كلية العلوم
قسم الإحصاء وبحوث العمليات

College of Science.
Department of Statistics & Operations Research

Second Midterm Exam
Academic Year 1443-1444 Hijri-First Semester

Exam Information معلومات الامتحان			
Course name	Modeling and Simulation	النمذجة والمحاكاة	اسم المقرر
Course Code	OPER 441	441 بحث	رمز المقرر
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Exam Time	1:00 pm		وقت الامتحان
Exam Duration	2.5 hours	ساعتان ونصف	مدة الامتحان
Classroom No.			رقم قاعة الاختبار
Instructor Name			اسم استاذ المقرر

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Student's Name		اسم الطالب
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Section No.		رقم الشعبة
Serial Number		الرقم التسلسلي

General Instructions: / **تعليمات عامة:**

- Your Exam consists of ☐ PAGES (except this paper)
- Keep your mobile and smart watch out of the classroom.
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- عدد صفحات الامتحان ☐ صفحة (باستثناء هذه الورقة)
- يجب إبقاء الهواتف والساعات الذكية خارج قاعة الامتحان.
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هذا الجزء خاص بأستاذ المادة

This section is ONLY for instructor

#	Course Learning Outcomes (CLOs)	Related Question (s)	Points	Final Score
1	Understanding the processes and steps for building a simulation model			
2	Implement an inverse cumulative distribution function based random variate generation algorithm			
3	Explain and implement the convolution algorithm for random variate generation			
4	Explain and implement the acceptance rejection algorithm for random variate generation			
5	Compute statistical quantities from simulation output			
6	Generate random numbers from any given distribution discrete or continuous			
7	Building simulation models from basic applications			
8				

EXAM COVER PAGE

Question #1:

The period of time (in months) between rainfalls in Riyadh city is modeled using the following pdf:

$$f(x) = 1.06\, \boldsymbol{e}^{\frac{-x}{2}} \quad ; \quad 1 \le x \le 4$$

Where random variable X is time between rainfalls in months.

a) Write the inverse transform for measuring the time between rainfalls.
b) Simulate the rainfalls (in months) in Riyadh city using all number given below (move by rows)
c) From simulated date, compute the average rainfall in Riyadh per year.
d) Assume that the period of time of each rain fall (in hours) is a Binomial distribution that last for a maximum of 3 hours with parameter (p=0.75). Simulate the duration of rainfall using the numbers below (move by columns).

Move by rows →				
	0.744	0.443	0.820	0.166
	0.256	0.542	0.844	0.936
	0.744	0.444	0.017	0.967

Solution

(a) Inverse transform between rainfalls (months)

$$\text{CDF} = 2(1.06)\left(\boldsymbol{e}^{\frac{-1}{2}} - \boldsymbol{e}^{\frac{-x}{2}}\right) \rightarrow F^{-1} = X(u) = -2\ln\left(\boldsymbol{e}^{\frac{-1}{2}} - \frac{u}{2(1.06)}\right)$$

(b) Simulate the next 4 rainfalls in Abha city: rainfall #n time **RFT(n) = RFT(n-1) +X(n)**

	u~U	X(n) mon	RFT(n) mon	Rainfall duration
Rain #1	0.744	2.73	2.73	**3**
Rain #2	0.443	1.84	4.57	**2**
Rain #3	0.82	3.03	7.60	**3**
Rain #4	0.166	1.28	8.88	**2**
Rain #5	0.256	1.44	10.32	**2**
Rain #6	0.542	2.09	12.42	**2**
Rain #7	0.844	3.14	15.56	**3**
Rain #8	0.936	3.60	19.16	**3**
Rain #9	0.744	2.73	21.89	**3**
Rain #10	0.444	1.85	23.73	**2**
Rain #11	0.017	1.03	24.76	**1**
Rain #12	0.967	3.79	28.55	**3**

(c) From simulated date, compute the average rainfall in Riyadh per year

Total number of rainfall from simulation = 11
Total period of simulation = 24.76 months
number of years in simulation = 24.76/12 = 2.063 years

X	**0**	**1**	**2**	**3**
p{X}	0.015625	0.140625	0.421875	0.421875
CDF{X}	0.015625	0.15625	0.578125	1

Question #2:

Consider the continuous random Y with the following pdf:

$$f_Y(y) = \begin{cases} 0, & y < 0 \\ 0.2y, & 0 \le y \le 1 \\ 0.1 + 0.1y, & 1 < y \le 2 \\ 0.25 + 0.025y, & 2 < y \le 4 \\ 0, & y > 4. \end{cases}$$

a) Write the cumulative distribution function of $f_Y(y)$ and compute the expected value of Y?

b) Write the Inverse transform for $f_Y(y)$?

c) Write the algorithm for generating 10 random numbers from $f_Y(y)$.

d) Let Y be the time (in hourse) for surgery in an Operations Room (OR) in K.A.N Hospital. The hospital has one Operations Room. Patients are transferred to the OR according to a Poisson Process with average time between arrivals equals to 5 hours. The operations Room work 24 hours per day. Define the random processes for the simulation model for the OR and apply it for 5 patients. Use the following U[0,1] numbers as needed. Starting simulations time is zero.

Move by rows →				
	0.744	0.443	0.820	0.166
	0.256	0.542	0.844	0.936
	0.744	0.444	0.017	0.967

(a)

$$\underset{0 \le y \le 1}{F_Y(y)} = \int_0^y 0.2t\,dt = \left[0.1\,t^2\right]_0^y = 0.1\,y^2 \quad ; \quad 0 \le y \le 1$$

$$\underset{1 \le y \le 2}{F_Y(y)} = \int_0^1 f_Y(y)\,dt + \int_1^y 0.1(1+t)\,dt = 0.1 + \left[0.1\,t + \frac{0.1}{2}\,t^2\right]_1^y$$

$$= 0.1\,y + \frac{0.1}{2}\,y^2 - 0.05 \quad ; \quad 1 \le y \le 2$$

$$\underset{2 < y \le 4}{F_Y(y)} = \int_0^2 f_Y(t)\,dt + \int_2^y (0.25 + 0.025t)\,dt = 0.35 + \left[0.25t + \frac{0.025}{2}t^2\right]_2^y$$

$$= \frac{0.025}{2}\,y^2 + 0.25y - 0.2 \quad ; \quad 2 < y \le 4$$

$$F_Y(y) = \begin{cases} 0.1\,y^2 & ; \quad 0 \le y \le 1 \\ \frac{0.1}{2}y^2 + 0.1y - 0.05 & ; \quad 1 < y \le 2 \\ \frac{0.025}{2}y^2 + 0.25y - 0.2 & ; \quad 2 < y \le 4 \end{cases}$$

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(b)

for $0 \le y \le 1$

$$u = 0.1\,y^2 \iff y = \sqrt{\frac{u}{0.1}}$$

$$\therefore\ 0 \le \sqrt{\frac{u}{0.1}} \le 1 \iff 0 \le u \le 0.1$$

for $2 \le y \le 4$

$$u = \frac{0.25y^2}{2} + 0.25y - 0.2$$

$$\iff y^2 + 20y - (16 + 80u) = 0$$

$$\therefore\ y = \frac{-20 \pm \sqrt{20^2 + 4(16+80u)}}{2}$$

$$y = -10 \pm \sqrt{116 + 80u} > 0$$

$$\Rightarrow y = -10 + \sqrt{116 + 80u}$$

$$\therefore\ 2 \le -10 + \sqrt{116 + 80u} \le 1$$

$$\vdots$$

$$0.35 \le u \le 1$$

for $1 \le u \le 2$

$$u = \frac{0.1}{2} y^2 + 0.1y - 0.05$$

$$\iff y^2 + 2y - (1 + 20u) = 0$$

$$\therefore\ y = \frac{-2 \pm \sqrt{4 + 4(1+20u)}}{2}$$

$$y = -1 \pm \sqrt{2 + 20u} > 0$$

$$\therefore\ y = -1 + \sqrt{2 + 20u}$$

$$1 \le y \le 2$$

$$1 \le -1 + \sqrt{2 + 20u} \le 2$$

$$2 \le \sqrt{2 + 20u} \le 3$$

$$4 \le 2 + 20u \le 9$$

$$2 \le 20u \le 7$$

$$0.1 \le u \le 0.35$$

(c) The algorithm

1. Let N = 1
2. Get u ~U[0,1]
3. If $0 \le u \le 0.1$ Then
    - $y = (10\,u)^{0.5}$
4. If $0.1 < u \le 0.35$ Then
    - $y = -1+(2+20\,u)^{0.5}$
5. If $0.35 < u \le 1$ Then
    - $y = -10+(116+80\,u)^{0.5}$
6. Let N = N +1
7. If $N \le 10$. Then GO TO Step 2
    - Else, STOP simulation

(d) The Simulation Model:

Random Process #1: Patient Arrival AT(n)
Let T(n) time between patients
T(n) ~Exp (1/5) ➔ T(n) = –5 ln(1–w) ; w ~U[0,1]

AT(n) = AT(n–1) + T(n)

Random Process #2: Y(n) is the operation duration for patient (n)

Patient #	u~U[0,1]	T(n) (hr)	AT(n) (hr)	u~U[0,1]	Y^{-1}	Operation Time Y (hr)	Leave OR
1	0.744	6.82	6.82	0.443	3	2.06	9.12
2	0.820	8.5	15.39	0.166	2	1.31	16.70
3	0.256	1.48	16.87	0.542	3	2.62	19.49
4	0.844	9.29	26.15	0.936	3	3.82	29.97
5	0.744	6.81	32.97	0.444	3	2.31	35.28

Question #3:

A car repair workshop manager wants to develop a simulation model. For one particular repair, the times to completion can be represented by the following distribution (x in days):

$$f(x) = \begin{cases} \frac{x}{8} - \frac{1}{4} & ; \quad 2 \le x \le 4 \\ \frac{10}{24} - \frac{x}{24} & ; \quad 4 \le x \le 10 \end{cases}$$

(a) Write the inverse transform to generate random numbers for repair time.

(b) Using U[0,1] random number in the following table, use the inverse transform in part (a) to determine the time of each car repair to compute the average speed of completion for this workshop (number of services completed per day). The workshop works daily from 8:00am to 8:00pm

(c) Write the algorithm for using the acceptance/rejection method to simulate random number from f(x). Use the same random numbers in the table to apply the algorithm.

	1	2	3	4	5	6	7	8	9	10
U[0,1]	0.138	0.776	0.911	0.259	0.458	0.343	0.105	0.940	0.188	0.343
Repair Time	3.486	6.721	7.933	4.036	4.899	4.384	3.296	8.303	3.734	4.384

Q#1: $f(x) = \begin{cases} \frac{x}{8} - \frac{1}{4} & ; 2 \le x \le 4 \\ \frac{1}{24}(10 - x) & ; 4 \le x \le 10 \end{cases}$

(a) CDF: $2 \le x \le 4$

$$F(x) = \int_2^x \frac{y}{8} - \frac{1}{4}\, dy$$

$$= \left[\frac{y^2}{16} - \frac{y}{4}\right]_2^x$$

$$= \frac{x^2}{16} - \frac{x}{4} + \frac{1}{4} \quad ; \quad 2 \le x \le 4$$

CDF: $4 \le x \le 10$

$$F(x) = \int_2^4 f(y)\, dy + \int_4^x \frac{1}{24}(10 - y)\, dy$$

$$= 1/4 + \frac{1}{24}\left[10y - \frac{1}{2}y^2\right]_4^x$$

$$= \frac{10}{24}x - \frac{1}{48}x^2 - \frac{52}{48}$$

for $4 \le x \le 4$

(b). <u>Inverse</u> $F(x) = u$ $\quad u \sim U[0,1]$

$$\frac{x^2}{16} - \frac{x}{4} + \frac{1}{4} = u$$

$$\Rightarrow \frac{x^2}{16} - \frac{x}{4} + \left(\frac{1}{4} - u\right) = 0$$

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{-(-1) \pm \sqrt{1 - 4(1-4u)(1/4)}}{2(1/4)}$$

$$\Rightarrow x = 2 \pm 2\sqrt{4u}$$

take $\boxed{X = 2 + 4\sqrt{u}}$

$$2 \le x \le 4$$

$$2 \le 2 + 4\sqrt{u} \le 4$$

$$0 \le 4\sqrt{u} \le 2$$

$$\Rightarrow \boxed{0 \le u \le 1/4}$$

Inverse $F(x) = u$

$$\frac{10}{24}x - \frac{1}{48}x^2 - \frac{52}{48} = u$$

$$\Leftrightarrow \frac{1}{48}x^2 - \frac{10}{24}x + \left(\frac{52}{48} + u\right) = 0$$

$$\therefore x = \frac{\left(\frac{10}{24}\right) \pm \sqrt{\left(\frac{10}{24}\right)^2 - 4\left(\frac{1}{48}\right)\left(\frac{52}{48} + u\right)}}{(2/48)}$$

$$\therefore x = 10 \pm 24\sqrt{\left(\frac{5}{12}\right)^2 - \frac{4}{48}\left(\frac{13}{12} + u\right)}$$

since $4 \le x \le 10$

$\Rightarrow$ Take $\boxed{x = 10 - 24\sqrt{\left(\frac{5}{12}\right)^2 - \frac{4}{48}\left(\frac{13}{12} + \right.}}$

and $\boxed{1/4 \le u \le 1}$

(C) Algorithm for using the acceptance/rejection method to simulate random number from $f(x)$. Use the same random numbers in the table to apply the algorithm.

G(x) = max {f(x)} = f(4) = (4/8)-(1/4) = 0.25 ; 2 <= X <= 10

C = 0.25*(10-2) = 4 → W(x) = 0.25/4 = 0.0625 ; 2 <= X <= 10 → W^{-1}= 2 + 8 u_1

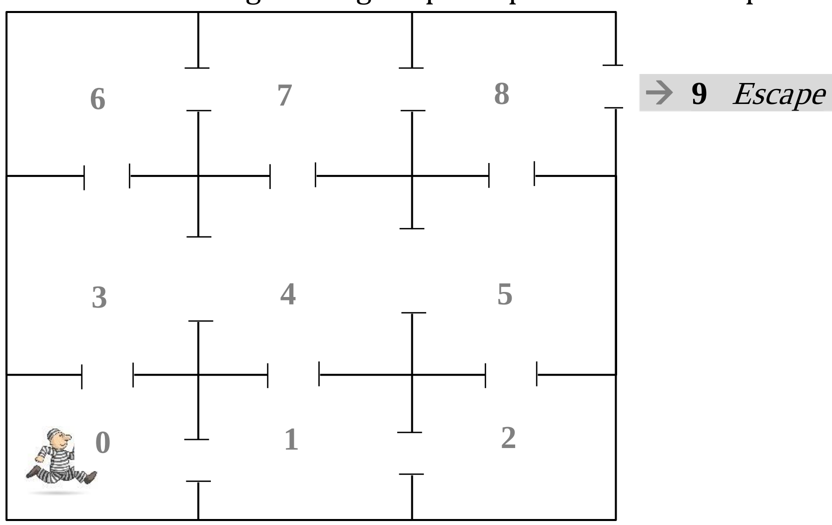
<u>Algorithm</u>

1. **Generate u1~ Y[0,1]**
2. **Get w → → W^{-1}= 2 + 8 u_1**
3. **Get f(w)**
4. **Generate u2~U[0,1]**
5. **If f(w)/0.25 >= u2 → accept w ; else reject and repeat 1**

	1	2	3	4	5	6	7	8	9	10
U[0,1]	0.138	0.776	0.911	0.259	0.458	0.343	0.105	0.94	0.188	0.343
w	3.104	**test**	9.288	**test**	5.664	**test**	2.84	**test**	3.504	**test**
f(w)1	0.138		0		0		0.105		0.188	
f(w)2	0.0000		0.0297		0.1807		0.0000		0.0000	
f/g	0.5520	Reject	0.1187		0.7227	Accept	0.4200	Accept	0.7520	Accept
Repair Time	**Reject**		**Reject**		**5.664**		**2.84**		**3.504**	

Question #4:

Consider an escaped prisoner who entered in a maze. The maze contains 8 chambers. If the prisoner enters any chamber he is equally likely to choose any door in the chamber (including the door he entered through). The prisoner has no time to waste, he has only 5 moves to escape out starting from chamber 0 before he gets caught up and put back into the prison.



a) Define all random processes and write the algorithm for generating the prisoner moves to escape.

b) Using your answer in (a), Simulate the path of the prisoner for 5 attempts in the following table

	Move-1	Move-2	Move-3	Move-4	Move-5	Move-6	Move-7	Move-8	Move-9	Move-10
Attept#1	0.338	0.765	0.976	0.107	0.154	0.684	0.901	0.715	0.013	0.228
Chambers	**1**	**4**	**5**	**2**	**1**	**4**	**7**	**8**	**5**	**2**
Attept#2	0.849	0.145	0.081	0.308	0.543	0.959	0.113	0.381	0.492	0.158
Chambers	**3**	**0**	**1**	**0**	**3**	**6**	**3**	**4**	**3**	**0**
Attept#3	0.731	0.487	0.396	0.144	0.982	0.180	0.631	0.802	0.712	0.507
Chambers	**3**	**4**	**3**	**0**	**3**	**0**	**3**	**6**	**7**	**6**

c) From the simulation data, what is the estimate number of moves to escape.

The algorithm

1. If Chamber# = 0 then
   - Get u ~U[0,1]
   - If u ≤ 0.5 Then Chamber# = 1
   - Else, Chamber# = 3
2. If Chamber# = 1 then
   - Get u ~U[0,1]
   - If u ≤ 0.33 Then Chamber# = 0
   - Else, u ≤ 0.66 Then Chamber# = 2
   - Else, Chamber# = 4

3. If Chamber# = 2 then
   - Get u ~U[0,1]
   - If u ≤ 0.5 Then Chamber# = 1
   - Else, Chamber# = 5
4. If Chamber# = 3 then
   - Get u ~U[0,1]
   - If u ≤ 0.33 Then Chamber# = 0
   - Else, u ≤ 0.66 Then Chamber# = 4
   - Else, Chamber# = 6
5. If Chamber# = 4 then
   - Get u ~U[0,1]
   - If u ≤ 0.25 Then Chamber# = 1
   - Else, u ≤ 0.5 Then Chamber# = 3
   - Else, u ≤ 0.75 Then Chamber# = 5
   - Else, Chamber# = 7
6. If Chamber# = 5 then
   - Get u ~U[0,1]
   - If u ≤ 0.33 Then Chamber# = 2
   - Else, u ≤ 0.66 Then Chamber# = 4
   - Else, Chamber# = 8
7. If Chamber# = 6 then
   - Get u ~U[0,1]
   - If u ≤ 0.5 Then Chamber# = 3
   - Else, Chamber# = 7
8. If Chamber# = 7 then
   - Get u ~U[0,1]
   - If u ≤ 0.33 Then Chamber# = 4
   - Else, u ≤ 0.66 Then Chamber# = 6
   - Else, Chamber# = 8
9. If Chamber# = 8 then
   - Get u ~U[0,1]
   - If u ≤ 0.33 Then Chamber# = 5
   - Else, u ≤ 0.66 Then Chamber# = 7
   - Else, Chamber# = 9

Question #5:

Airplanes land on a small airport according to time between airplanes follows Erlang distribution with parameters r=3 and rate λ=5 airplanes per day. Also, the airplanes depart at random from the same airport according to Weibull distribution with parameter α=3 and β=0.5 for the time between air planes

$$f(x) = \frac{\alpha}{\beta^{\alpha}} x^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}} \quad ; \quad x \geq 0$$

Assume that the airport works 18 hours.

1. Give a random number for total number of air planes landed in the airport on one working day using the following U[0,1] numbers ***as needed***. *(Answer on the back of the page)*

	1	2	3	4	5	6	7	8	9	10
U[0,1]	0.171	0.023	0.879	0.305	0.696	0.415	0. 721	0.901	0.188	0.051

Algorithm:

1. Let TBL(n): Time between landing airplanes (days) ~ Er (r=3 and rate λ=5 plan/day)
2. Let LT(n) : the landing time of plane (n) → LT(n) = LT(n-1) + TBL(n)
3. Get TBL(n) using convulsion:
   - Get u_1, u_2, u_3 ~U[0,1]
   - TBL(n) = $T_1 + T_2 + T_3$; T_i ~ Exp (λ plane/day)
   - TBL(n) = -(1/5) [ln(1-u_1) + ln(1-u_2) + ln(1-u_3)]
   - LT(n) = LT(n-1) + TBL(n)
   - Repeat if LT(n) <= 1 day

	1	2	3	4	5	6	7	8	9	10
U1	0.171	0.305	0.721	0.051	0.696	0.415	0.188	0.023	0.696	0.901
U2	0.023	0.696	0.901	0.171	0.305	0.721	0.051	0.696	0.415	0.188
U3	0.879	0.415	0.188	0.023	0.696	0.901	0.171	0.305	0.721	0.051
TBL(n) (day)	**0.465**	**0.418**	**0.759**	**0.053**	**0.549**	**0.825**	**0.090**	**0.316**	**0.601**	**0.515**
LT(n)*(day)*	**0.465**	**0.883**	**1.642**	**1.695**	**2.244**	**3.069**	**3.159**	**3.474**	**4.075**	**4.589**

2. Give a random generation for the time of ***the last airplane departed*** from the airport on one day using the following U[0,1] numbers ***as needed***.

	1	2	3	4	5	6	7	8	9	10
U[0,1]	0.815	0.636	0.563	0.923	0.925	0.605	0. 971	0.023	0.879	0.305

Algorithm:

1. Let TBD(n): Time between departing airplanes (days) ~ Weibull (α=3 and β=0.5)
2. Let DT(n) : the landing time of plane (n) → DT(n) = DT(n-1) + TBD(n)
3. Get TBD(n) using Inverse:
   - Get u_1~U[0,1]
   - TBD(n) = $0.5[-\ln(1-U)]^{\frac{1}{3}}$
   - DT(n) = DT(n-1) + TBD(n)

- Repeat if DT(n) <= 1 day

	1	2	3	4	5	6	7	8	9	10
U[0,1]	0.815	0.636	0.563	0.923	0.925	0.605	0.971	0.023	0.879	0.305
TBD(n) (day)	**0.595**	**0.502**	**0.469**	**0.684**	**0.687**	**0.488**	**0.762**	**0.143**	**0.642**	**0.357**
DT(n)(day)	**0.595**	**1.097**	**1.567**	**2.251**	**2.938**	**3.425**	**4.187**	**4.330**	**4.972**	**5.329**

3. Assume that the percentage of departing airplanes from the airport is 44.5%. Make a discrete event simulation run of the airport for 12 hours. Write the simulation algorithm for this system and use it with the following U[0,1] as needed. *(Answer on the back of the page)*

Algorithm:

1. Let Sim Time= 0
2. Get u ~U[0,1]
   - If u ≤ 0.445 Then "Airplane Departing" and Go to Step 3
   - Else, "Airplane Arriving" and Go To Step 4
3. If "Airplane Departing" then
   - Get u_1~U[0,1]
   - TBD(n) = $0.5[-\ln(1-U)]^{\frac{1}{3}}$
   - DT(n) = DT(n-1) + TBD(n)
   - Update Sim Time:

     Sim Time = Clock Time + DT(n)
4. If "Airplane Arriving" then
   - Get u_1, u_2, u_3 ~U[0,1]
   - TBL(n) = T_1+ T_2+ T_3; $T_i \sim$ Exp (λ plane/day)
   - TBL(n) = -(1/5) [ln(1-u_1) + ln(1-u_2) + ln(1-u_3)]
   - LT(n) = LT(n-1) + TBL(n)
   - Update Sim Time:

     Sim Time = Clock Time + LT(n)

event#	U	Event type	U1	U2	U3	TBD/TBL	Sim Time day	Sim Time hrs
1	0.248	**Dep**	0.817			**0.60**	0.597	**10.7**
2	0.968	**Land**	0.465	0.668	0.482	**0.48**	1.074	**19.3**
3	0.876	**Land**	0.86	0.694	0.732	**0.89**	1.967	**35.4**
4	0.639	**Land**	0.002	0.546	0.695	**0.40**	2.363	**42.5**
5	0.035	**Dep**	0.243			**0.33**	2.689	**48.4**