111 ريض - حساب التكامل
الفصل الدراسي الثاني 1443 هـ
حل الاختبار الفصلي
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السؤال الأول (12 درجة) :

(1) استخدم مجموع ريمان لحساب التكامل المحدد $\int_0^2 (4x-3)\,dx$.

الحل : $[a,b]=[0,2]$ و $f(x)=4x-3$.

$$\Delta_x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

$$x_k = a + k\,\Delta_x = 0 + k\left(\frac{2}{n}\right) = \frac{2k}{n}$$

$$f(x_k) = 4\left(\frac{2}{n}\right) - 3 = \frac{8k}{n} - 3$$

$$R_n = \sum_{k=1}^{n} f(x_k)\ \Delta_x = \sum_{k=1}^{n}\left(\frac{8k}{n}-3\right)\ \left(\frac{2}{n}\right) = \sum_{k=1}^{n}\left(\frac{16k}{n^2}-\frac{6}{n}\right)$$

$$\sum_{k=1}^{n}\frac{16k}{n^2} - \sum_{k=1}^{n}\frac{6}{n} = \frac{16}{n^2}\sum_{k=1}^{n}k - \frac{6}{n}\sum_{k=1}^{n}1 = \frac{16}{n^2}\ \frac{n(n+1)}{2} - \frac{6}{n}\,(n) = 8\left(\frac{n+1}{n}\right) - 6$$

$$\int_0^2 (4x-3)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[8\left(\frac{n+1}{n}\right)-6\right] = 8(1)-6 = 2$$

(2) جد $F'(x)$ إذا كانت $F(x) = \int_{e^{2x}}^{2^{\cos x}} \frac{1}{\sqrt{2-t^2}}\,dt$

الحل :

$$F'(x) = \frac{d}{dx}\int_{e^{2x}}^{2^{\cos x}} \frac{1}{\sqrt{2-t^2}}\,dt$$

$$= \frac{1}{\sqrt{2-\left(2^{\cos x}\right)^2}}\ 2^{\cos x}\,(-\sin x)\ln 2 - \frac{1}{\sqrt{2-\left(e^{2x}\right)^2}}\ e^{2x}(2)$$

$$= \frac{-2^{\cos x}\sin x\ \ln 2}{\sqrt{2-2^{2\cos x}}} - \frac{2\,e^{2x}}{\sqrt{2-e^{4x}}}$$

احسب $\frac{dy}{dx}$ فيمايلي :

$$y = 5^{2x} \ \cosh^{-1}\left(e^{\sqrt{x}}\right) \quad (3)$$

الحل :

$$\frac{dy}{dx} = \left(5^{2x}\ (2)\ln 5\right) \ \cosh^{-1}\left(e^{\sqrt{x}}\right) + 5^{2x}\left[\frac{1}{\sqrt{\left(e^{\sqrt{x}}\right)^2 - 1}}\ e^{\sqrt{x}}\ \frac{1}{2\sqrt{x}}\right]$$

$$= 5^{2x}\ 2\ln 5\ \cosh^{-1}\left(e^{\sqrt{x}}\right) + \frac{e^{\sqrt{x}}}{2\sqrt{x}\ \sqrt{e^{2\sqrt{x}} - 1}}$$

$$y = \sin^{-1} x\ \log_6 |8 + \tan 2x| \quad (4)$$

الحل :

$$\frac{dy}{dx} = \left(\frac{1}{\sqrt{1 - x^2}}\right)\log_6 |8 + \tan 2x| + \sin^{-1} x\left(\frac{\sec^2(2x)\ (2)}{8 + \tan 2x}\ \frac{1}{\ln 6}\right)$$

$$= \frac{\log_6 |8 + \tan 2x|}{\sqrt{1 - x^2}} + \frac{2\sec^2(2x)\ \sin^{-1} x}{(8 + \tan 2x)\ \ln 6}$$

$$y = (\tan x)^{\sec x} \quad (5)$$

الحل :

$$y = (\tan x)^{\sec x} \implies \ln|y| = \ln\left|(\tan x)^{\sec x}\right| = \sec x\ \ln|\tan x|$$

بإشتقاق الطرفين

$$\frac{y'}{y} = (\sec x \tan x)\ \ln|\tan x| + \sec x\ \left(\frac{\sec^2 x}{\tan x}\right)$$

$$y' = y\ \left[\sec x \tan x\ \ln|\tan x| + \frac{\sec^3 x}{\tan x}\right]$$

$$y' = (\tan x)^{\sec x}\ \left[\sec x \tan x\ \ln|\tan x| + \frac{\sec^3 x}{\tan x}\right]$$

$$y = \coth^{-1}\left(\frac{1}{x}\right) \quad (6)$$

الحل :

$$\frac{dy}{dx} = \frac{-1}{1 - \left(\frac{1}{x}\right)^2}\ \left(\frac{-1}{x^2}\right) = \frac{1}{x^2\left[1 - \frac{1}{x^2}\right]} = \frac{1}{x^2 - 1}$$

السؤال الثاني (18 درجة) : أحسب التكاملات التالية

$$(1)\quad \int \frac{\sec^2(\ln x)}{x}\,dx$$

الحل :

$$\int \frac{\sec^2(\ln x)}{x}\,dx = \int \sec^2(\ln x)\ \left(\frac{1}{x}\right)\ dx = \tan(\ln x) + c$$

باستخدام القانون

$$\int \sec^2(f(x))\ f'(x)\,dx = \tan(f(x)) + c$$

$$(2)\quad \int \left(7 - x^2\right)^3\ x\,dx$$

الحل :

$$\int \left(7 - x^2\right)^3\ x\,dx = \frac{1}{-2}\int \left(7 - x^2\right)^3\ (-2x)\,dx = \frac{1}{-2}\ \frac{(7 - x^2)^4}{4} + c$$

باستخدام القانون $\int [f(x)]^n\ f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c$ حيث $n \neq -1$.

$$(3)\quad \int_0^1 x\ 10^{2-x^2}\,dx$$

الحل :

$$\int_0^1 x\ 10^{2-x^2}\,dx = \frac{1}{-2}\int_0^1 10^{2-x^2}(-2x)\,dx = \frac{1}{-2}\left[\frac{10^{2-x^2}}{\ln 10}\right]_0^1$$

$$= \frac{1}{-2}\left[\frac{10^{2-(1)^2}}{\ln 10} - \frac{10^{2-(0)^2}}{\ln 10}\right] = \frac{1}{-2}\left[\frac{10^1}{\ln 10} - \frac{10^2}{\ln 10}\right] = \frac{1}{-2}\left(\frac{-90}{\ln 10}\right) = \frac{45}{\ln 10}$$

باستخدام القانون $\int a^{f(x)}\ f'(x)\,dx = \frac{a^{f(x)}}{\ln a} + c$.

$$(4)\quad \int \frac{x-1}{x+2}\,dx$$

الحل :

$$\int \frac{x-1}{x+2}\,dx = \int \frac{(x+2)-3}{x+2}\,dx = \int \left(\frac{x+2}{x+2} - \frac{3}{x+3}\right)\,dx$$

$$= \int \left(1 - \frac{3}{x+2}\right)\, dx = \int 1\, dx - \int \frac{3}{x+2}\, dx = \int 1\, dx - 3\int \frac{1}{x+2}\, dx$$

$$= x - 3\ln|x+2| + c$$

باستخدام القانون $\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + c$.

(5) $\int \frac{e^{2x}}{\sqrt{9+e^{4x}}}\, dx$

الحل :

$$\int \frac{e^{2x}}{\sqrt{9+e^{4x}}}\, dx = \int \frac{e^{2x}}{\sqrt{(3)^2+(e^{2x})^2}}\, dx = \frac{1}{2}\int \frac{2\, e^{2x}}{\sqrt{(3)^2+(e^{3x})^2}}\, dx$$

$$= \frac{1}{2}\sinh^{-1}\left(\frac{e^{2x}}{3}\right) + c$$

باستخدام القانون

$$\int \frac{f'(x)}{\sqrt{a^2+[f(x)]^2}}\, dx = \sinh^{-1}\left(\frac{f(x)}{a}\right) + c$$

(6) $\int \frac{\tanh(\sqrt{x})}{\sqrt{x}}\, dx$

الحل :

$$\int \frac{\tanh(\sqrt{x})}{\sqrt{x}}\, dx = \int \tanh\left(\sqrt{x}\right)\ \left(\frac{1}{\sqrt{x}}\right)\, dx = 2\int \tanh\left(\sqrt{x}\right)\ \left(\frac{1}{2\sqrt{x}}\right)\, dx$$

$$= 2\ln\left|\cosh\left(\sqrt{x}\right)\right| + c$$

باستخدام القانون

$$\int \tanh(f(x))\ f'(x)\, dx = \ln|\cosh(f(x))| + c$$

(7) $\int \frac{3x}{x^4+9}\, dx$

الحل :

$$\int \frac{3x}{x^4+9}\, dx = 3\int \frac{x}{(x^2)^2+(3)^2}\, dx = 3\ \left(\frac{1}{2}\right)\int \frac{2x}{(x^2)^2+(3)^2}\, dx$$

$$= \frac{3}{2}\,\frac{1}{3}\ \tan^{-1}\left(\frac{x^2}{3}\right) + c = \frac{1}{2}\ \tan^{-1}\left(\frac{x^2}{3}\right) + c$$

باستخدام القانون

$$.\int \frac{f'(x)}{a^2 + [f(x)]^2}\,dx = \frac{1}{a}\ \tan^{-1}\left(\frac{f(x)}{a}\right) + c$$

(8) $\int (x+1)\,\sqrt{x-1}\,dx$

الحل : بوضع $\sqrt{x-1} = u \implies x-1 = u^2 \implies x = u^2+1 \implies x+1 = u^2+2$

$$dx = 2u\,du$$

$$\int (x+1)\,\sqrt{x-1}\,dx = \int \left(u^2+2\right)\ u\,2u\,du = 2\int u^2\left(u^2+2\right)\ du$$

$$= 2\int \left(u^4 + 2u^2\right)\ du = 2\left(\frac{u^5}{5} + 2\,\frac{u^3}{3}\right) + c$$

$$= \frac{2}{5}\left(\sqrt{x-1}\right)^5 + \frac{4}{3}\left(\sqrt{x-1}\right)^3 + c = \frac{2}{5}\ (x+1)^{\frac{5}{2}} + \frac{4}{3}\ (x+1)^{\frac{3}{2}} + c$$

(9) $\int e^{-2x}\cosh x\,dx$

الحل :

$$\int e^{-2x}\cosh x\,dx = \int e^{-2x}\left(\frac{e^{x}+e^{-x}}{2}\right)\ dx = \int \left(\frac{e^{-x}+e^{-3x}}{2}\right)\ dx$$

$$= \int \left(\frac{e^{-x}}{2} + \frac{e^{-3x}}{2}\right)\ dx = \frac{1}{2}\,(-1)\int e^{-x}\ (-1)\,dx + \frac{1}{2}\,\frac{1}{-3}\int e^{-3x}(-3)\,dx$$

$$= \frac{-1}{2}\,e^{-x} - \frac{1}{6}\,e^{-3x} + c = -\frac{e^{-x}}{2} - \frac{e^{-3x}}{6} + c$$

باستخدام القانون $\int e^{f(x)}\ f'(x)\,dx = e^{f(x)} + c$