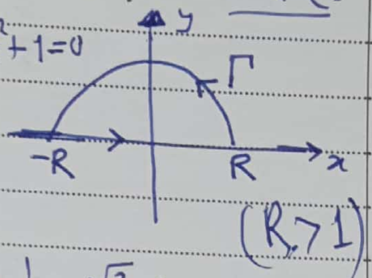
لا يكتب في هذا الهامش

ج ٣: احسب التكامل $\oint_C \frac{dz}{z^4+z^2+1}$ ، حيث C نصف الدائرة كما في الشكل:

$z = \pm\frac{1}{2} \pm \frac{\sqrt{3}}{2} i \Leftarrow z^4+z^2+1=0$



نقط $\underbrace{\frac{1}{2}+\frac{\sqrt{3}}{2}i}_{z_1}$ و $\underbrace{-\frac{1}{2}+\frac{\sqrt{3}}{2}i}_{z_2}$ داخل C ،

$$Res(f,z_1) = \lim_{z\to z_1} \left\{(z-z_1)\cdot\frac{1}{z^4+z^2+1}\right\} = -\frac{1}{4}-\frac{\sqrt{3}}{12}i$$

$$Res(f,z_2) = \lim_{z\to z_2} \left\{(z-z_2)\cdot\frac{1}{z^4+z^2+1}\right\} = \frac{1}{4}-\frac{\sqrt{3}}{12}i$$

$$\oint_C \frac{dz}{z^4+z^2+1} = \int_{-R}^{R} \frac{dx}{x^4+x^2+1} + \int_\Gamma \frac{dz}{z^4+z^2+1} \quad \cdots\cdots (*)$$

$\underline{\underline{\int_\Gamma \frac{dz}{z^4+z^2+1}}}$: $|z^4+z^2+1| \geq |z|^4-|z|^2-1$

$= R^4-R^2-1$ on Γ

$\Rightarrow \left|\frac{1}{z^4+z^2+1}\right| \leq \frac{1}{R^4-R^2-1}$ on Γ

$$\left|\int_\Gamma \frac{dz}{z^4+z^2+1}\right| \leq \frac{1}{R^4-R^2-1}\cdot\pi R$$

نأخذ النهاية عندما $R\to\infty$:

$$\Rightarrow \frac{\pi R}{R^4-R^2-1} \to 0 \Rightarrow \left|\int_\Gamma \frac{dz}{z^4+z^2+1}\right| \to 0 \Rightarrow \int_\Gamma \frac{dz}{z^4+z^2+1} \to 0 \text{ as } R\to\infty$$

نرجع إلى (*):

من نظرية الرواسب:

$$2\pi i[Res(f,z_1)+Res(f,z_2)] = \int_{-R}^{R} \frac{dx}{x^4+x^2+1} + \int_\Gamma \frac{dz}{z^4+z^2+1}$$

$$2\pi i\left(-\frac{1}{4}-\frac{\sqrt{3}}{12}i+\frac{1}{4}-\frac{\sqrt{3}}{12}i\right) = \int_{-R}^{R} \frac{dx}{x^4+x^2+1} + \int_\Gamma \frac{dz}{z^4+z^2+1}$$

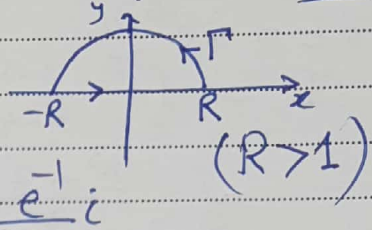
نأخذ النهاية: $R \to \infty$:

$$2\pi i\left(-\frac{\sqrt{3}}{6}i\right) = \int_{-\infty}^{\infty} \frac{dx}{x^4+x^2+1} + 0$$

$$\frac{\sqrt{3}}{3}\pi = \int_{-\infty}^{\infty} \frac{dx}{x^4+x^2+1}$$

$$\Rightarrow \frac{\sqrt{3}}{6}\pi = \int_0^\infty \frac{dx}{x^4+x^2+1} \quad \left(\text{لأن } \frac{1}{x^4+x^2+1} \text{ دالة زوجية}\right)$$

ج٣ : اعتبر التكامل : $\oint_C \frac{e^{iz}}{(z^2+1)^5}dz$ حيث C نصف الدائرة كما في الشكل :



الدالة المكاملة لها القطبين $z=\pm i$ ، مرتبة 5
والذي داخل C هو : i

$$Res(f,i)=\lim_{z\to i}\left[(z-i)^5\cdot\frac{e^{iz}}{(z^2+1)^5}\right]=\frac{-e^{-1}}{32}i$$

$$\oint_C \frac{e^{iz}}{(z^2+1)^5}dz=\int_{-R}^{R}\frac{e^{ix}}{(x^2+1)^5}dx+\int_\Gamma \frac{e^{iz}}{(z^2+1)^5}dz \quad ------ (*)$$

$\underline{\int_\Gamma \frac{e^{iz}}{(z^2+1)^5}dz}$: $|z^2+1|\geq |z|^2-1$

$= R-1$ on Γ

$$\left|\frac{1}{(z^2+1)^5}\right|\leq\frac{1}{(R^2-1)^5} \text{ on } \Gamma$$

كذلك لاحظ أن : $|e^{iz}|=e^{-y}\leq 1$ عندما $y\geq 0$

$$\Rightarrow \left|\frac{e^{iz}}{(z^2+1)^5}\right|\leq\frac{1}{(R^2-1)^5}$$

$$\Rightarrow \left|\int_\Gamma \frac{e^{iz}}{(z^2+1)^5}dz\right|\leq\frac{1}{(R^2-1)^5}\pi R$$

نأخذ النهاية عندما $R\to\infty$:

$$\frac{\pi R}{(R^2-1)^5}\to 0 \Rightarrow \left|\int_\Gamma \frac{e^{iz}}{(z^2+1)^5}dz\right|\to 0 \Rightarrow \int_\Gamma \frac{e^{iz}}{(z^2+1)^5}dz\to 0 \text{ as } R\to\infty$$

نرجع إلى (*) :

من نظرية الرواسب :

$$2\pi i[Res(f,i)]=\int_{-R}^{R}\frac{e^{ix}}{(x^2+1)^5}dx+\int_\Gamma\frac{e^{iz}}{(z^2+1)^5}dz$$

$$2\pi i\left(\frac{-e^{-1}}{32}\right)i=\int_{-R}^{R}\frac{e^{ix}}{(x^2+1)^5}dx+\int_\Gamma\frac{e^{iz}}{(z^2+1)^5}dz$$

نأخذ : $R\to\infty$:

$$\frac{\pi}{16e}=\int_{-\infty}^{\infty}\frac{e^{ix}}{(x^2+1)^5}dx+0$$

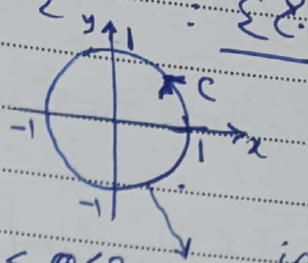
$$=\int_{-\infty}^{\infty}\frac{\cos(x)}{(x^2+1)^5}dx+i\int_{-\infty}^{\infty}\frac{\sin(x)}{(x^2+1)^5}dx$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{\cos(x)}{(x^2+1)^5}dx=\frac{\pi}{32e} \Rightarrow \int_{0}^{\infty}\frac{\cos(x)}{(x^2+1)^5}dx=\frac{\pi}{32e}$$

(لأن $\frac{\cos(x)}{(x^2+1)^5}$ دالة زوجية)

$z = e^{i\theta} \Rightarrow \cos(3\theta) = \frac{z^3+z^{-3}}{2}$, $\cos(2\theta) = \frac{z^2+z^{-2}}{2}$: ع.ع

$dz = i e^{i\theta} d\theta$



$0 \le \theta \le 2\pi$, $z = e^{i\theta}$

$$I = \int_0^{2\pi} \frac{\cos^2(3\theta)}{5-4\cos(2\theta)}\, d\theta$$

$$= \oint_C \frac{\left(\frac{z^3+z^{-3}}{2}\right)^2}{5-4\cdot\frac{z^2+z^{-2}}{2}} \cdot \frac{dz}{i\cdot z}$$

$$= \frac{i}{4}\oint_C \frac{z^{12}+2z^6+1}{z^5(2z^4-5z^2-2)}\, dz$$

$$= \frac{i}{4}\cdot 2\pi i \sum Res$$

$2z^4-5z^2-2=0$

$\Rightarrow z_{1,2} = \pm\sqrt{\frac{5+\sqrt{41}}{2}}$

$$= -\frac{\pi}{2}\left[Res(f,0)+Res(f,z_3)+Res(f,z_4)\right]$$

$z_{3,4} = \pm i\sqrt{\frac{\sqrt{41}-5}{2}}$

z_3, z_4 داخل C (قطبان من الرتبة 1)

$z=0$ داخل C
من الرتبة 5

$$= -\frac{\pi}{2}\left[-\frac{3}{4}\right]$$

$$= \frac{3\pi}{8}$$