EX1.1: Let X be normal random variable with distribution N(θ,1). Let $X_1, X_2, \ldots, X_{16}$ be 16 copies of X. Test the hypothesis $H_\circ: \theta \leq 1 \; vs \; H_1: \theta > 1$ by γ_{UMP} with size $\alpha_{UMP} = 0.05$

Solution

$$f(x;\theta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\theta)^2} \qquad -\infty < x < \infty$$

$f(x;\theta)$ Belongs to the class of exponential family

$$f(x;\theta) = e^{-\frac{1}{2}\log 2\pi - \frac{1}{2}(x-\theta)^2}$$

$a(\theta) = -\frac{1}{2}\theta^2 \quad b(x) = -\frac{1}{2}\log(2\pi) - \frac{1}{2}x^2 \qquad c(\theta) = \theta \qquad d(x) = x$

Since $c(\theta)$ is increasing function of then γ_{UMP} reject H₀ if $\sum d(x_i) = \sum x_i > k$ or $\bar{X} > C$

$$0.05 = P(\bar{X} > C|\theta = 1) = P(\bar{X} > \frac{C-1}{\frac{1}{4}} = P(Z > 4(C-1))$$

Thus

$$4(C-1) = 1.645 \rightarrow C = 1.41125 \; and \; K = 22.58$$

EX1.2: Let X be gamma random variable with distribution Gamma(5,θ). Let $X_1, X_2, \ldots, X_6$ be 6 copies of X. Test the hypothesis $H_\circ: \theta \geq \frac{1}{2}\ vs\ H_1: \theta < \frac{1}{2}$ by γ_{UMP} with size $\alpha = 0.05$

Solution

$$f(x;\theta) = \frac{\theta^5}{\Gamma(5)} x^{5-1} e^{-\theta x} \qquad 0 < x < \infty$$

$f(x;\theta)$ belongs to the class of exponential family

$$f(x;\theta) = e^{5\log\theta - \log\Gamma(5) + 4\log x - \theta x}$$

$a(\theta) = 5\log\theta \quad b(x) = 4\log x - \log\Gamma(5) \qquad c(\theta) = -\theta \qquad d(x) = x$

Since $c(\theta)$ is decreasing function of then γ_{MP} reject H₀ if $S = \sum d(x_i) = \sum x_i > k$

$$X_i \sim Gamma(5,\theta) \quad then \sum X_i \sim Gamma(5\times 6,\theta) \quad and\ U = 2\theta S \sim \chi^2_{2(5\times 6)}$$

To find k

$$0.05 = P\left(S > k \middle| \theta = \frac{1}{2}\right) = P\left(U > 2\theta k \middle| \theta = \frac{1}{2}\right) = P(U > k)$$

K=79.08

let $\underline{X} = (X_1, X_2, \dots, X_n)$ random sample from $N(\theta, \sigma^2)$ where σ^2 known
Find γ_{GLR} and α for $H_\circ: \theta = \theta_\circ$ vs $H_1: \theta \neq \theta_\circ$

Maximum likelihood estimator of θ is $\bar{X}$

γ_{GLR} reject $H_\circ$ if $\lambda = \frac{\ell(\underline{X};\theta_\circ)}{\ell(\underline{X};\bar{X})} < k$

$$\lambda = \frac{\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n exp\left(-\frac{1}{2}\frac{\sum(X_i - \theta_\circ)^2}{\sigma^2}\right)}{\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n exp\left(-\frac{1}{2}\frac{\sum(X_i - \bar{X})^2}{\sigma^2}\right)}$$

$$\lambda = exp\left(-\frac{1}{2}\frac{\sum(X_i - \theta_\circ)^2 - (X_i - \bar{X})^2}{\sigma^2}\right)$$

$$\lambda = exp\left(-\frac{1}{2}\frac{\sum(X_i - \theta_\circ - \bar{X} + \bar{X})^2 - (X_i - \bar{X})^2}{\sigma^2}\right)$$

$$\lambda = exp\left(-\frac{1}{2}\frac{\sum(\bar{X} - \theta_\circ)^2 + (X_i - \bar{X})^2 - (X_i - \bar{X})^2}{\sigma^2}\right)$$

$$\lambda = exp\left(-\frac{1}{2}\frac{n(\bar{X} - \theta_\circ)^2}{\sigma^2}\right)$$

i.e

$$exp\left(-\frac{1}{2}\frac{n(\bar{X} - \theta_\circ)^2}{\sigma^2}\right) < k$$

$$-\frac{1}{2}\frac{n(\bar{X} - \theta_\circ)^2}{\sigma^2} < \ln k$$

$$\frac{n(\bar{X} - \theta_\circ)^2}{\sigma^2} > -2\ln k = d^2$$

Then γ_{GLR} reject $H_\circ$ if

$$\alpha = P(\lambda < k|\theta_\circ)$$

$$= P\left(\frac{n(\bar{X} - \theta_\circ)^2}{\sigma^2} > d^2\right)$$

$$= P\left(\frac{\bar{X} - \theta_\circ}{\sigma/\sqrt{n}} > d\right) + P\left(\frac{\bar{X} - \theta_\circ}{\sigma/\sqrt{n}} < -d\right)$$

When $H_\circ$ true then $\frac{\overline{X}-\theta_\circ}{\sigma/\sqrt{n}} \sim N(0,1)$ and $d = z_{1-\frac{\alpha}{2}}$

We accept $H_\circ$ if

$$\frac{\overline{X}-\theta_\circ}{\sigma/\sqrt{n}} \in \left(-z_{1-\frac{\alpha}{2}}, z_{1-\frac{\alpha}{2}}\right)$$

let $\underline{X} = (X_1, X_2, \dots, X_n)$ random sample from $N(\theta, \sigma^2)$ where σ^2 known

Find γ_{CI} and α for $H_\circ: \theta = \theta_\circ$ vs $H_1: \theta \neq \theta_\circ$

$$100(1-\alpha)\ C.I.is$$

$$\left(T_1(\underline{X}), T_1(\underline{X})\right) = \left(\overline{X} - z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}, \overline{X} + z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right)$$

Then we accept $H_\circ$ if

$$\theta_\circ \in \left(\overline{X} - z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}, \overline{X} + z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right) \Leftrightarrow \frac{\overline{X}-\theta_\circ}{\sigma/\sqrt{n}} \in \left(-z_{1-\frac{\alpha}{2}}, z_{1-\frac{\alpha}{2}}\right)$$