

# MATH203 Calculus

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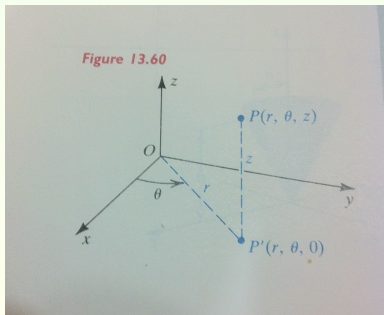
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# Cylindrical coordinates

## Theorem

The rectangular coordinates  $(x, y, z)$  and the cylindrical coordinates  $(r, \theta, z)$  of a point  $P$  are related as follows:

$$\begin{aligned}x &= r \cos \theta, & y &= r \sin \theta, & \tan \theta &= \frac{y}{x}, \\r^2 &= x^2 + y^2, & z &= z\end{aligned}$$



# Cylindrical coordinates

## Example 1

Change the equation to cylindrical coordinates.

(i)  $z^2 = x^2 + y^2$ ,      (ii)  $x^2 - y^2 - z^2 = 1$

## Triple Integral using cylindrical coordinates

$$\iiint_Q f(r, \theta, z) dV = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(r_k, \theta_k, z_k) \Delta V_k$$

where  $\Delta V_k = \bar{r} \Delta r_k \Delta \theta_k \Delta z_k$

## Nice Region

If  $Q = \{(r, \theta, z) : a \leq r \leq b, c \leq \theta \leq d, m \leq z \leq n\}$ , then

$$\iiint_Q f(r, \theta, z) dV = \int_m^n \int_c^d \int_a^b f(r, \theta, z) r dr d\theta dz$$

# Cylindrical coordinates

## Complicated Region

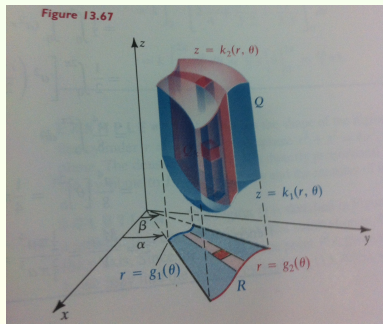
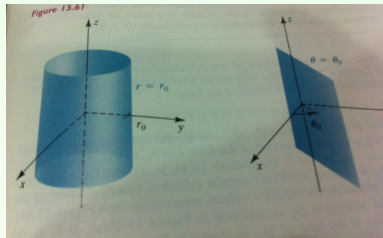
If  $Q = \{(r, \theta, z) : (r, \theta) \in R, K_1(r, \theta) \leq z \leq K_2(r, \theta)\}$ , then

$$\iiint_Q f(r, \theta, z) dV = \iint_R \left[ \int_{k_1(r, \theta)}^{k_2(r, \theta)} f(r, \theta, z) dz \right] dA$$

then,

$$\iiint_Q f(r, \theta, z) dV = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} \int_{k_1(r, \theta)}^{k_2(r, \theta)} f(r, \theta, z) r dz dr d\theta$$

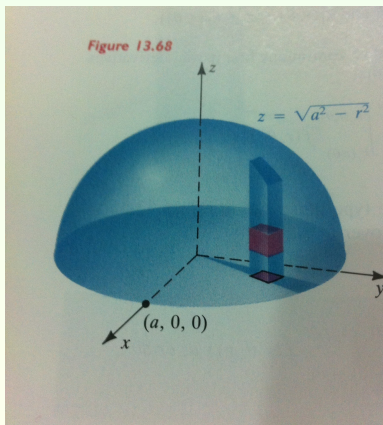
# Cylindrical coordinates



# Cylindrical coordinates

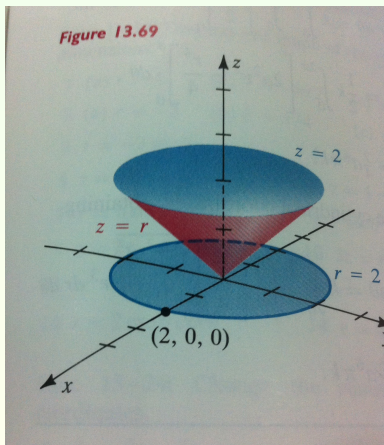
## Examples

(1) Find the centroid of solid  $Q$  as shown in Figure, where  $z = \sqrt{a^2 - r^2}$ .



# Cylindrical coordinates

(2) By using cylindrical coordinates, find the mass of solid  $Q$  bounded by the cone  $z = \sqrt{x^2 + y^2}$  and  $z = 2$ , where the density at  $(x, y, z)$  is  $\delta = k(x^2 + y^2 + z^2)$ .



# Cylindrical coordinates

(3) A solid has the shape of the region  $Q$  that lies inside the cylinder  $r = a$ , within the sphere  $r^2 + z^2 = 4a^2$  and above the  $xy$ -plane. The density at a point  $\delta(x, y, z) = kz$ . Find the mass and the moment of inertia  $I_z$  of the solid.

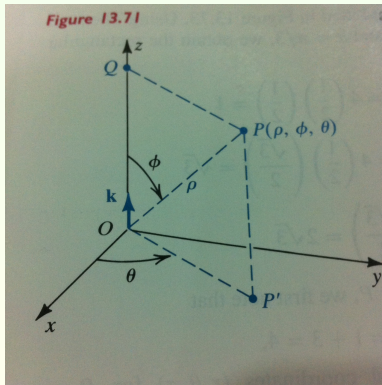


# Spherical coordinates

## Theorem

The rectangular coordinates  $(x, y, z)$  and the spherical coordinates  $(\rho, \phi, \theta)$  of a point  $P$  are related as follows:

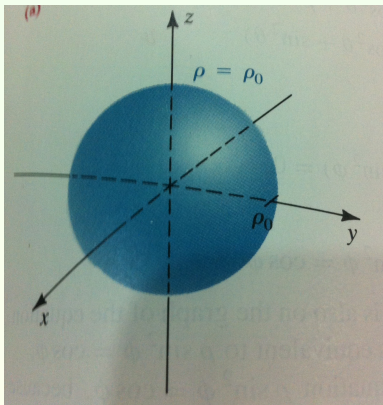
- (1)  $x = \rho \sin \phi \cos \theta$ ,  $y = \rho \sin \phi \sin \theta$ ,  $z = \rho \cos \phi$
- (2)  $\rho^2 = x^2 + y^2 + z^2$



# Some important graphs

(1)  $\rho = c, c > 0$

graph is a sphere of radius  $\rho = c$ , with center  $(0, 0, 0)$ .



# Some important graphs

(2)  $\phi = c, 0 < c < \frac{\pi}{2}$

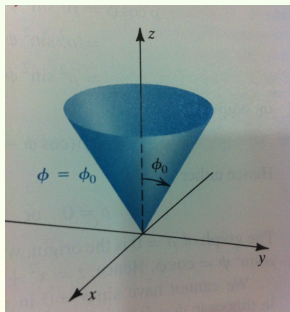
graph is a half-cone opening upward, with center  $(0, 0, 0)$ .

(i)  $\phi = 0$ , graph is non-negative part of  $z$ -axis.

(ii)  $\phi = \frac{\pi}{2}$ , graph is  $xy$ -plane.

(iii)  $\phi = \pi$ , graph is non-positive half of  $z$ -axis.

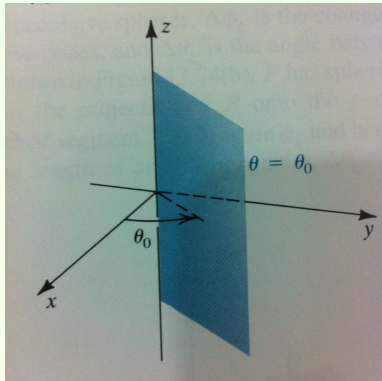
(iv)  $\frac{\pi}{2} < \phi < \pi$  graph is a half-cone opening downwards, with center  $(0, 0, 0)$ .



# Some important graphs

(1)  $\theta = c$

graph is a half plane containing the  $z$ -axis.



# Spherical coordinates

## Example 1

1- Change the spherical coordinates to (a) rectangular coordinates (b) cylindrical coordinates.

(i)  $(4, \frac{\pi}{6}, \frac{\pi}{3})$ ,      (ii)  $(1, \frac{3\pi}{4}, \frac{2\pi}{3})$

2- Change the cylindrical coordinates to (a) rectangular coordinates (b) spherical coordinates.

(i)  $(4, -\frac{\pi}{6}, 6)$

## Example 2

Find an equation in the spherical coordinates, whose graph is the paraboloid  $z = x^2 + y^2$ .

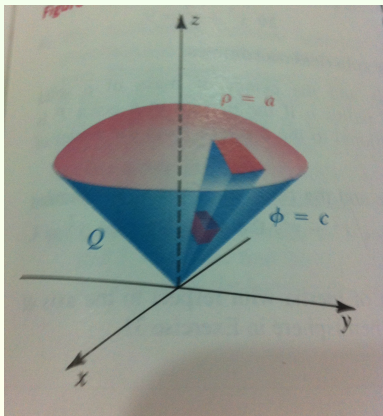
## Triple Integral using spherical coordinates

$$\iiint_Q f(\rho, \phi, \theta) dV = \int_m^n \int_c^d \int_a^b f(\rho, \phi, \theta) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

# Spherical coordinates

## Examples

(1) Find the volume and the centroid of solid  $Q$ , as shown in Figure, that bounded above by the sphere  $\rho = a$  and below by cone  $\phi = c$ , where  $0 < c < \frac{\pi}{2}$ .



# Spherical coordinates

## Examples

(2) Use spherical coordinates to derive the formula for the volume of sphere centered at the origin and with radius  $a$ .

(3) Find the volume of sphere that lies inside the sphere  $x^2 + y^2 + z^2 = 2$  and outside  $z^2 = x^2 + y^2$ .

(4) Use spherical coordinates to express region between the sphere  $x^2 + y^2 + z^2 = 1$  and the cone  $z = \sqrt{x^2 + y^2}$ .

(5) Find the integral of  $f(x, y, z) = e^{(x^2+y^2+z^2)^{\frac{3}{2}}}$  in the region  $R = \{x \geq 0, y \geq 0, z \geq 0, x^2 + y^2 + z^2 = 1\}$ .