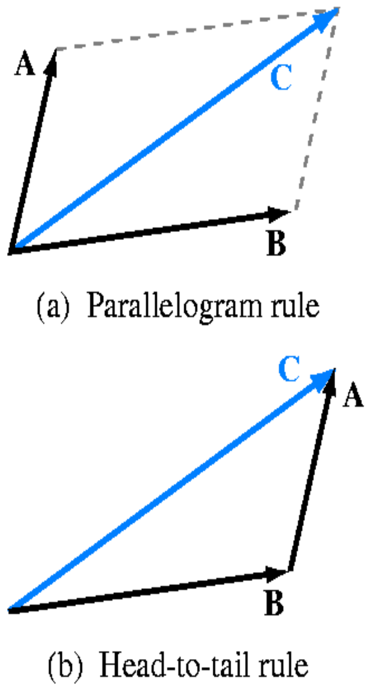
Normal Vectors

- When we wish to describe direction we use *normalized* vectors.
- We often need to *normalize* a vector:

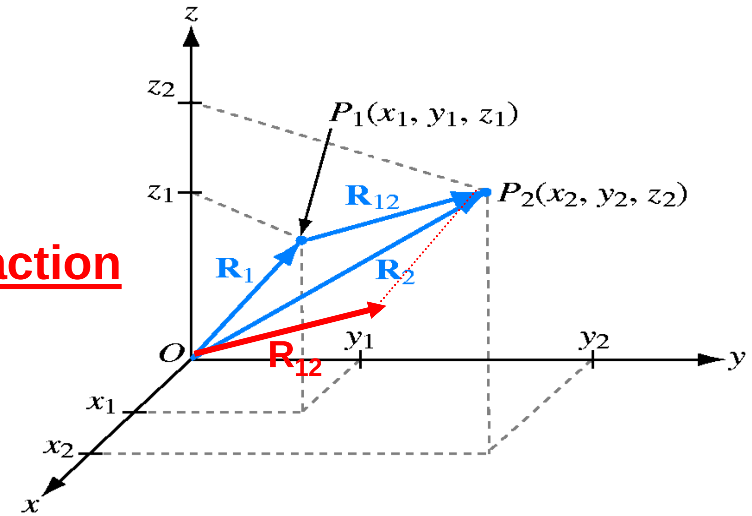
$$\mathbf{v}' = \frac{\mathbf{v}}{\left\|\mathbf{v}\right\|} = \frac{1}{\sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}}\mathbf{v}$$

Vector Addition and Subtraction

Addition



Subtraction



$$\overrightarrow{R_{12}} = \overrightarrow{P_1P_2} = \vec{R}_2 - \vec{R}_1$$

$$\vec{R}_{12} = \hat{x}(x_2 - x_1) + \hat{y}(y_2 - y_1) + \hat{z}(z_2 - z_1)$$

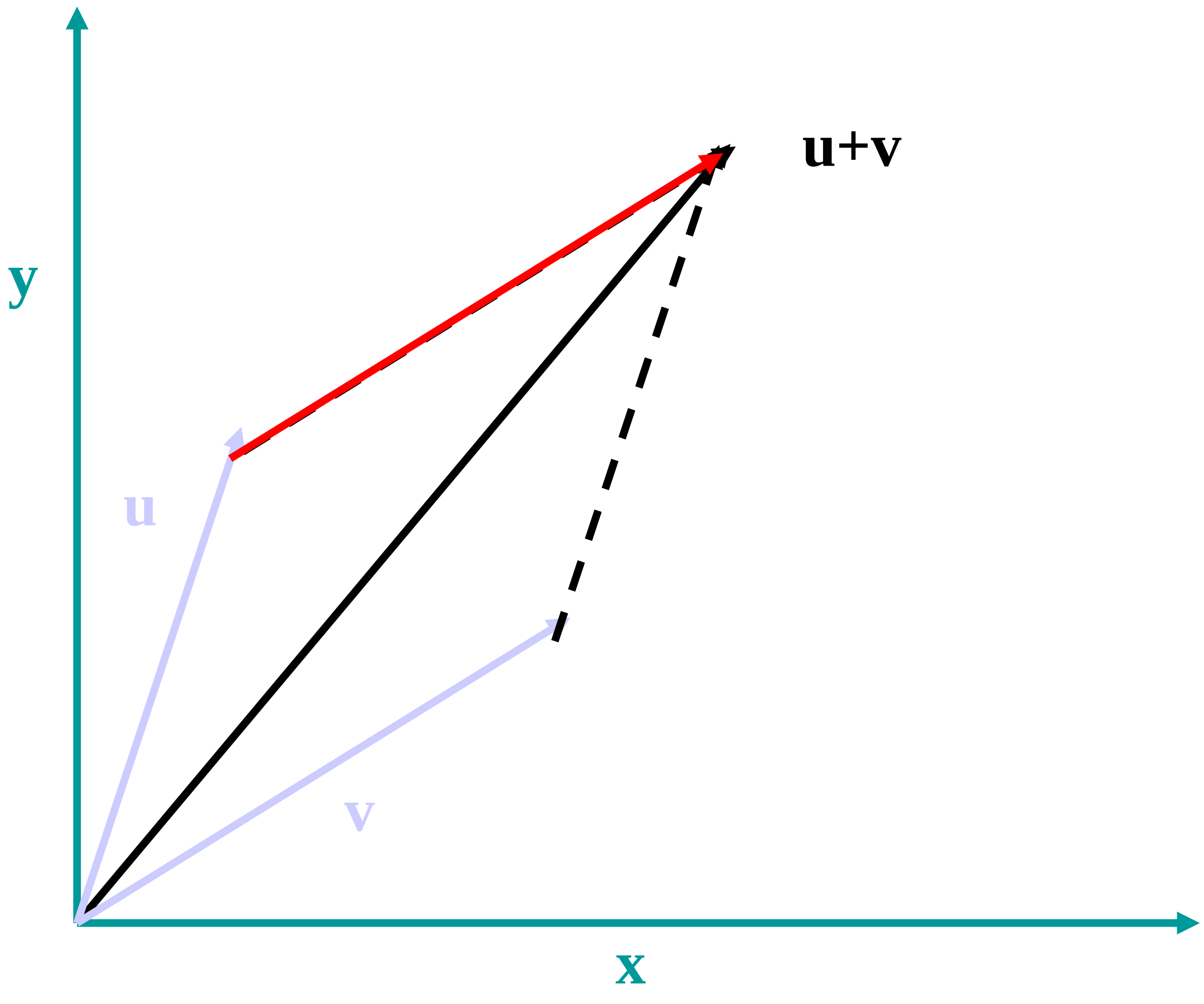
$$\left|\vec{R}_{12}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$\vec{A} = \hat{a}A = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z$$

$$\vec{B} = \hat{b}B = \hat{x}B_x + \hat{y}B_y + \hat{z}B_z$$

$$\vec{C} = \hat{c}C = \vec{A} \pm \vec{B} = \hat{a}A \pm \hat{b}B = \hat{x}(\underbrace{A_x \pm B_x}_{C_x}) + \hat{y}(\underbrace{A_y \pm B_y}_{C_y}) + \hat{z}(\underbrace{A_z \pm B_z}_{C_z})$$

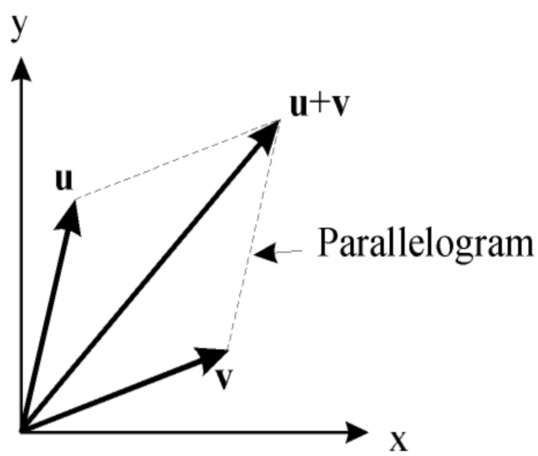
Vector Addition

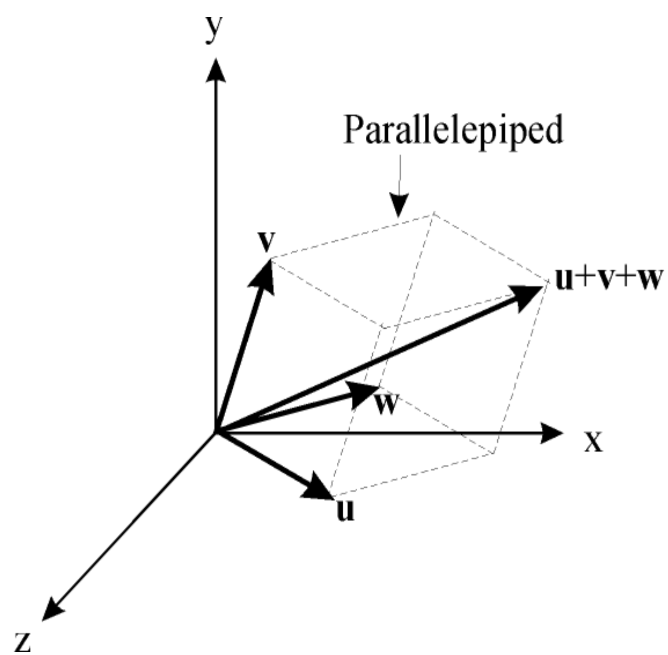


Vector Addition

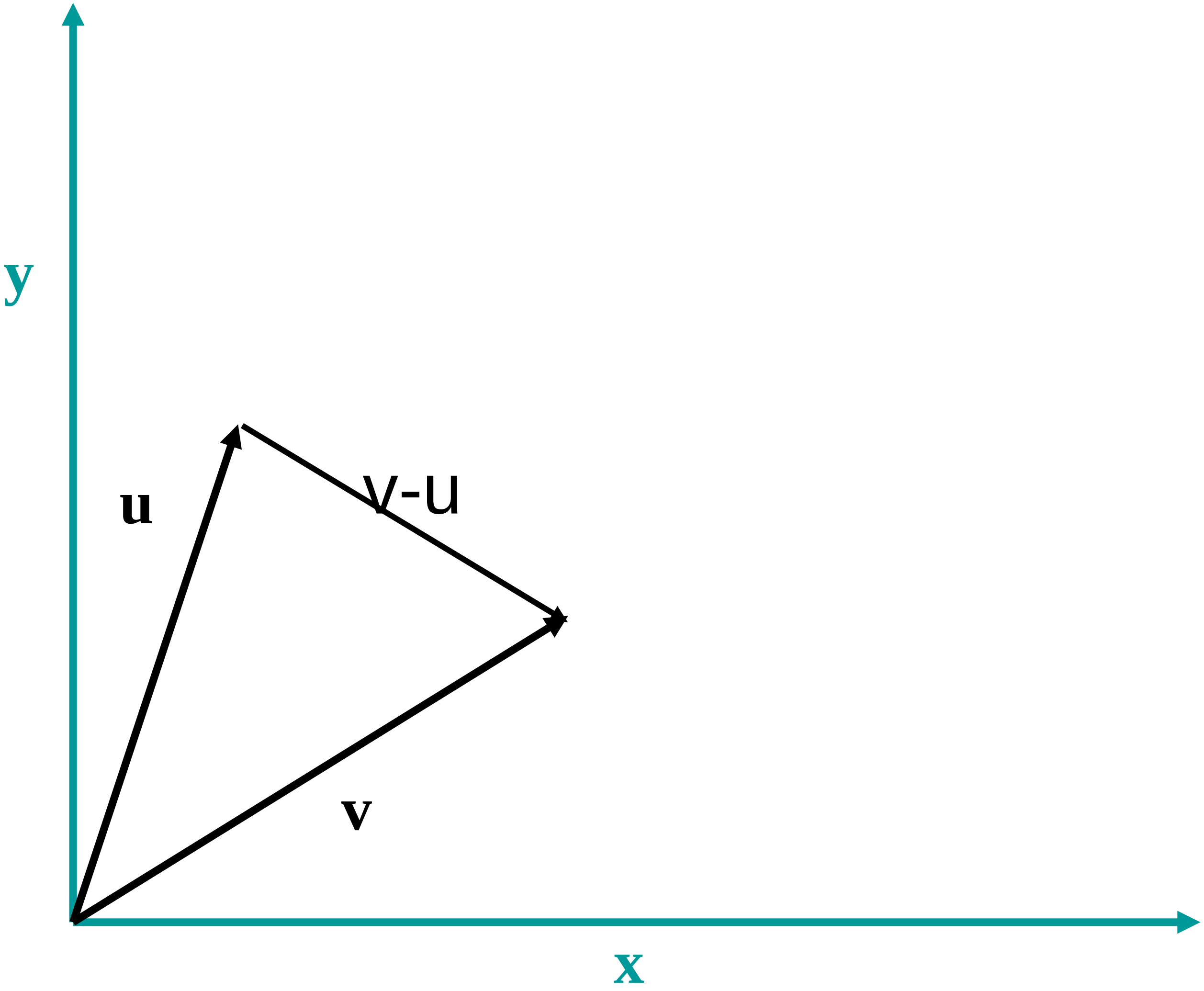
- Addition of vectors follows the *parallelogram* law in 2D and the *parallelepiped* law in higher dimensions:

$$\underset{\mathbf{u}}{\begin{bmatrix}1\\2\\3\end{bmatrix}}+\underset{\mathbf{v}}{\begin{bmatrix}4\\5\\6\end{bmatrix}}=\underset{\mathbf{u+v}}{\begin{bmatrix}5\\7\\9\end{bmatrix}}$$



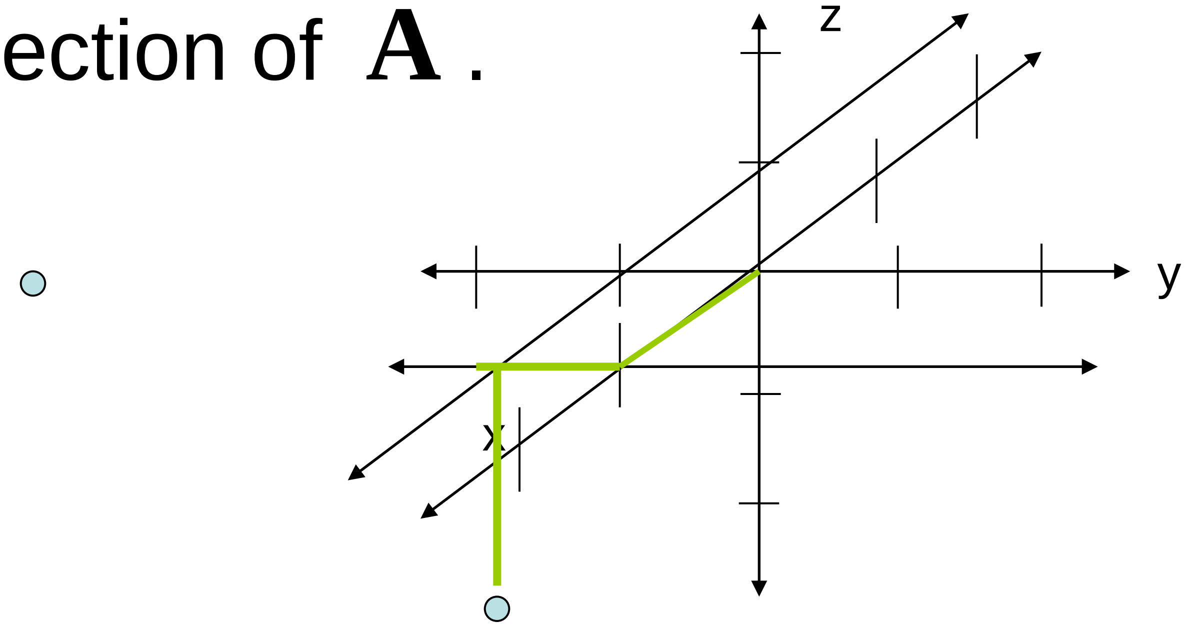


Vector Subtraction



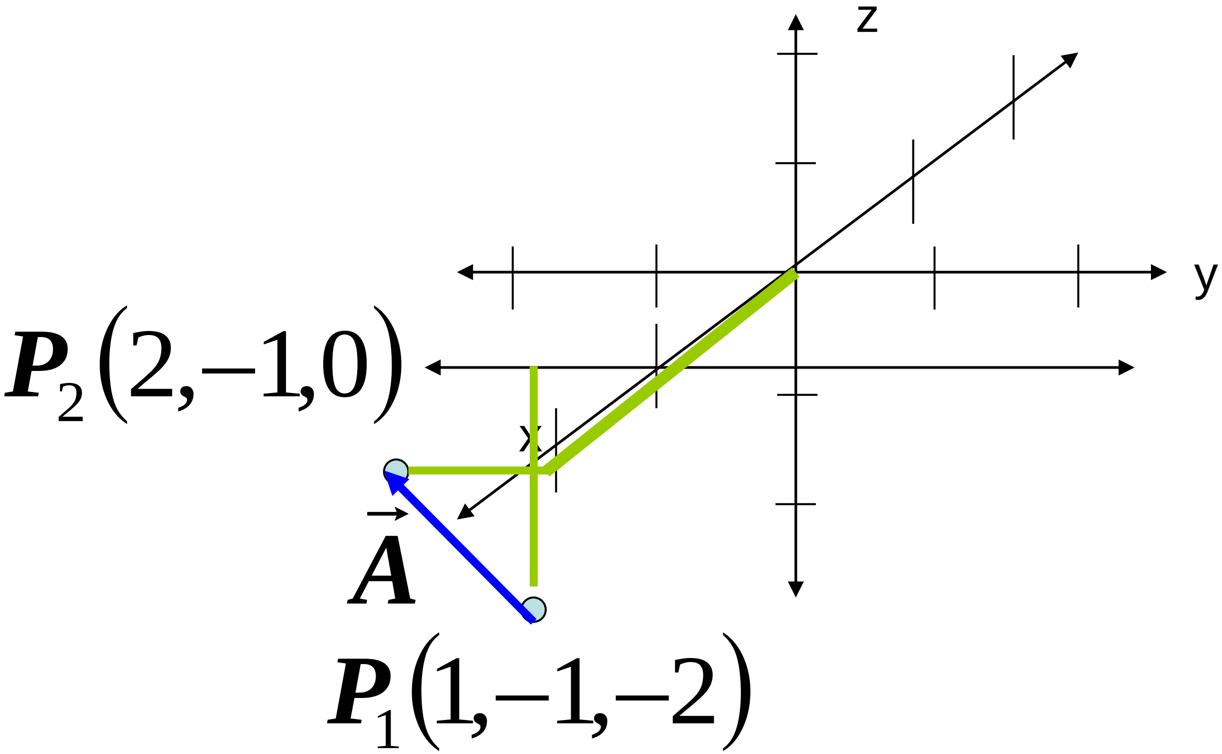
Problem 3-1

- Vector $\vec{\mathbf{A}}$ starts at point (1,-1,-2) and ends at point (2,-1,0). Find a unit vector in the direction of $\vec{\mathbf{A}}$.



$$\vec{\mathbf{A}} = \hat{\mathbf{x}}(2-1) + \hat{\mathbf{y}}(-1-(-1)) + \hat{\mathbf{z}}(0-(-2))$$

Problem 3-1



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$$\vec{\boldsymbol{A}} = \hat{\boldsymbol{x}} + \hat{\boldsymbol{z}}2$$

$$\left\|\vec{\mathbf{A}}\right\| = \sqrt{1+4} = \sqrt{5} = 2.24$$

$$\hat{\mathbf{a}} = \frac{\vec{\mathbf{A}}}{\left|\vec{\mathbf{A}}\right|} = \frac{\hat{\mathbf{x}} + \hat{\mathbf{z}}2}{2.24} = \hat{\mathbf{x}}0.45 + \hat{\mathbf{z}}0.89$$

Vector Multiplication

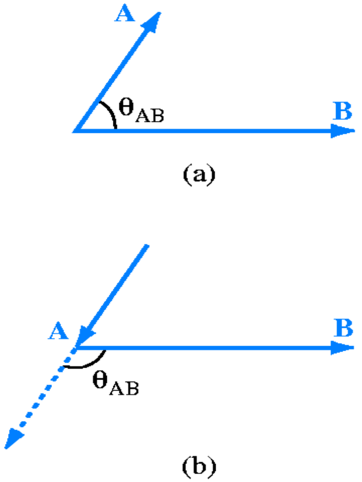
1- Simple Product
2- Scalar or Dot Product
3- Vector or Cross Product

1- Simple Product

$$\vec{B} = k\vec{A} = \hat{a}kA = \hat{x}(kA_x) + \hat{y}(kA_y) + \hat{z}(kA_z)$$

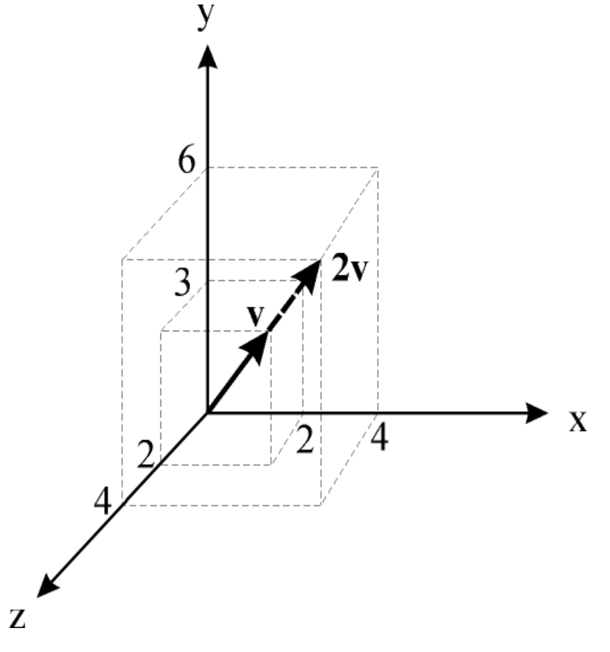
Scalar or Dot Product

$$\vec{A} \cdot \vec{B} = AB\cos\Theta_{AB}$$



Vector Multiplication by a Scalar

- Each vector has an associated *length*
- Multiplication by a scalar scales the vectors length appropriately (but does *not* affect direction):



Dot Product

- Dot product (*inner product*) is defined as:

$$\mathbf{u} \cdot \mathbf{v} = \sum_i u_i v_i$$

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta$$

Dot Product

- Note:

$$\mathbf{u} \cdot \mathbf{u} = u_1^2 + u_2^2 + u_3^2 = \|\mathbf{u}\|^2$$

- Therefore we can redefine magnitude in terms of the dot-product operator:

$$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$$

- Dot product operator is *commutative* & *distributive*.

Problem 3.2

- Given vectors:

$$\vec{\mathbf{A}} = \hat{\mathbf{x}}2 - \hat{\mathbf{y}}3 + \hat{\mathbf{z}}$$

$$\vec{\mathbf{B}} = \hat{\mathbf{x}}2 - \hat{\mathbf{y}} + \hat{\mathbf{z}}3$$

$$\vec{\mathbf{C}} = \hat{\mathbf{x}}4 + \hat{\mathbf{y}}2 - \hat{\mathbf{z}}2$$

  – show that $\vec{\mathbf{C}}$ is perpendicular to both $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$.

Recall

$$\hat{\mathbf{x}} \bullet \hat{\mathbf{x}} = |x||x| \cos 0^\circ = 1$$

$$\hat{\mathbf{y}} \bullet \hat{\mathbf{y}} = |y||y| \cos 0^\circ = 1$$

$$\hat{\mathbf{z}} \bullet \hat{\mathbf{z}} = |z||z| \cos 0^\circ = 1$$

Also, recall

$$\hat{\mathbf{x}} \bullet \hat{\mathbf{y}} = \hat{\mathbf{x}} \bullet \hat{\mathbf{z}} = \hat{\mathbf{y}} \bullet \hat{\mathbf{z}} = 0$$

Similarly

$$\vec{A} \bullet \vec{B} = 0$$

If the angle between the two vectors is 90°.

Problem 3.2

$$\vec{\mathbf{A}} \bullet \vec{\mathbf{C}} = \left(\hat{\mathbf{x}}2 - \hat{\mathbf{y}}3 + \hat{\mathbf{z}}\right) \bullet \left(\hat{\mathbf{x}}4 + \hat{\mathbf{y}}2 - \hat{\mathbf{z}}2\right) = 8 - 6 - 2 = 0$$

$$\vec{\mathbf{B}} \bullet \vec{\mathbf{C}} = \left(\hat{\mathbf{x}}2 - \hat{\mathbf{y}} + \hat{\mathbf{z}}3\right) \bullet \left(\hat{\mathbf{x}}4 + \hat{\mathbf{y}}2 - \hat{\mathbf{z}}2\right) = 8 - 2 - 6 = 0$$

$$\vec{A}\cdot\vec{B} = AB\cos\Theta_{AB}$$

$$\vec{A}\cdot\vec{B} = (\hat{x}A_x + \hat{y}A_y + \hat{z}A_z)\cdot(\hat{x}B_x + \hat{y}B_y + \hat{z}B_z)$$

$$\hat{x}\cdot\hat{x} = \hat{y}\cdot\hat{y} = \hat{z}\cdot\hat{z} = 1$$

$$\hat{x}\cdot\hat{y} = \hat{y}\cdot\hat{z} = \hat{z}\cdot\hat{x} = 0$$

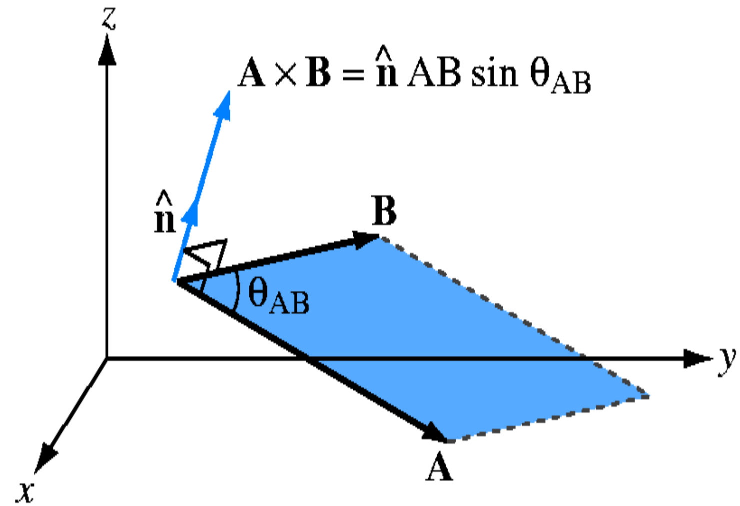
$$A\cdot B = A_x B_x + A_y B_y + A_z B_z$$

$$A = \left|\vec{A}\right| = \sqrt[+]{\vec{A}\cdot\vec{A}} \qquad \Theta_{AB} = \cos^{-1}\left[\frac{\vec{A}\cdot\vec{B}}{\left|\vec{A}\right|\left|\vec{B}\right|}\right]$$

$$\begin{cases} \vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A} & \text{Commutative property} \\ \vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C} & \text{Distributive property} \end{cases}$$

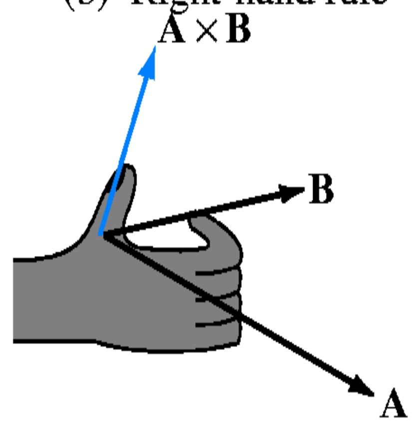
Vector or Cross Product

$$\vec{A}\times\vec{B}=\hat{n}AB\sin\theta_{AB}$$



(a) Cross product

(b) Right-hand rule



$$\vec{A}\times\vec{B}=(\hat{x}A_x+\hat{y}A_y+\hat{z}A_z)\times(\hat{x}B_x+\hat{y}B_y+\hat{z}B_z)$$

$$\hat{x}\times\hat{y}=\hat{z}\qquad\hat{y}\times\hat{z}=\hat{x}\qquad\hat{z}\times\hat{x}=\hat{y}$$

$$\hat{x}\times\hat{x}=\hat{y}\times\hat{y}=\hat{z}\times\hat{z}=0$$

$$\boxed{\vec{A}\times\vec{B}=\hat{x}(A_yB_z-A_zB_y)+\hat{y}(A_zB_x-A_xB_z)+\hat{z}(A_xB_y-A_yB_x)}$$

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Show

$$\boxed{\vec{A}\times\vec{B}=\hat{x}(A_yB_z-A_zB_y)+\hat{y}(A_zB_x-A_xB_z)+\hat{z}(A_xB_y-A_yB_x)}$$

$$\vec{\mathbf{A}}=\hat{\mathbf{x}}A_x+\hat{\mathbf{y}}A_y+\hat{\mathbf{z}}A_z$$

$$\vec{\mathbf{B}}=\hat{\mathbf{x}}B_x+\hat{\mathbf{y}}B_y+\hat{\mathbf{z}}B_z$$

Recall

$$\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}} \qquad \hat{\mathbf{y}} \times \hat{\mathbf{x}} = -\hat{\mathbf{z}}$$

$$\hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}} \qquad \hat{\mathbf{z}} \times \hat{\mathbf{y}} = -\hat{\mathbf{x}}$$

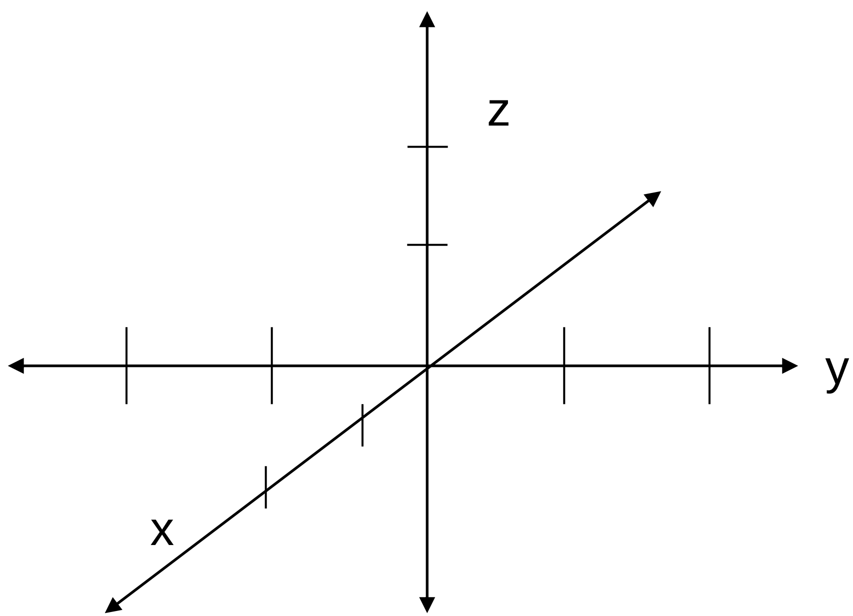
$$\hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}} \qquad \hat{\mathbf{x}} \times \hat{\mathbf{z}} = -\hat{\mathbf{y}}$$

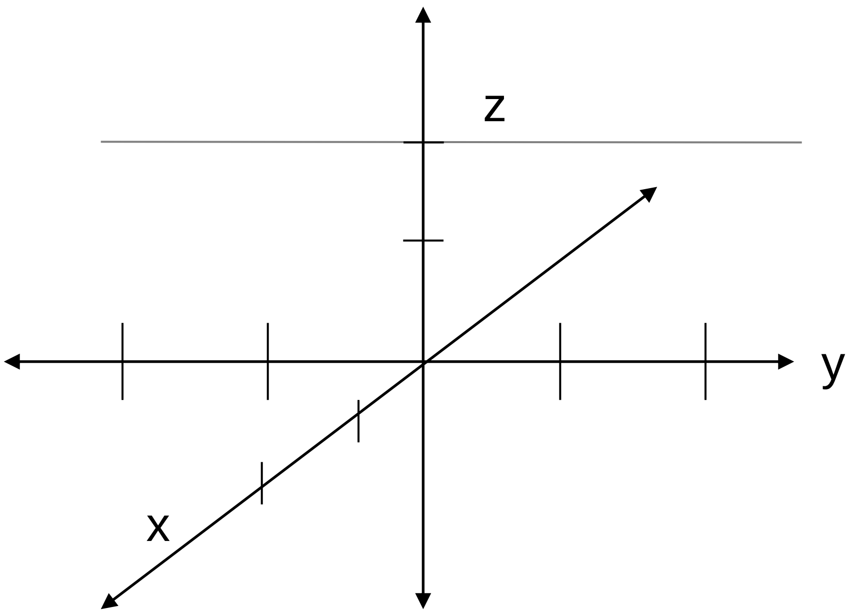
$$
\begin{aligned}
\vec{\mathbf{A}}\times\vec{\mathbf{B}} &= \left(\hat{\mathbf{x}}A_x+\hat{\mathbf{y}}A_y+\hat{\mathbf{z}}A_z\right)\times\left(\hat{\mathbf{x}}B_x+\hat{\mathbf{y}}B_y+\hat{\mathbf{z}}B_z\right)\\
&= \hat{\mathbf{z}}\left(A_xB_y\right)-\hat{\mathbf{y}}\left(A_xB_z\right)-\hat{\mathbf{z}}\left(A_yB_x\right)+\hat{\mathbf{x}}\left(A_yB_z\right)+\hat{\mathbf{y}}\left(A_zB_x\right)-\hat{\mathbf{x}}\left(A_zB_y\right)\\
&= \hat{\mathbf{x}}\left(A_yB_z-A_zB_y\right)+\hat{\mathbf{y}}\left(A_zB_x-A_xB_z\right)+\hat{\mathbf{z}}\left(A_xB_y-A_yB_x\right)
\end{aligned}
$$

$$
\boxed{\vec{A}\times\vec{B}=\hat{x}(A_yB_z-A_zB_y)+\hat{y}(A_zB_x-A_xB_z)+\hat{z}(A_xB_y-A_yB_x)}
$$

Problem 3.3

- In the Cartesian coordinate system, the three corners of a triangle are P1(0,2,2), P2(2,-2,2), and P3(1,1,-2). Find the area of the triangle.


z
y
x


z
y
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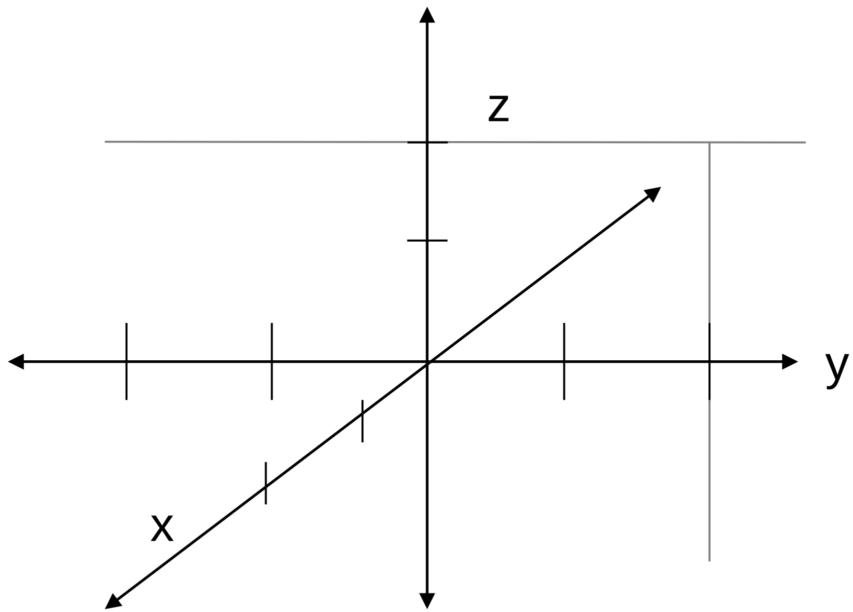
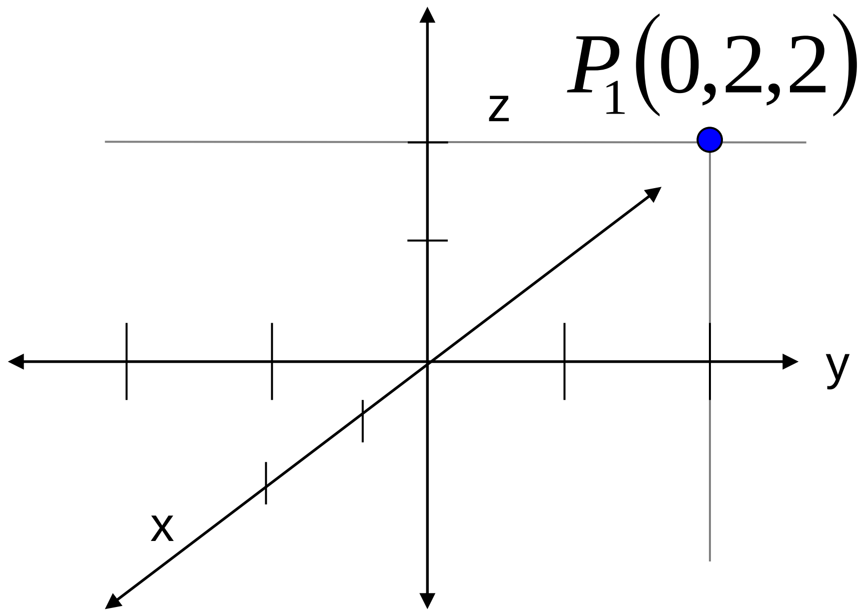
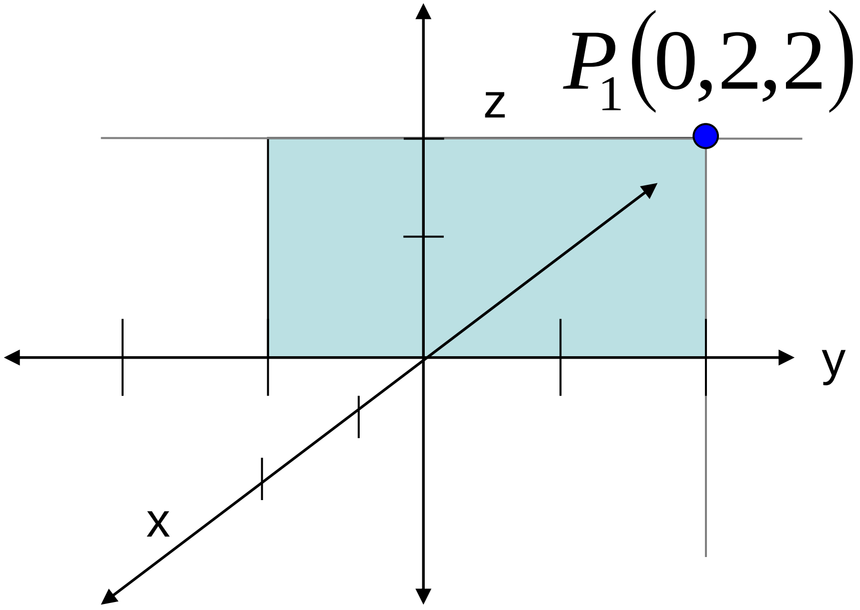
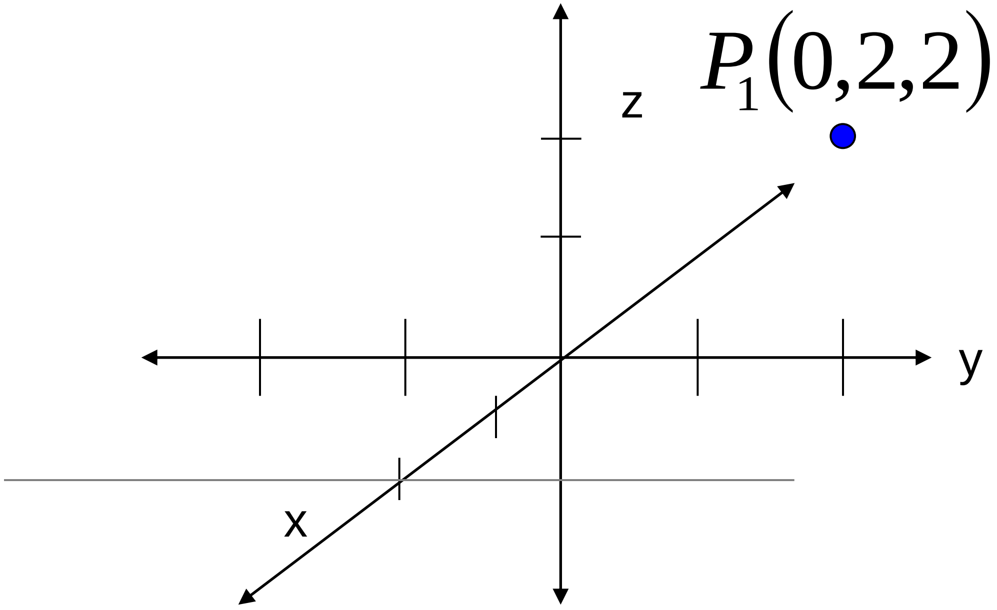

z
y
x

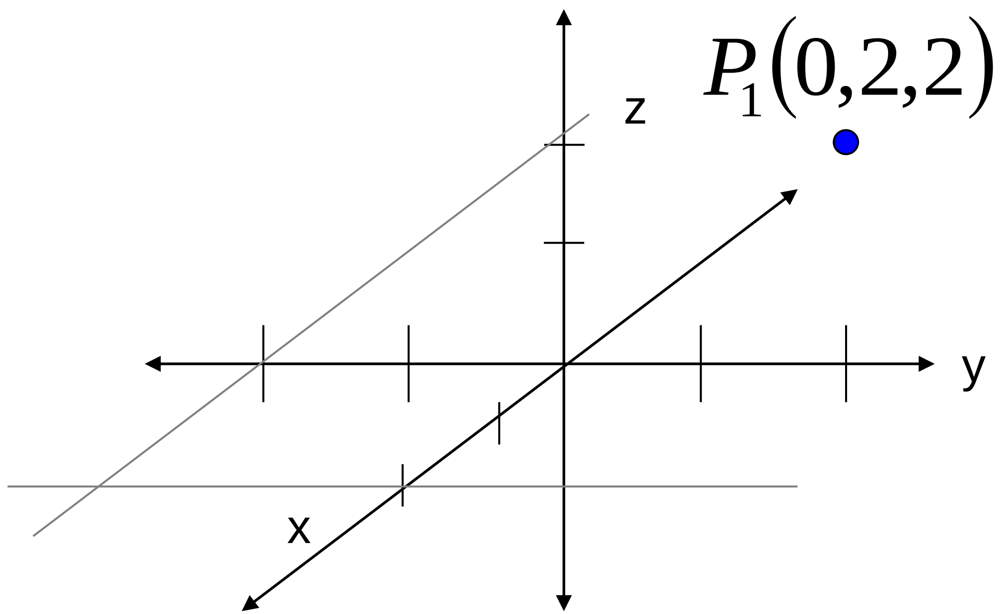

$P_1(0,2,2)$
z
y
x



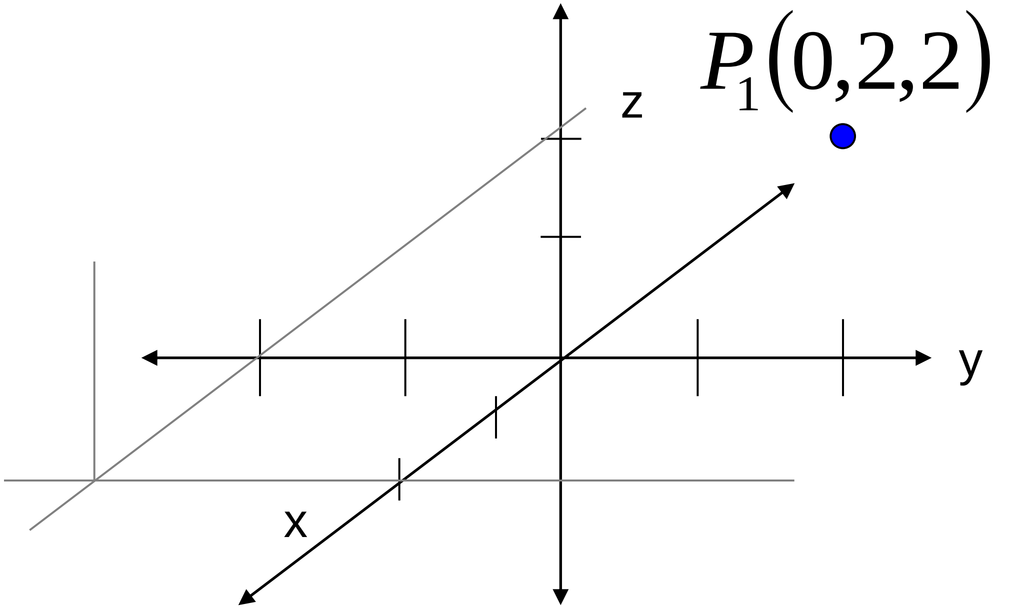
$P_1(0,2,2)$
z
y
x



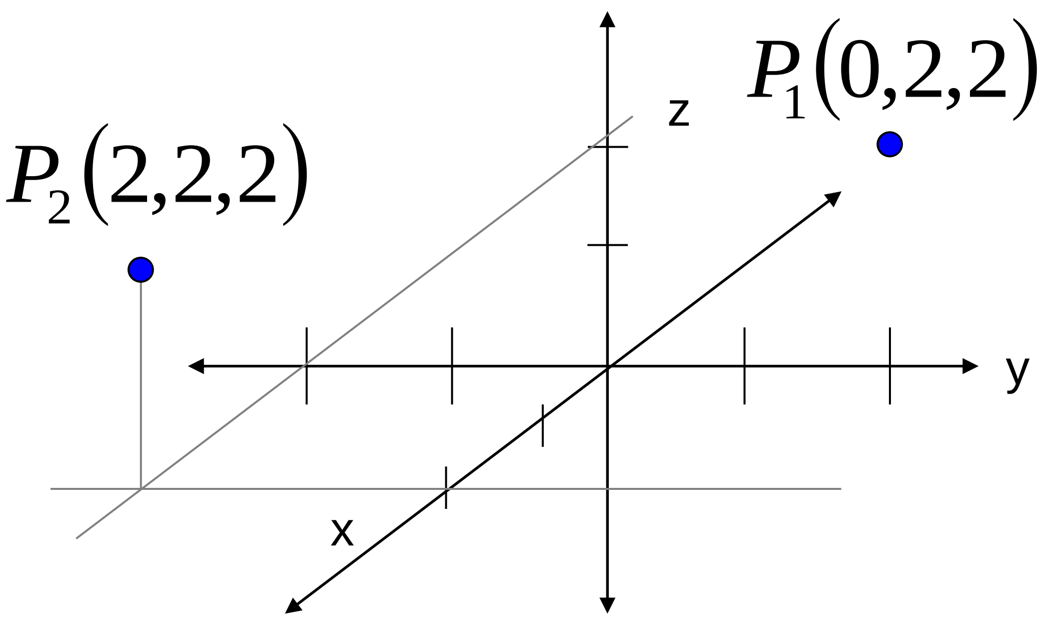
$P_1(0,2,2)$
z
y
x



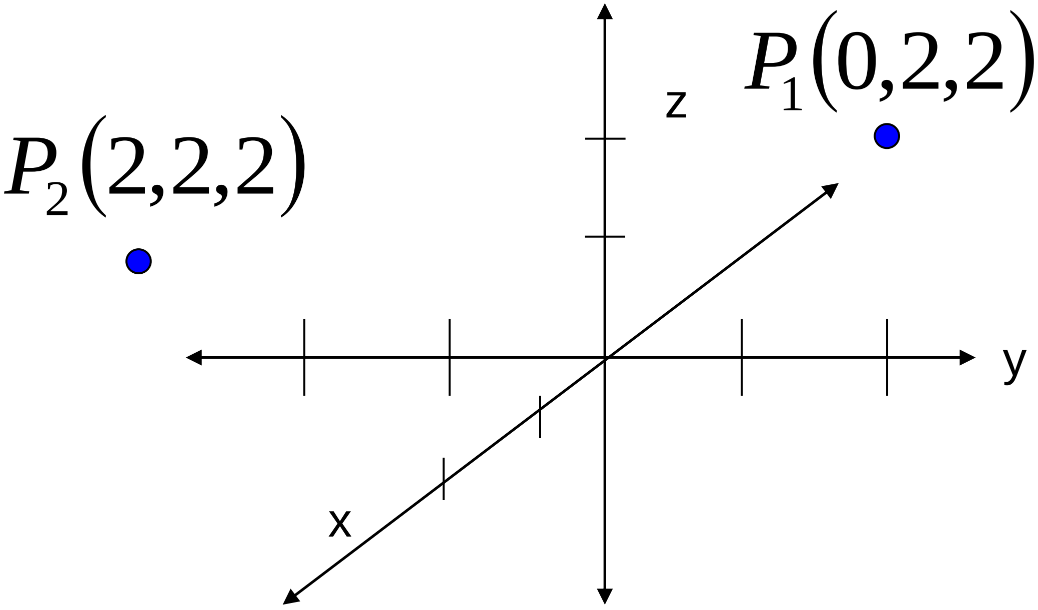
$P_1(0,2,2)$
z
y
x



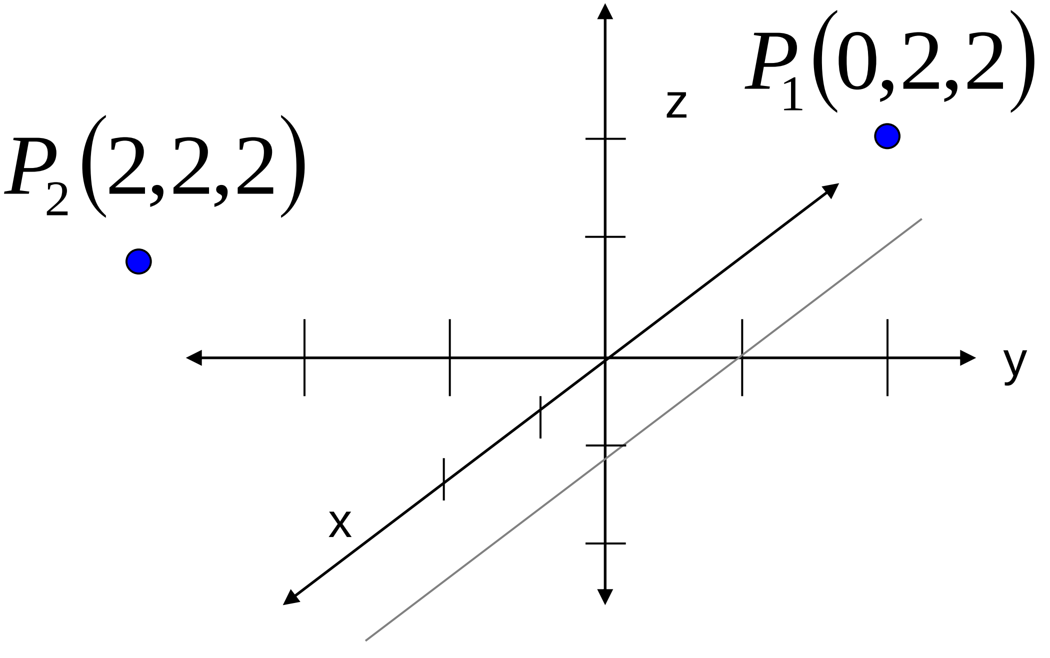
$P_1(0,2,2)$
z
y
x

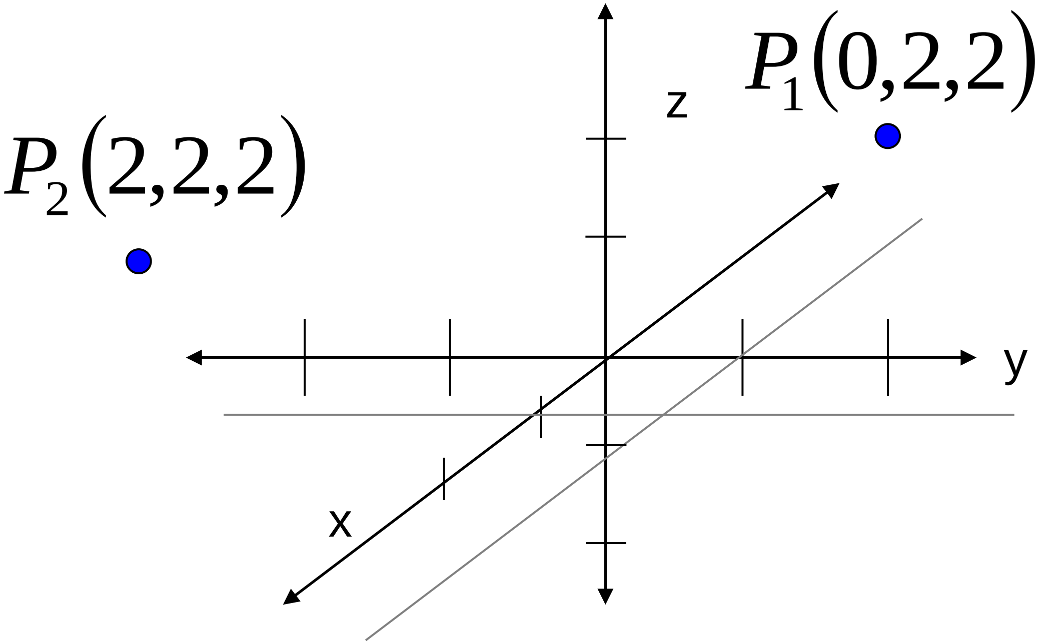


$P_2(2,2,2)$
$P_1(0,2,2)$
z
y
x

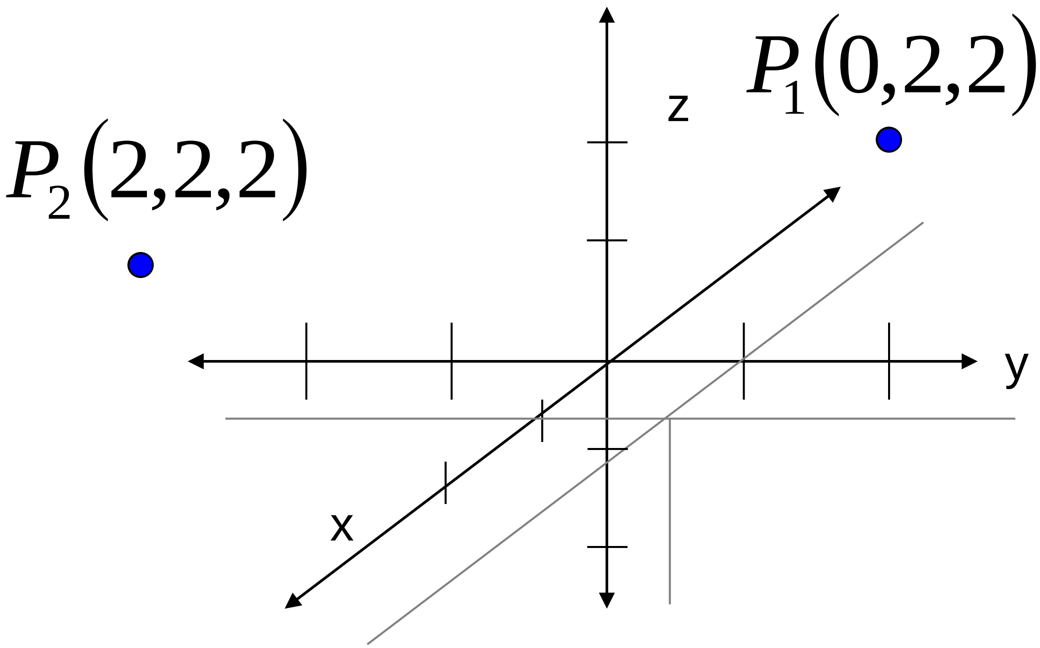


$P_1(0,2,2)$
$P_2(2,2,2)$
z
y
x

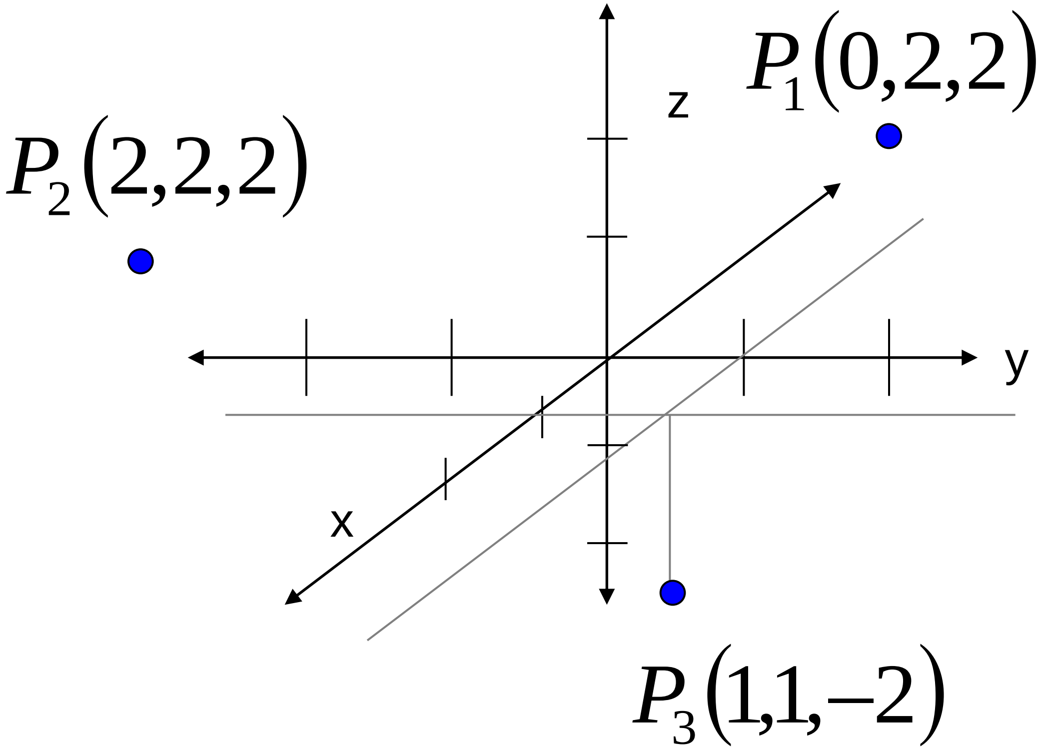

$P_1(0,2,2)$
$P_2(2,2,2)$
z
y
x

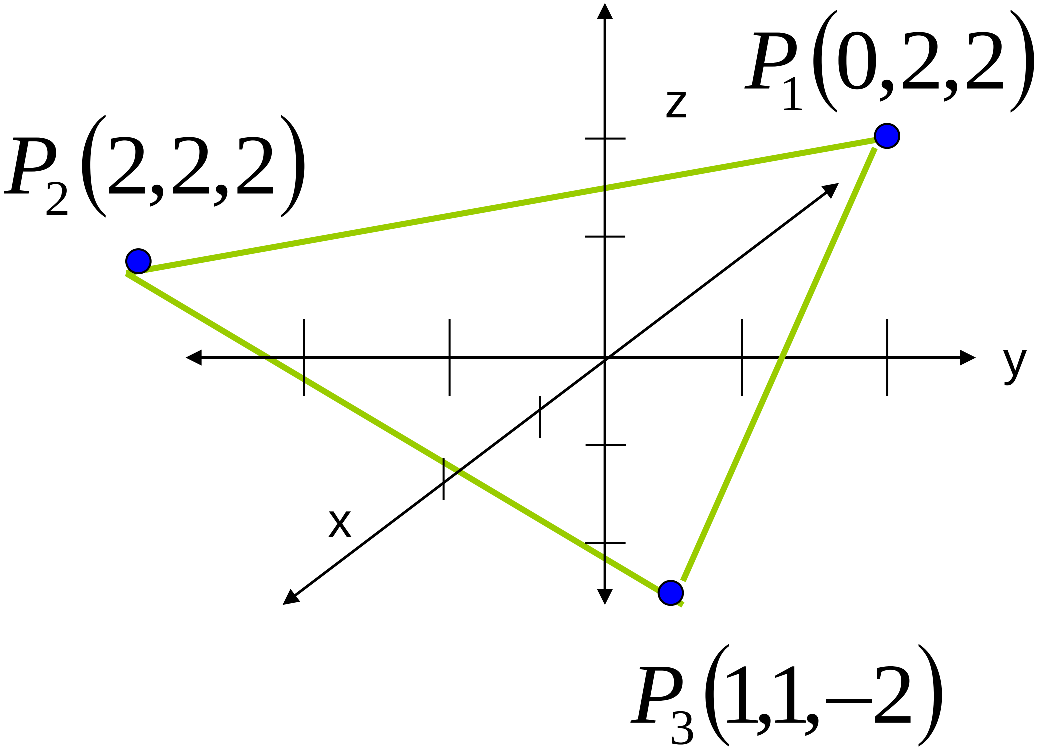


$P_2(2,2,2)$
z
$P_1(0,2,2)$
y
x



$P_2(2,2,2)$
$P_1(0,2,2)$
z
y
x


$P_1(0,2,2)$
z
$P_2(2,2,2)$
y
x
$P_3(1,1,-2)$



$P_1(0,2,2)$
$P_2(2,2,2)$
$P_3(1,1,-2)$
z
y
x

Let

$$\vec{\mathbf{B}} = \overrightarrow{P_1 P_2} = \hat{\mathbf{x}}2 - \hat{\mathbf{y}}4$$

and

$$\vec{\mathbf{C}} = \overrightarrow{P_1 P_3} = \hat{\mathbf{x}} - \hat{\mathbf{y}} - \hat{\mathbf{z}}4$$

represent two sides of the triangle.

Since the magnitude of the cross product is the area of the parallelogram, half of this magnitude is the area of the triangle.

$$A = \frac{1}{2}\left\|\vec{\mathbf{B}} \times \vec{\mathbf{C}}\right\| = \frac{1}{2}\left\|\left(\hat{\mathbf{x}}2 - \hat{\mathbf{y}}4\right) \times \left(\hat{\mathbf{x}} - \hat{\mathbf{y}} - \hat{\mathbf{z}}4\right)\right\|$$

$$= \frac{1}{2}\left\|2\cancelto{0}{\left(\hat{\mathbf{x}} \times \hat{\mathbf{x}}\right)} - 2\cancelto{\hat{\mathbf{z}}}{\left(\hat{\mathbf{x}} \times \hat{\mathbf{y}}\right)} - 8\cancelto{-\hat{\mathbf{y}}}{\left(\hat{\mathbf{x}} \times \hat{\mathbf{z}}\right)} - 4\cancelto{-\hat{\mathbf{z}}}{\left(\hat{\mathbf{y}} \times \hat{\mathbf{x}}\right)} + 4\cancelto{0}{\left(\hat{\mathbf{y}} \times \hat{\mathbf{y}}\right)} + 16\cancelto{\hat{\mathbf{x}}}{\left(\hat{\mathbf{y}} \times \hat{\mathbf{z}}\right)}\right\|$$

$$= \frac{1}{2}\left\|-2\hat{\mathbf{z}} + 8\hat{\mathbf{y}} + 4\hat{\mathbf{z}} + 16x\right\| = \frac{1}{2}\left\|16x + 8\hat{\mathbf{y}} + 2\hat{\mathbf{z}} +\right\|$$

$$= \frac{1}{2}\sqrt{16^2 + 8^2 + 2^2} = \frac{1}{2}\sqrt{256 + 64 + 4} = \frac{1}{2}\sqrt{324} = \frac{1}{2}(18) = 9$$

- The area of the triangle is 9 sq. units.

Scalar and Vector Triple Products

$$\vec{A}\times\vec{B}\times\vec{C} \quad ? \quad \boxed{\left(\vec{A}\times\vec{B}\right)\times\vec{C}} \neq \boxed{\vec{A}\times\left(\vec{B}\times\vec{C}\right)}$$

$$\vec{A}\cdot\vec{B}\cdot\vec{C} \quad ? \quad \vec{A}\cdot\left(\vec{B}\cdot\vec{C}\right) \quad ? \quad \left(\vec{A}\cdot\vec{B}\right)\cdot\vec{C} \quad ?$$

$$\vec{A}\times\vec{B}\cdot\vec{C} \quad ? \quad \vec{A}\times\left(\vec{B}\cdot\vec{C}\right) \quad ? \quad \boxed{\left(\vec{A}\times\vec{B}\right)\cdot\vec{C}}$$

$$\vec{A}\cdot\vec{B}\times\vec{C} \quad ? \quad \boxed{\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)} \quad \left(\vec{A}\cdot\vec{B}\right)\times\vec{C} \quad ?$$

Scalar Triple Product

$$\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)=\vec{B}\cdot\left(\vec{C}\times\vec{A}\right)=\vec{C}\cdot\left(\vec{A}\times\vec{B}\right)=\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Vector Triple Product

$$\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B})$$

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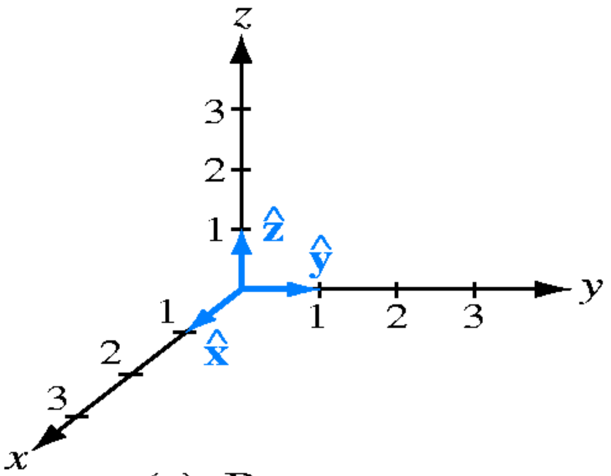
Summary

$$\vec{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z$$

$$A = \left|\vec{A}\right| = \sqrt[+]{A_x^2 + A_y^2 + A_z^2}$$

$$\hat{a} = \frac{\vec{A}}{A} = \frac{\hat{x}A_x + \hat{y}A_y + \hat{z}A_z}{\sqrt[+]{A_x^2 + A_y^2 + A_z^2}}$$



(a) Base vectors

Addition

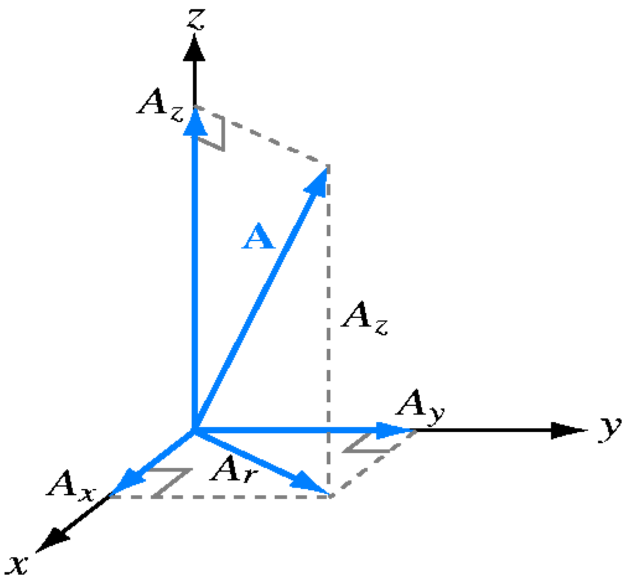
Subtraction

Vector Multiplication

1- Simple Product
2- Scalar or Dot Product
3- Vector or Cross Product

Scalar Triple Product

Vector Triple Product

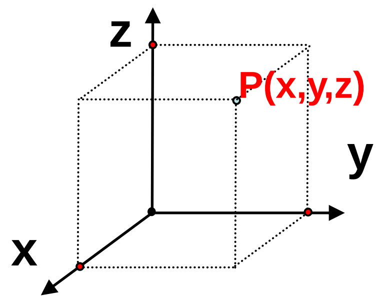


(b) Components of **A**

Orthogonal Coordinate Systems: (coordinates mutually perpendicular)

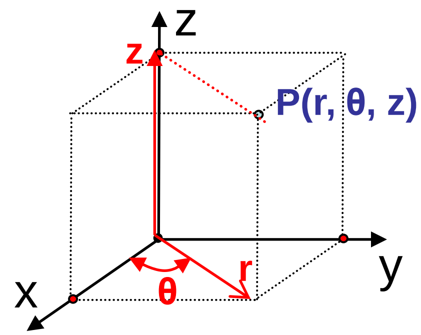
Cartesian Coordinates

P (x,y,z)



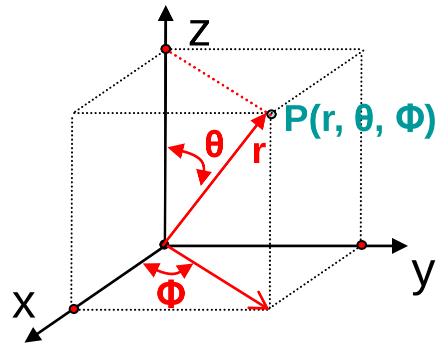
Cylindrical Coordinates

P (r, Θ, z)

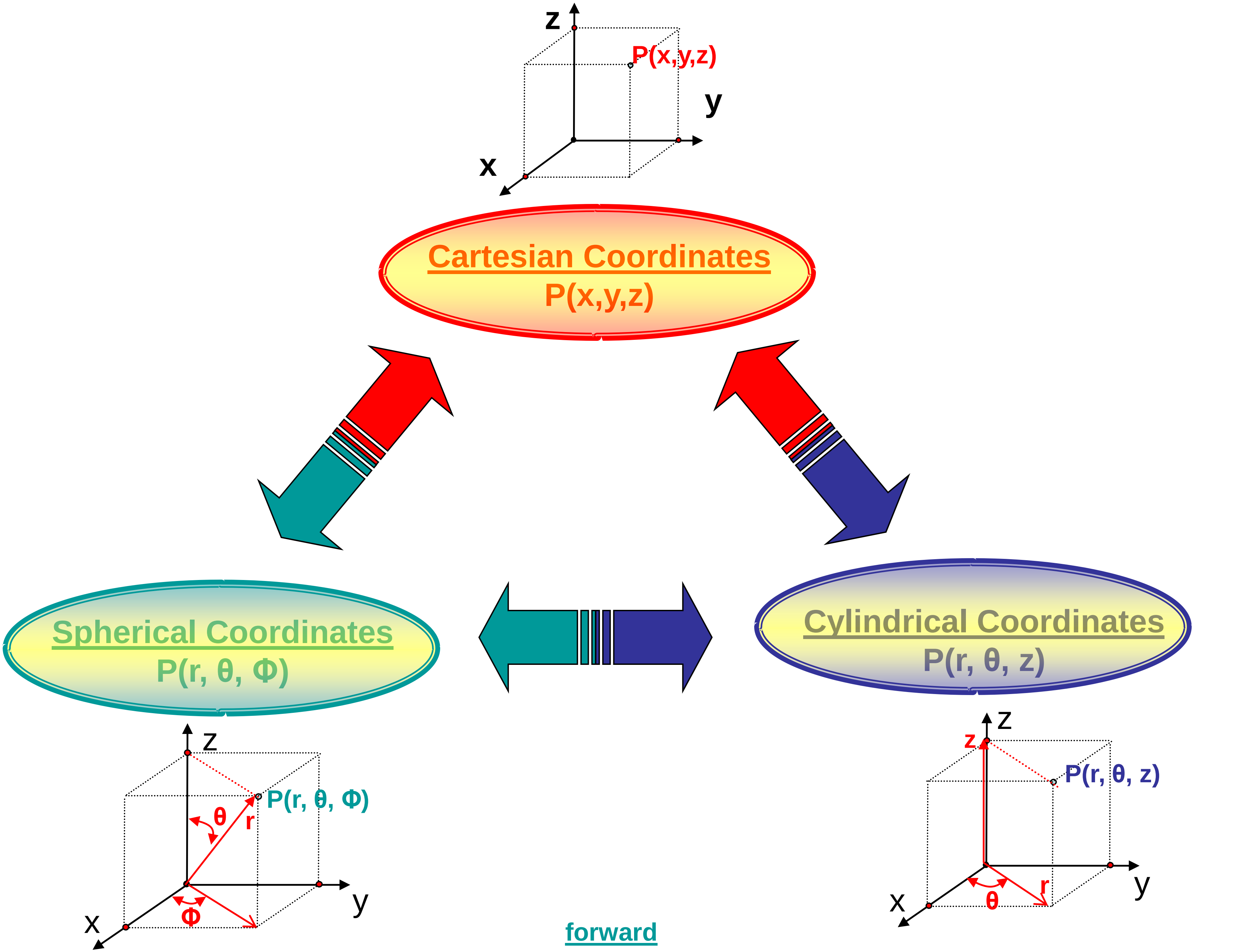


Spherical Coordinates

P (r, Θ, Φ)



-**Parabolic Cylindrical Coordinates (u,v,z)**
-**Paraboloidal Coordinates (u, v, Φ)**
-**Elliptic Cylindrical Coordinates (u, v, z)**
-**Prolate Spheroidal Coordinates (ξ, η, φ)**
-**Oblate Spheroidal Coordinates (ξ, η, φ)**
-**Bipolar Coordinates (u,v,z)**
-**Toroidal Coordinates (u, v, Φ)**
-**Conical Coordinates (λ, μ, ν)**
-**Confocal Ellipsoidal Coordinate (λ, μ, ν)**
-**Confocal Paraboloidal Coordinate (λ, μ, ν)**

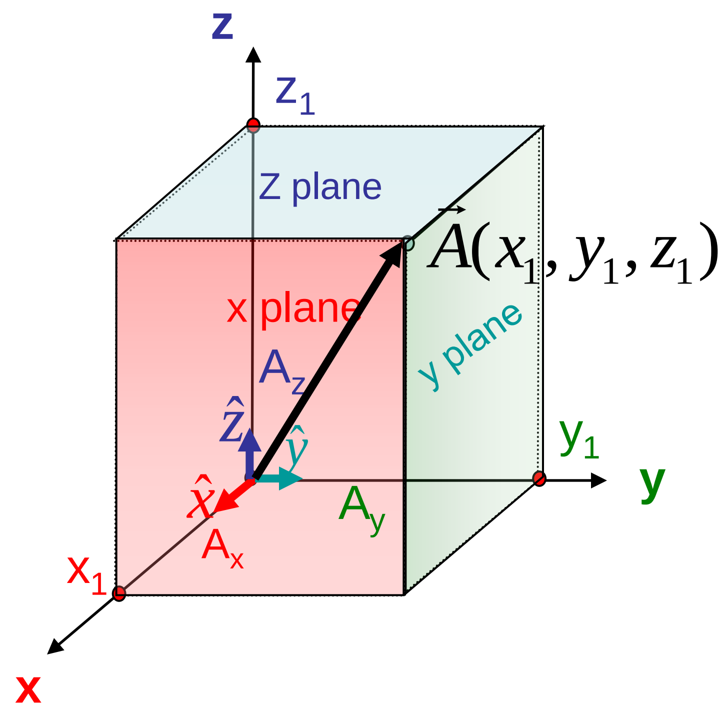


Cartesian Coordinates
P(x,y,z)
Spherical Coordinates
P(r, θ, Φ)
Cylindrical Coordinates
P(r, θ, z)
z
P(x,y,z)
y
x
z
P(r, θ, Φ)
θ
r
Φ
x
y
z
z
P(r, θ, z)
r
θ
x
y

forward

Cartesian Coordinates

(x, y, z)



Vector representation

$$\vec{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z$$

Magnitude of A

$$\left|\vec{A}\right| = \sqrt[+]{\vec{A} \cdot \vec{A}} = \sqrt[+]{A_x^2 + A_y^2 + A_z^2}$$

Position vector A

$$\hat{x}x_1 + \hat{y}y_1 + \hat{z}z_1$$

Base vector properties

$$\hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1$$

$$\hat{x} \cdot \hat{y} = \hat{y} \cdot \hat{z} = \hat{z} \cdot \hat{x} = 0$$

$$\hat{x} \times \hat{y} = \hat{z}$$

$$\hat{y} \times \hat{z} = \hat{x}$$

$$\hat{z} \times \hat{x} = \ddot{y}$$

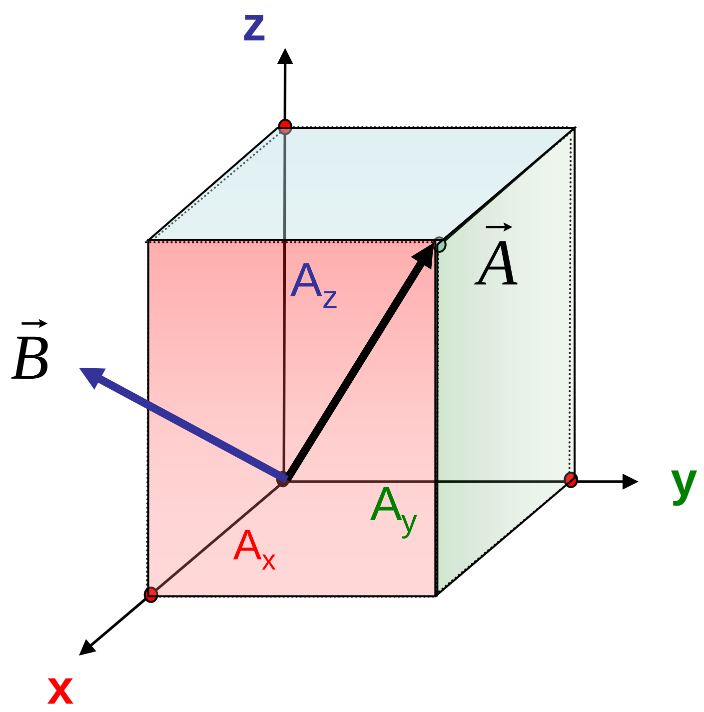
Cartesian Coordinates

Dot product:

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

Cross product:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$



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Cartesian Coordinates

Differential quantities:

Length:

$$d\vec{l} = \hat{x}dx + \hat{y}dy + \hat{z}dz$$

Area:

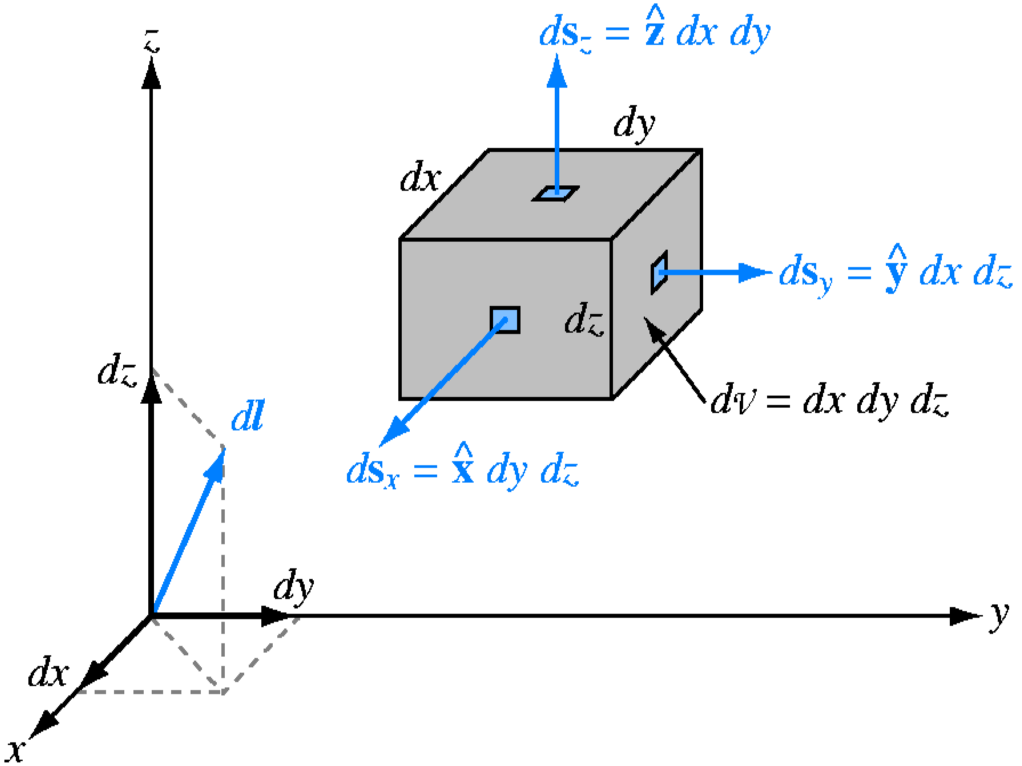
$$d\vec{s}_x = \hat{x}dydz$$

$$d\vec{s}_y = \hat{y}dxdz$$

$$d\vec{s}_z = \hat{z}dxdy$$

Volume:

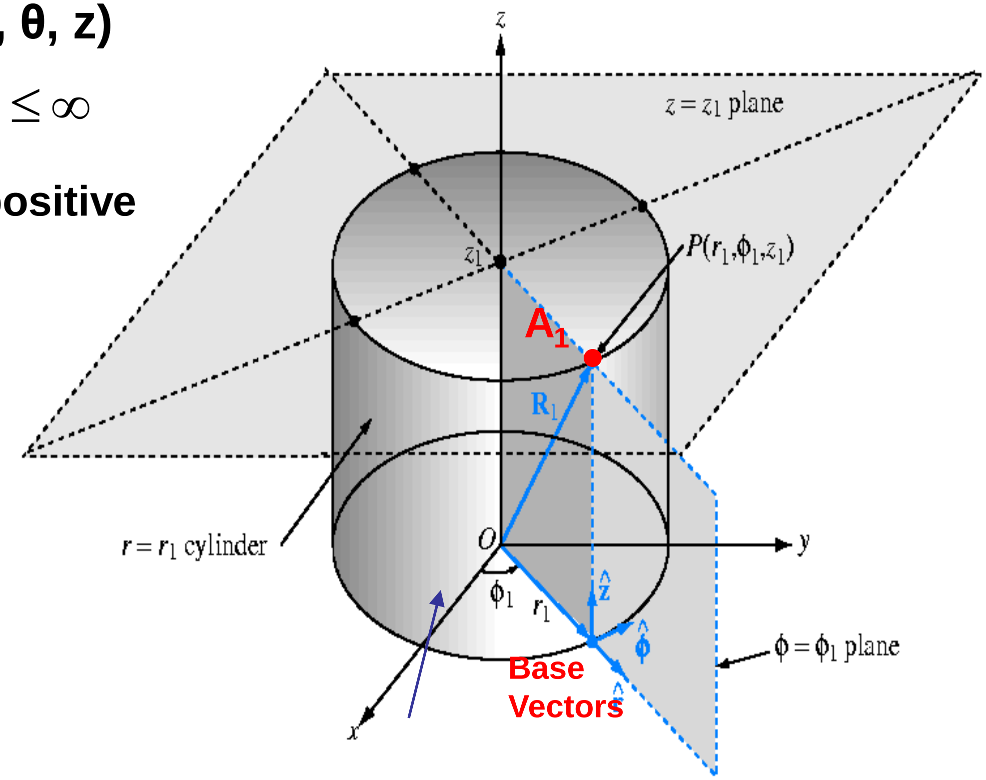
$$dv = dxdydz$$



Cylindrical Coordinates

(r, θ, z)

- **r radial distance in x-y plane** $0 \le r \le \infty$
- **Φ azimuth angle measured from the positive x-axis** $0 \le \Phi < 2\pi$
- **z** $-\infty < z < \infty$



Vector representation

$$\vec{A} = \hat{a}\left|\vec{A}\right| = \hat{r}A_r + \hat{\Phi}A_\Phi + \hat{z}A_z$$

Magnitude of A

$$\left|\vec{A}\right| = \sqrt[+]{\vec{A}\cdot\vec{A}} = \sqrt[+]{A_r^2 + A_\Phi^2 + A_z^2}$$

Base vector properties

$$\hat{r}\times\hat{\Phi} = \hat{z}, \qquad \hat{\Phi}\times\hat{z} = \hat{r}, \qquad \hat{z}\times\hat{r} = \hat{\Phi}$$

Position vector A

$$\hat{r}r_1 + \hat{z}z_1$$

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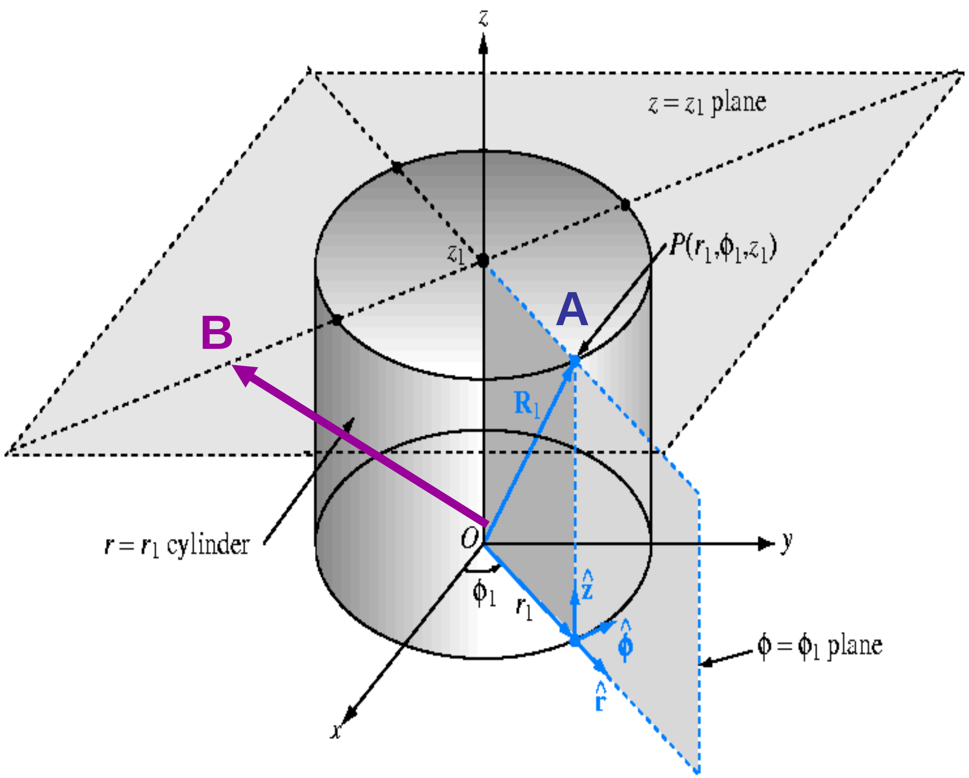
Cylindrical Coordinates

Dot product:

$$\vec{A}\cdot\vec{B} = A_r B_r + A_\phi B_\phi + A_z B_z$$

Cross product:

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{r} & \hat{\phi} & \hat{z} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$$



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Differential quantities:

Length:

$$d\vec{l} = \hat{r}dr + \hat{\Phi}rd\Phi + \hat{z}dz$$

Area:

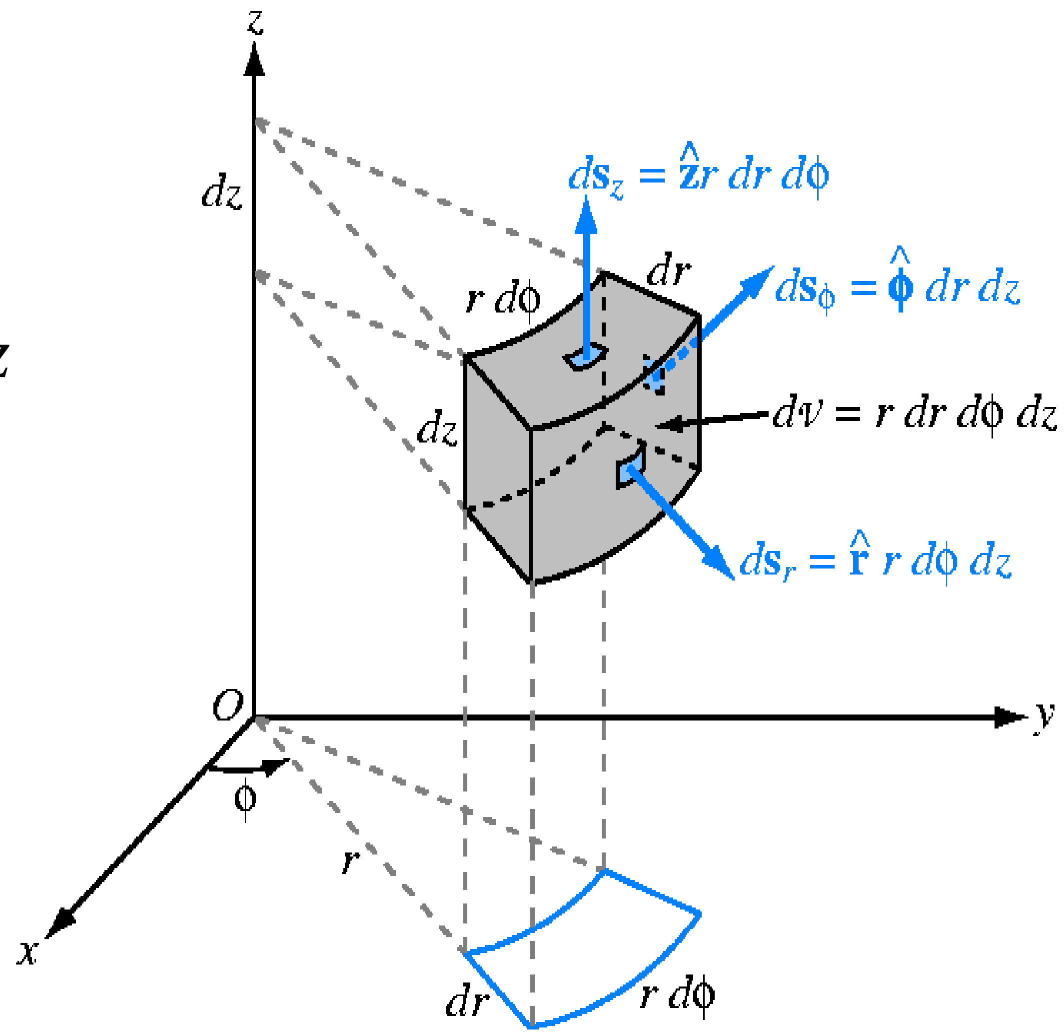
$$d\vec{s}_r = \hat{r}rd\Phi dz$$

$$d\vec{s}_\Phi = \hat{\Phi}drdz$$

$$d\vec{s}_z = \hat{z}rdrd\Phi$$

Volume:

$$dv = rdrd\Phi dz$$



Spherical Coordinates

(R, θ, Φ)

Vector representation

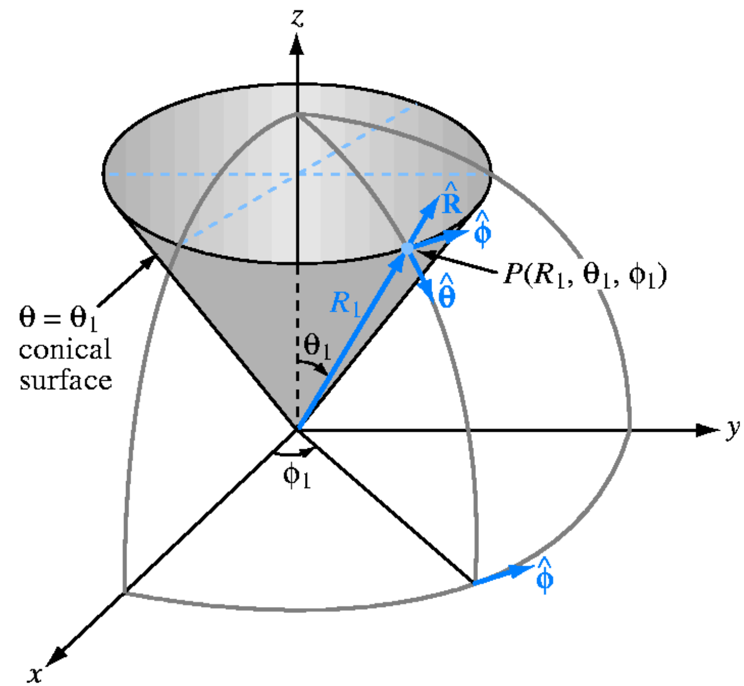
$$\vec{A} = \hat{R}A_R + \hat{\theta}A_\theta + \hat{\phi}A_\phi$$

Magnitude of A

$$\left|\vec{A}\right| = \sqrt[+]{\vec{A}\cdot\vec{A}} = \sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}$$

Position vector A

$$\hat{R}R_1$$



Base vector properties

$$\hat{R}\times\hat{\Theta} = \hat{\Phi}, \qquad \hat{\Theta}\times\hat{\Phi} = \hat{R}, \qquad \hat{\Phi}\times\hat{R} = \hat{\Theta}$$

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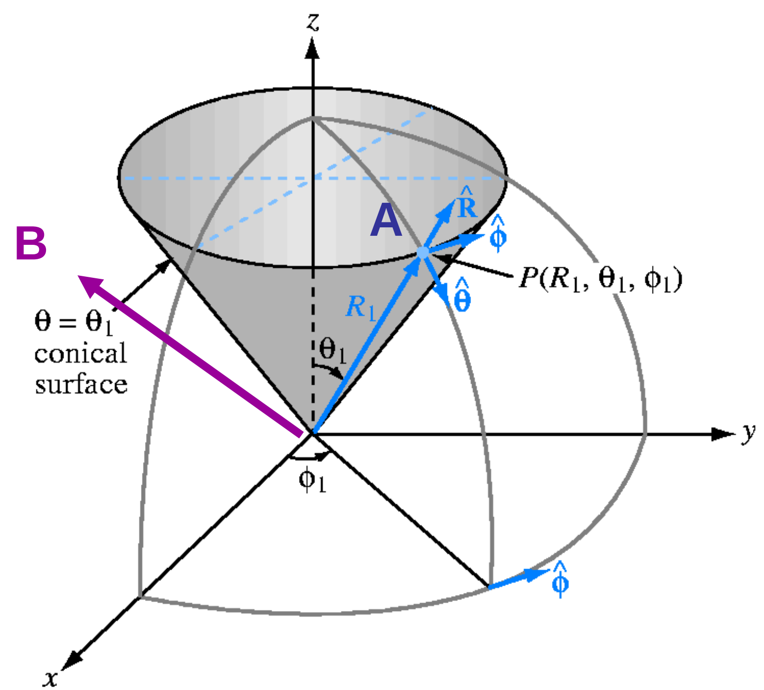
Spherical Coordinates

Dot product:

$$\vec{A}\cdot\vec{B} = A_R B_R + A_\theta B_\theta + A_\phi B_\phi$$

Cross product:

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{R} & \hat{\theta} & \hat{\phi} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$$



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Spherical Coordinates

Differential quantities:

$$dl_R = dR$$
$$dl_\Theta = Rd\Theta$$
$$dl_\Phi = R\sin\Theta d\Phi$$

Length:

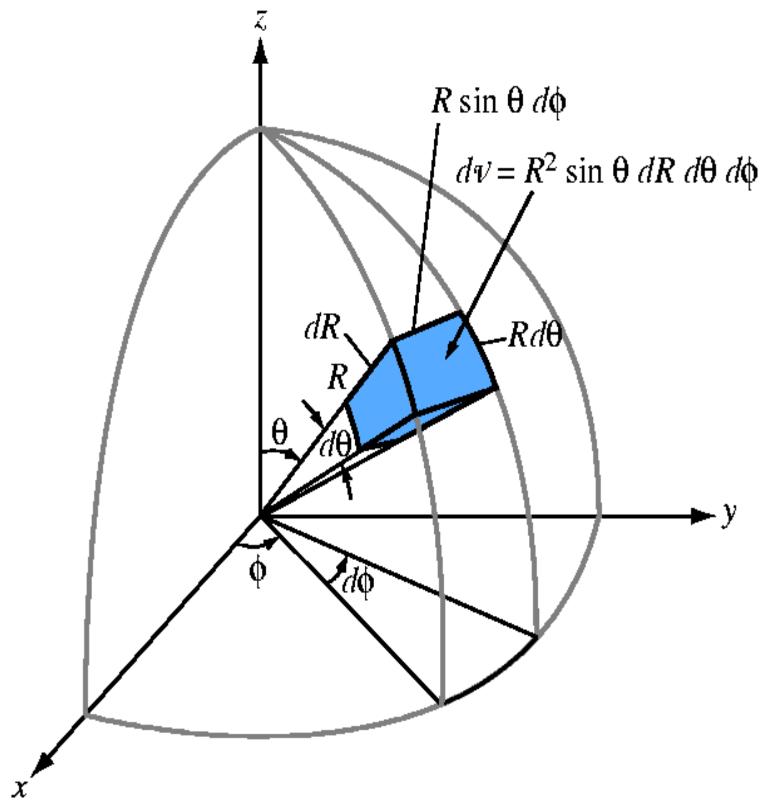
$$d\vec{l} = \hat{R}dl_R + \hat{\Theta}dl_\Theta + \hat{\Phi}dl_\Phi$$
$$= \hat{R}dR + \hat{\Theta}Rd\Theta + \hat{\Phi}R\sin\Theta d\Phi$$

Area:

$$d\vec{s}_R = \hat{R}dl_\Theta dl_\Phi = \hat{R}R^2\sin\Theta d\Theta d\Phi$$
$$d\vec{s}_\Theta = \hat{\Theta}dl_R dl_\Phi = \hat{\Theta}R\sin\Theta dRd\Phi$$
$$d\vec{s}_\Phi = \hat{\Phi}dl_R dl_\Theta = \hat{\Phi}RdRd\Theta$$

Volume:

$$dv = R^2\sin\Theta dRd\Theta d\Phi$$



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Cartesian to Cylindrical Transformation

$$A_r = A_x \cos\phi + A_y \sin\phi$$

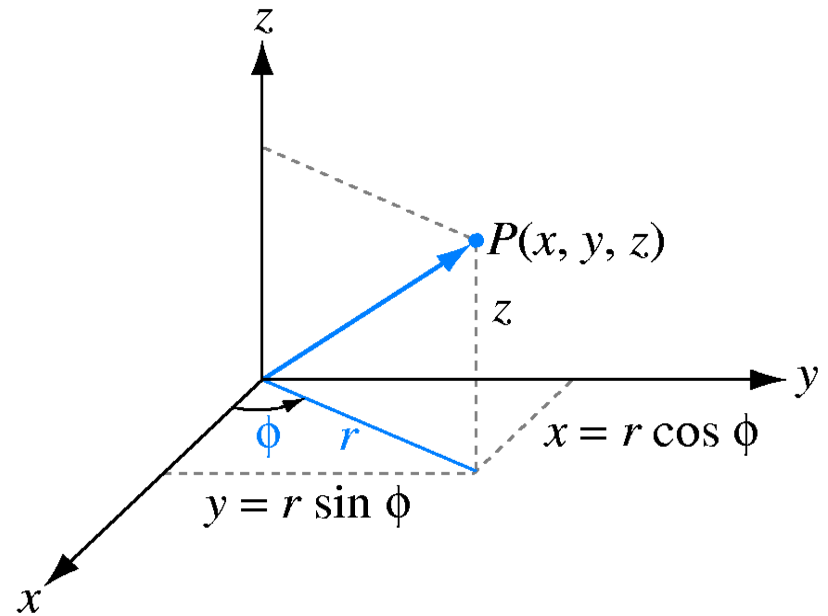
$$A_\phi = -A_x \sin\phi + A_y \cos\phi$$

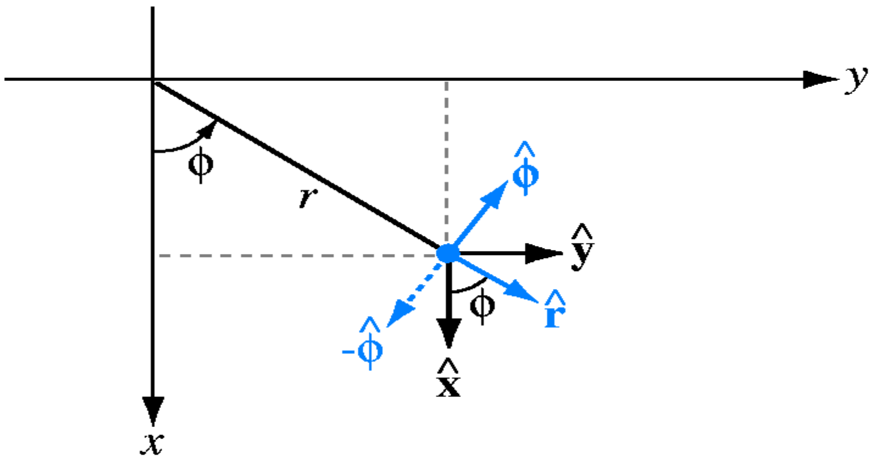
$$A_z = A_z$$

$$r = \sqrt[+]{x^2 + y^2}$$

$$\phi = \tan^{-1}(y/x)$$

$$z = z$$





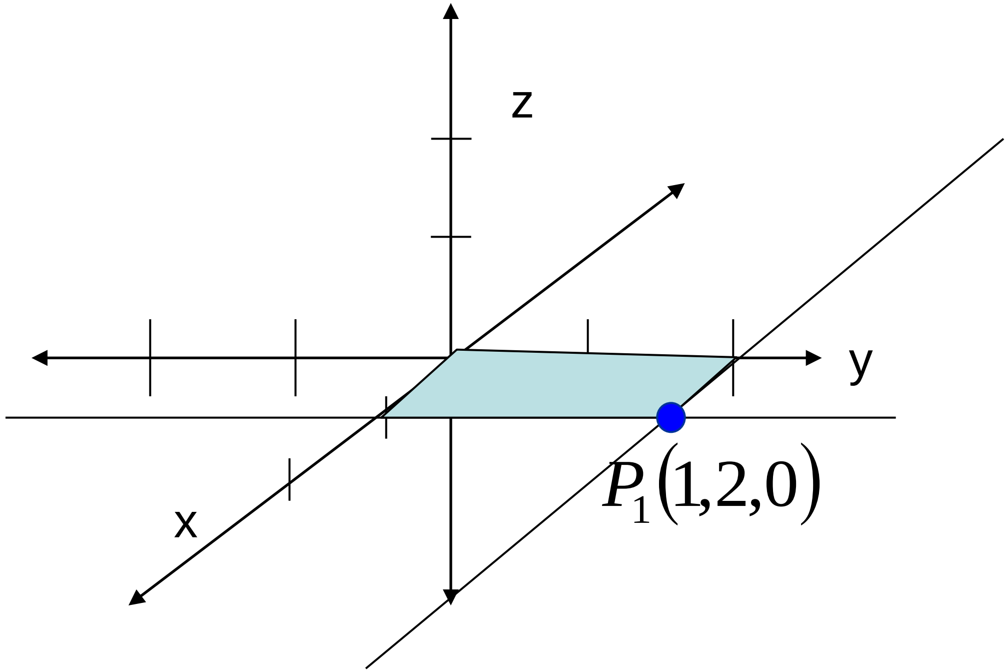
$$\hat{r} = \hat{x}\cos\phi + \hat{y}\sin\phi$$

$$\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$$

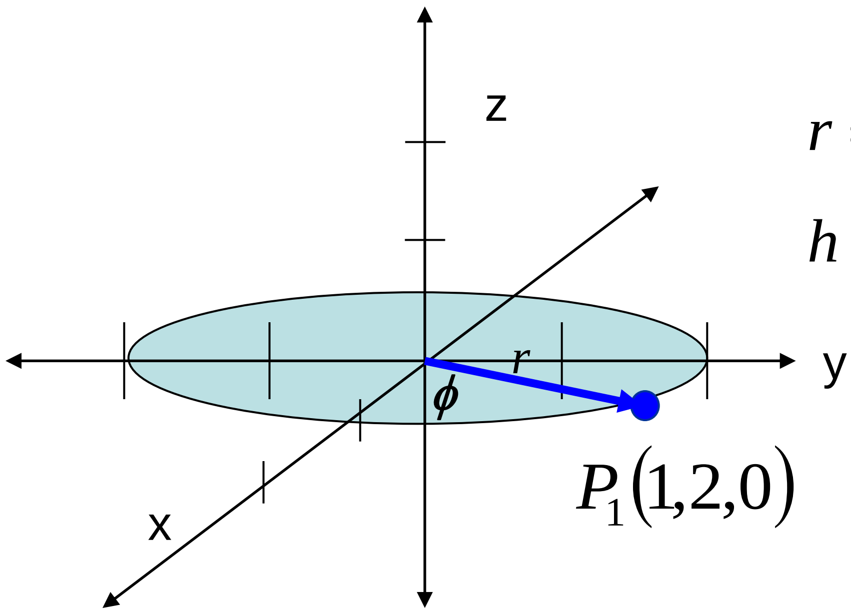
$$\hat{z} = \hat{z}$$

Back

Convert the coordinates of $P_1(1,2,0)$ from the Cartesian to the Cylindrical and Spherical coordinates.



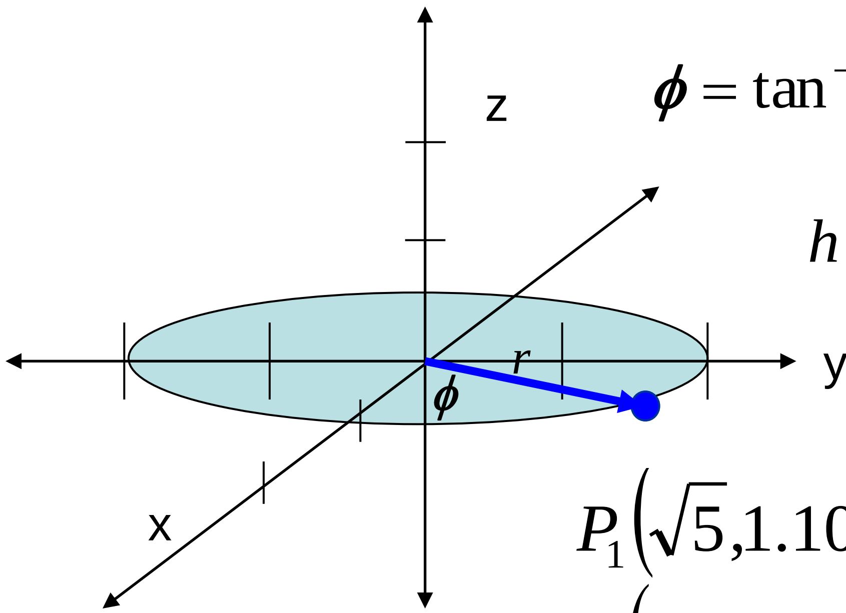
Convert the coordinates of $P_1(1,2,0)$ from the Cartesian to the Cylindrical coordinates.



$$r = \sqrt{x^2 + y^2} = \sqrt{4+1} = \sqrt{5}$$

$$h = 0$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{2}{1}\right) = 1.107\ rad$$



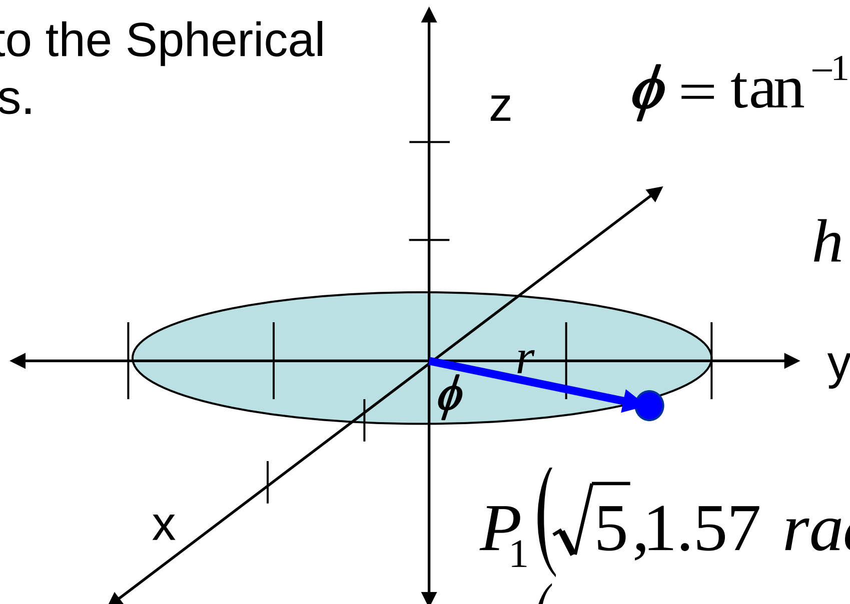
$$r = \sqrt{x^2 + y^2} = \sqrt{4+1} = \sqrt{5}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{2}{1}\right) = 1.107\ rad$$

$$h = 0$$

$$P_1\left(\sqrt{5}, 1.107\ rad, 0\right)$$
$$\approx \left(2.24, 63.4^\circ, 0\right)$$

Convert the coordinates of *P1(1,2,0)* from the Cartesian to the Spherical coordinates.



$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{4+1+0} = \sqrt{5}$$

$$\boldsymbol{\phi} = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{2}{1}\right) = 1.107\ rad$$

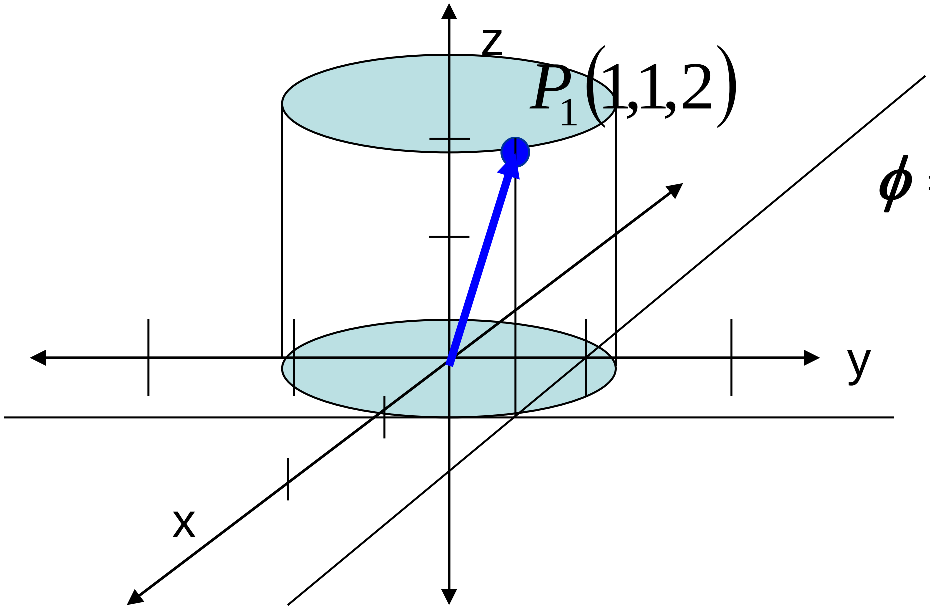
$$h = 0$$

$$P_1\left(\sqrt{5}, 1.57\ rad, 1.107\ rad\right)$$

$$\approx \left(2.24, 90^\circ, 63.4^\circ\right)$$

$$\boldsymbol{\theta} = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right) = \tan^{-1}\left(\frac{\sqrt{5}}{0}\right) = 1.57\ rad$$

Convert the coordinates of $P_3(1,1,2)$ from the Cartesian to the Cylindrical and Spherical coordinates.



$$r = \sqrt{x^2 + y^2} = \sqrt{1+1} = \sqrt{2}$$

$$h = 2$$

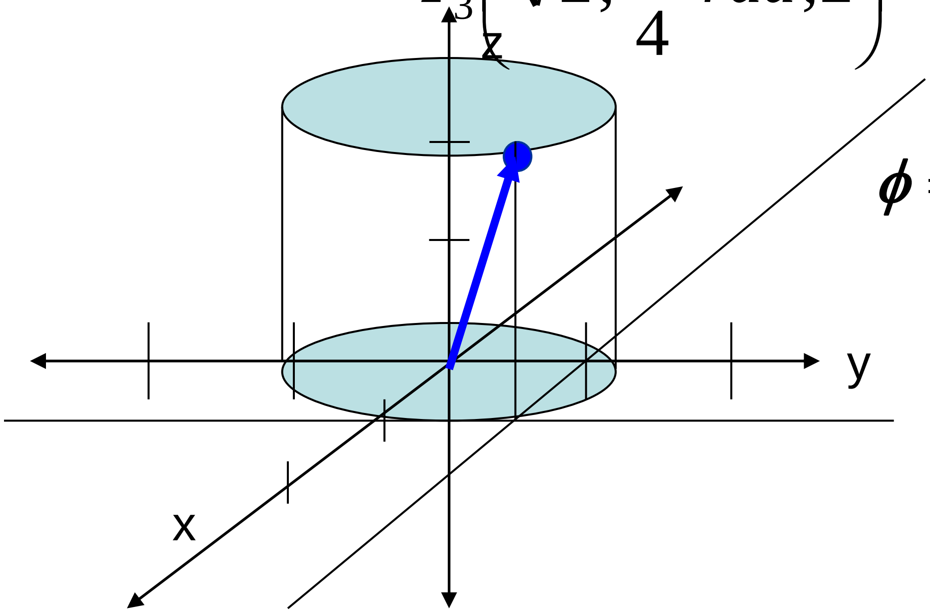
$$\phi = \tan^{-1}\left(\frac{1}{x}\right) = \tan^{-1}\left(\frac{1}{1}\right) = 0.785\ rad$$

Convert the coordinates of $P_3(1,1,2)$ from the Cartesian to the Cylindrical coordinates.

$$r = \sqrt{x^2 + y^2} = \sqrt{1+1} = \sqrt{2}$$

$$P_3\left(\sqrt{2}, \frac{\pi}{4}\ rad, 2\right) \quad h = 2$$

$$\phi = \tan^{-1}\left(\frac{1}{x}\right) = \tan^{-1}\left(\frac{1}{1}\right) = 0.785\ rad$$



Convert the coordinates of $P_3(1,1,2)$ from the Cartesian to the Spherical coordinates.

$$r=\sqrt{x^2+y^2+z^2}=\sqrt{1^2+1^2+2^2}=\sqrt{6}$$

$$\theta=\tan^{-1}\left(\frac{\sqrt{1^2+1^2}}{2}\right)=\tan^{-1}\left(\frac{\sqrt{2}}{2}\right)=0.616\ rad$$

$$\phi=\tan^{-1}\left(\frac{1}{1}\right)=\tan^{-1}(1)=\frac{\pi}{4}\ rad$$