The Riemann Integral

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Introduction to Riemann Integral

Definition

A finite ordered set $\sigma = \{x_0, \ldots, x_n\}$ is called a partition of the interval $[a, b]$ if $a = x_0 < \ldots < x_n = b$. The interval $[x_j, x_{j+1}]$ is called the j^{th} subinterval of σ.

Definition

Let $f : [a, b] \longrightarrow \mathbb{R}$ be a bounded function. Define

$$M_j = \sup_{x \in [x_j, x_{j+1}]} f(x), \qquad m_j = \inf_{x \in [x_j, x_{j+1}]} f(x),$$

$$U(f, \sigma) = \sum_{j=0}^{n-1} M_j (x_{j+1} - x_j) \tag{1}$$

$$L(f, \sigma) = \sum_{j=0}^{n-1} m_j (x_{j+1} - x_j). \tag{2}$$

$U(f, \sigma)$ and $L(f, \sigma)$ are called respectively the upper sum and the lower sum of f on the partition σ. Note that $L(f, \sigma) \leq U(f, \sigma)$.

Proposition

If σ_1 is finer than σ_2 and $f : [a, b] \longrightarrow \mathbb{R}$ is a bounded function, then

$$L(f, \sigma_2) \leq L(f, \sigma_1) \leq U(f, \sigma_1) \leq U(f, \sigma_2) \tag{3}$$

Proposition

If $f : [a, b] \longrightarrow \mathbb{R}$ is a bounded function and σ_1, σ_2 are two partitions of the interval $[a, b]$, then $L(f, \sigma_1) \leq U(f, \sigma_2)$.

Definition

Let $f : [a, b] \longrightarrow \mathbb{R}$ be a bounded function. If we denote $K([a, b])$ the set of partitions of $[a, b]$, then we define the upper integral of f on the interval $[a, b]$ by

$$U(f) = \inf_{\sigma \in K([a,b])} U(f, \sigma)$$

and the lower integral of f on the interval $[a, b]$ by

$$L(f) = \sup_{\sigma \in K([a,b])} L(f, \sigma)$$

Definition

Let $f : [a, b] \longrightarrow \mathbb{R}$ be a bounded function. We say that f is Riemann integrable on the interval $[a, b]$ if $U(f) = L(f)$.
If f is Riemann integrable on the interval $[a, b]$, we denote $\int_a^b f(x)dx = U(f) = L(f)$ which called the integral of f on the interval $[a, b]$.
The set of Riemann integrable functions on the interval $[a, b]$ is denoted by $\mathscr{R}([a, b])$.

Criterions of Riemann Integrability

Theorem

[Riemann's Criterion]
Let $f : [a, b] \longrightarrow \mathbb{R}$ be a bounded function. The following statements are equivalent
i) f is Riemann-integrable.
ii) $\forall \varepsilon > 0$; there exists a partition σ such that $U(f, \sigma) - L(f, \sigma) \leq \varepsilon$.

Definition

If $\sigma = \{x_0, \dots, x_n\}$ is a partition of the interval $[a, b]$, we define the norm of σ by

$$||\sigma|| = \sup_{0 \le j \le n-1} x_{j+1} - x_j.$$

Theorem

[Darboux's Criterion]
Let $f : [a, b] \longrightarrow \mathbb{R}$ be a bounded function. The following statements are equivalent
i) f is Riemann-integrable.
ii) For all $\varepsilon > 0$; there exists $\delta > 0$ such that for all partition of the interval $[a, b]$ such that if $||\sigma|| \le \delta$ then $U(f, \sigma) - L(f, \sigma) \le \varepsilon$.

Definition

Let $\sigma = \{x_0, \dots, x_n\}$ be a partition of the interval $[a, b]$. We say that $\alpha = \{\alpha_0, \dots, \alpha_{n-1}\}$ is a mark of σ if $\forall 0 \leq j \leq n-1$, $\alpha_j \in [x_j, x_{j+1}]$.
We define

$$U(f, \sigma, \alpha) = \sum_{j=0}^{n-1} f(\alpha_j)(x_{j+1} - x_j)$$

called the Riemann sum of f on σ with respect to the mark α.
As particular case, if f is Riemann integrable on the interval $[a, b]$, the sequence S_n defined by

$$S_n = \frac{b-a}{n} \sum_{k=1}^{n} f(a + k\frac{b-a}{n})$$

$\int^b$

Properties of the Riemann Integral

Properties

1. Linearity $\int_a^b \alpha(f+\beta g)(x)dx = \alpha\int_a^b f(x)dx + \beta\int_a^b g(x)dx.$
2. If $f \geq 0$, then $\int_a^b f(x)dx \geq 0.$
3. If $f \leq g$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx.$
4. $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx.$
5. If $m \leq f(x) \leq M$, for all $x \in [a, b]$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

Theorem

Let $f : [a, b] \longrightarrow [c, d]$ be a Riemann integrable function and $\varphi : [c, d] \longrightarrow \mathbb{R}$ be a continuous function. Then $\varphi \circ f$ is Riemann integrable.

Theorem

Let $f : [a, b] \longrightarrow [c, d]$ be a Riemann integrable function, then the function F defined by

$$F(x) = \int_a^x f(t)dt$$

is continuous.
If f is continuous at the point c, then F is differentiable at c and $F'(c) = f(c)$.

Remarks

1. The integral of a non negative function Riemann-integrable is a non negative real number.
2. If f is Riemann-integrable on $[a, b]$, then

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx \leq (b-a)\sup_{x\in[a,b]} |f(x)|.$$

Corollary

IF f is Riemann-integrable on $[a, b]$, then the function $F(x) = \int_a^x f(t)\ dt$ is continuous on $[a, b]$.

Proof

$F(x) - F(y) = \int_y^x f(t)\, dt$. Since f is bounded on $[a, b]$, there exist $M > 0$ such that $|F(x) - F(y)| \leq M|x - y|$. □

Corollary

Let f be a Riemann-integrable function on $[a, b]$. If $m = \inf_{x \in [a,b]} f(x)$ and $M = \sup_{x \in [a,b]} f(x)$, there exist $\lambda \in [m, M]$ such that

$$\frac{1}{b-a} \int_a^b f(x)\, dx = \lambda.$$

Proof

We have $m(b-a) \leq \int_a^b f(x)\, dx \leq M(b-a)$, then $\frac{1}{b-a} \int_a^b f(x)\, dx \in [m, M]$.

Corollary

[First Mean Value Formula]
Let f and g be two Riemann-integrable functions on an interval $[a, b]$. We assume that f is continuous and g has a constant sign on $[a, b]$. then there exist $c \in [a, b]$ such that

$$\int_a^b f(x) g(x) dx = f(c) \int_a^b g(x)\, dx.$$

Theorem

[The Cauchy-Schwarz Inequality]
Let f and g be two Riemann-integrable functions on an interval $[a, b]$, then

$$\Big(\int_a^b f(x)g(x)\ dx\Big)^2 \leq \int_a^b f^2(x)\ dx \int_a^b g^2(x)\ dx.$$

Corollary

[Minkowsky Inequality]
Let f and g be two Riemann-integrable functions on an interval $[a, b]$, then

$$\Big(\int_a^b (f(x)+g(x))^2\ dx\Big)^{\frac{1}{2}} \leq \Big(\int_a^b f^2(x)\ dx\Big)^{\frac{1}{2}} + \Big(\int_a^b g^2(x)\ dx\Big)^{\frac{1}{2}}.$$

Remarks

Let f and g be two non negative Riemann-integrable functions on an interval $[a, b]$ and let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.
1) For $x, y \in \mathbb{R}^+$

$$xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q. \tag{4}$$

Indeed, the inequality is trivial if $x = 0$ or $y = 0$. We take $xy > 0$. The Logarithmic function is concave, then $\ln\left(\frac{1}{p}x^p + \frac{1}{q}y^q\right) \geq \ln(xy)$. The result follows by taking the exponential of two sides.

2) We have

$$\int_a^b f(x)g(x)dx \leq \frac{\lambda^p}{p}\int_a^b f^p(x)dx + \frac{\lambda^{-q}}{q}\int_a^b g^q(x)dx, \quad \forall \lambda > 0 \tag{5}$$

If we replace x by $\lambda f(x)$ and y by $\frac{1}{\lambda}g(x)$ in the inequality (**??**), we deduce the inequality (**??**).

3) We deduce also that if $\int_a^b f^p(x)dx = 0$, then $\int_a^b f(x)g(x)dx = 0$.

Indeed if $\int_a^b f^p(x)dx = 0$, then $\int_a^b f(x)g(x)dx \leq \frac{\lambda^{-q}}{q}\int_a^b g^q(x)dx$ for all $\lambda > 0$. If we take the limit $(\lambda \to +\infty)$, we deduce that $\int_a^b f(x)g(x)dx = 0$.

Theorem

[Hölder Inequality for the integrals]
Let f and g be two non negative Riemann-integrable functions on an interval $[a, b]$. Then for all $p, q > 1$, where $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$\int_a^b f(x)g(x)\ dx \leq \Big(\int_a^b f^p(x)\ dx\Big)^{\frac{1}{p}}\Big(\int_a^b g^q(x)\ dx\Big)^{\frac{1}{q}}.$$

Proof

If $\int_a^b f^p(x)\ dx = 0$ or $\int_a^b g^q(x)\ dx = 0$, the inequality results from the previous remark.
If $\int_a^b f^p(x)\ dx \neq 0$ and $\int_a^b g^q(x)\ dx \neq 0$, we set $f_1(x) = \dfrac{f(x)}{(\int_a^b f^p(t)\ dt)^{1/p}}$

and $g_1(x) = \dfrac{g(x)}{(\int_a^b g^q(t)\ dt)^{1/q}}$. We have

$\int_a^b f_1^p(x)\ dx = \int_a^b g_1^q(x)\ dx = 1$. From the inequality (**??**), we have $f_1 g_1 \leq \frac{1}{p} f_1^p + \frac{1}{q} g_1^q$. We integrate on the interval $[a, b]$, we have

Theorem

[Second Mean Value Formula] Let f be non negative continuous function and decreasing on the interval $[a, b]$ and let g be Riemann-integrable function on $[a, b]$. Then there exists $c \in [a, b]$ such that

$$\int_a^b f(x)g(x)\ dx = f(a)\int_a^c g(x)\ dx.$$

Proof

We set $G(x) = \int_a^x g(t)\ dt$. The function G is continuous on $[a, b]$. We denote $m = \inf_{x\in[a,b]} G(x)$ and $M = \sup_{x\in[a,b]} G(x)$. To prove the theorem it suffices to prove that $mf(a) \leq \int_a^b f(x)g(x)\ dx \leq Mf(a)$. Let $\sigma_n = (x_0 = a, \dots, x_n)$ be a partition of $[a, b]$ such that $x_{i+1} - x_i = \dfrac{b-a}{n}$, $x_j = a + j\frac{b-a}{n}$. We set $\lambda_i = \dfrac{G(x_{i+1}) - G(x_i)}{x_{i+1} - x_i}$.

$$\lim_{n\to+\infty} \sum_{i=0}^{n-1} (x_{i+1} - x_i)(fg)(x_i) = \int_a^b f(x)g(x)\ dx.$$

$$
\begin{aligned}
\left|\sum_{i=0}^{n-1}(x_{i+1}-x_i)f(x_i)(g(x_i)-\lambda_i)\right| &\leq f(a)\sum_{i=0}^{n-1}(x_{i+1}-x_i)(M_i-m_i)\\
&= f(a)(U(g,\sigma_n)-L(g,\sigma_n) \underset{n\to+\infty}{\longrightarrow} 0,
\end{aligned}
$$

with $M_i = \sup_{t\in]x_i,x_{i+1}[} g(t)$ and $m_i = \inf_{t\in]x_i,x_{i+1}[} g(t)$. It results that

$$\lim_{n\to+\infty}\sum_{i=0}^{n-1} f(x_i)(G(x_{i+1})-G(x_i)) = \int_a^b f(x)g(x)\,dx.$$

$$\sum_{i=0}^{n-1} f(x_i)(G(x_{i+1}) - G(x_i)) = \sum_{i=0}^{n-1} f(x_i)G(x_{i+1}) - \sum_{i=0}^{n-1} f(x_i)G(x_i)$$
$$= \sum_{i=0}^{n-1}(f(x_{i-1}) - f(x_i))G(x_i) + f(x_{n-1})G(l$$

Since f is decreasing and non negative, we deduce

$$m\Big[f(x_{n-1}) + \sum_{i=0}^{n-1}(f(x_{i-1}) - f(x_i))\Big] \leq \sum_{i=0}^{n-1} f(x_i)(G(x_{i+1}) - G(x_i))$$
$$\leq M\Big[f(x_{n-1}) + \sum_{i=0}^{n-1}(f(x_{i-1}) - f(x$$

Then

Corollary

Let f be a monotone continuous function on an interval $[a, b]$ and let g be a Riemann-integrable function, then there exist $c \in [a, b]$ such that

$$\int_a^b f(x)g(x)\ dx = f(a)\int_a^c g(x)\ dx + f(b)\int_c^b g(x)\ dx.$$

Proof

We can assume that f is increasing. We use the previous Theorem to the functions $h(x) = f(b) - f(x)$ and g. $\square$

Theorem

[The Fundamental Theorem of Calculus]
Let $f : [a, b] \longrightarrow \mathbb{R}$ be a differentiable function and f' is Riemann integrable, then

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Theorem

[Taylor Formula with integral Reminder]
Let f be function of class $\mathcal{C}^{n+1}$ defined on an interval I in $\mathbb{R}$. For a and x in I, we have

$$f(x) = f(a) + \sum_{k=1}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) + \int_a^x \frac{(x-t)^n}{(n)!} f^{(n+1)}(t)dt.$$

Improper Integrals

Definition

1. Let f be a piecewise continuous function on the interval $[a, b[$, where $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{+\infty\}$.
   We say that the integral of f on the interval $[a, b[$ is convergent if the function $F(x) = \displaystyle\int_a^x f(t)dt$ defined on $[a, b[$ has a finite limit when x tends to b $(x < b)$. This limit is called the improper integral of f on $[a, b[$ and will be denoted by: $\displaystyle\int_a^b f(x)dx$.

Definition

1. Let f a piecewise continuous function on the interval $]a, b]$, where $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R}$.
We say that the integral of f on the interval $]a, b]$ is convergent if the function $G(x) = \int_x^b f(t)dt$ defined on $]a, b]$ has a finite limit when x tends to a $(x > a)$. This limit is called the improper integral of f on $]a, b]$ and will be denoted by: $\int_a^b f(x)dx$.

Definition

1. Let f be a piecewise continuous function on the interval $]a, b[$, where $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{+\infty\}$.
   We say that the integral of f on the interval $]a, b[$ is convergent if the integral of f is convergent on $]a, c]$ and on $[c, b[$ for any c in $]a, b[$.
2. Let f be a piecewise continuous function on an interval I.
   The function is called integrable on I (or the integral is absolutely convergent) if the integral of $|f|$ on the interval I is convergent.

Example

1. The integral of the function $f(t) = \dfrac{\sin t}{t}$ is convergent on $]0,1]$. The same for the function $g(t) = \sin \dfrac{1}{t}$ on $]0,1]$.
2. Let $\alpha \in \mathbb{R}$ and $a \in \mathbb{R}_+^*$. The integral $\int_a^{+\infty} \frac{dx}{x^\alpha}$ is convergent if and only if $\alpha > 1$ and the integral $\displaystyle\int_0^a \frac{dx}{x^\alpha}$ is convergent if and only if $\alpha < 1$.
3. $\displaystyle\int_0^{+\infty} \frac{dx}{1+x}$ is divergent, $\displaystyle\int_0^{+\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}$, $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}} = 2$.

Use of a Primitive

Let f be a function defined and piecewise continuous on an interval $I =]a, b[$. If F is a primitive of f on I, then $\int_a^b f(x)dx = F(b-) - F(a+)$ if $F(b-) - F(a+)$ is finite. $(F(b-) = \lim\limits_{x \to b, x < b} F(x)$ and $F(a+) = \lim\limits_{x \to a, x > a} F(x))$.

Change of Variables

Theorem

Let $\varphi\colon]a,b[\longrightarrow]\alpha,\beta[$ be a bijection of class $\mathcal{C}^1$ and let $f\colon]\alpha,\beta[\longrightarrow\mathbb{R}$ be a Riemann integrable function. Then

$\displaystyle\int_\alpha^\beta f(x)dx$ converges if and only if $\displaystyle\int_a^b f\circ\varphi(x)\varphi'(x)dx$ converges ,

and we have

$$\int_\alpha^\beta f(x)dx = \int_a^b f\circ\varphi(x)\varphi'(x)dx.$$

Example

Let $\beta \in \mathbb{R}$ and $a \in]1, +\infty[$. We set for $x \geq a$

$$F_\beta(x) = \int_a^x \frac{dt}{t(\ln t)^\beta}.$$

If $u = \ln t$, we get
$F_1(x) = \ln(\ln x) - \ln(\ln a)$ and for $\beta \neq 1$;
$F_\beta(x) = \int_{\ln a}^{\ln x} \frac{du}{u^\beta} = \frac{1}{1-\beta}[\frac{1}{(\ln x)^{\beta-1}} - \frac{1}{(\ln a)^{\beta-1}}]$. Thus the integral $\int_a^{+\infty} \frac{dx}{x(\ln x)^\beta}$ is convergent if and only if $\beta > 1$.

Integration by parts

Let $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{+\infty\}$ and u, v two real functions of class $\mathcal{C}^1$ on $[a, b[$. Assume that $\lim_{x \to b^-} u(x)v(x) = B$ exists in $\mathbb{R}$. Then $\int_a^b u(x)v'(x)dx$ converges if and only if $\int_a^b u'(x)v(x)dx$ converges and one has in the case of convergence

$$\int_a^b u(x)v'(x)dx = B - \int_a^b u'(x)v(x)dx - u(a)v(a).$$

The Cauchy Test

Theorem

[The Cauchy Test]
Let f be a piecewise continuous function on $[a, b[$, $b \in \mathbb{R} \cup \{+\infty\}$.
$\int_a^b f(x)dx$ converges if and only if

$$\forall \varepsilon > 0, \exists\ c \ \text{ tel que } \forall x, y \in]c, b[;\ \left|\int_x^y f(t)dt\right| \leq \varepsilon.$$

(we can take f a locally Riemann integrable function).

Remarks

1. This criterion is the Cauchy criterion of the existence of the limit of functions.
2. Let $a < b$ be two real numbers and $f : [a, b[\longrightarrow \mathbb{R}$ a bounded function. If f is piecewise continuous on $[a, b[$, then the integral of f on $[a, b[$ is convergent.

Comparison Test

Theorem

Let f be a non negative locally Riemann integrable function on $[a, b[$. The integral $\int_a^x f(t)dt$ converges if and only if there exists $M > 0$ such that $\forall x \in [a, b[;\ \int_a^x f(t)dt \leq M$.

Corollary

Let f and g be two non negative locally Riemann integrable functions on $[a, b[$. We assume that $f(t) \leq g(t); \forall t \in [a, b[$. Then

If $\int_a^b g(x)dx$ converges; the integral $\int_a^b f(x)dx$ converges.

If $\int_a^b f(x)dx$ diverges, the integral $\int_a^b g(x)dx$ diverges.

Corollary

Let f be a non negative locally Riemann integrable function on the interval $[a, b[$ and let $\mathcal{E} = \{(x_n)_n \in [a, b[;\ \lim_{n\to+\infty} x_n = b\}$. For any $x \in [a, b[$, we define $F(x) = \int_a^x f(t)dt$. Then following properties are equivalent

a) The integral of f on $[a, b[$ is convergent.
b) $\{F(x);\ x \in [a, b[\}$ is bounded.
c) For any sequence $(x_n)_n \in \mathcal{E}$, the sequence $(F(x_n))_n$ is convergent.
d) There exists a sequence $(x_n)_n \in \mathcal{E}$ such that the sequence $(F(x_n))_n$ is convergent.

Example

$f(t) = e^{-t^2}, t \in [0, +\infty[$, one has $f(t) \leq e^{-t}$ and $\int_0^{+\infty} e^{-x} dx = 1$
thus $\int_0^{+\infty} e^{-x^2} dx$ converges.
$\int_0^{\frac{\pi}{2}} \frac{dx}{\sin\ x}$ diverges because $\frac{1}{\sin x} \geq \frac{1}{x}\ \forall\ x\ \in]0, \frac{\pi}{2}]$

Proposition

Let I be an interval and $f\colon I \longrightarrow \mathbb{R}^+$ a non negative locally Riemann integrable function. The integral of f on I converges if and only if there exists an increasing sequence of intervals $([a_n, b_n])_n$ which covers I and a real $M \geq 0$ such that $\displaystyle\int_{a_n}^{b_n} f(x)dx \leq M$, for any $n \in \mathbb{N}$. In this case

$$\int_I f(x)dx = \sup_{n\in\mathbb{N}} \int_{a_n}^{b_n} f(x)dx.$$

Definition

Let f be a locally Riemann integrable function on an interval I. The integral of f on I is called absolutely convergent if the integral of $|f|$ on I is convergent.

Proposition

Let f be a locally Riemann integrable function on the interval $[a, b[$.

1. If the integral $\int_a^b f(x)dx$ is absolutely convergent, then $\int_a^b f(x)dx$ is convergent.
2. If there exists a non negative piecewise continuous function g on $[a, b[$, such that $\int_a^b g(x)dx$ converges and $|f(x)| \leq g(x)$, then $\int_a^b f(x)dx$ is absolutely convergent.

Remark

If $\int_a^b f(x)dx$ is convergent, then $\int_a^b f(x)dx$ is not in general absolutely convergent.

Consider the function $\frac{\sin x}{x}$ on the interval $[1, +\infty[$.

By integration by parts, $\int_1^s \frac{\sin x}{x} dx = \cos 1 - \frac{\cos s}{s} - \int_1^s \frac{\cos x}{x^2} dx$;

this shows that the integral of the function $\frac{\sin x}{x}$ is convergent on $[1, +\infty[$.

$$\begin{aligned}\int_{\pi}^{n\pi} \frac{|\sin x|}{x} dx &= \sum_{k=1}^{n-1} \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x} dx \\ &\geq \sum_{k=1}^{n-1} \frac{1}{(k+1)\pi} \int_{k\pi}^{(k+1)\pi} |\sin x| dx \\ &= \sum_{k=1}^{n-1} \frac{2}{(k+1)\pi}\end{aligned}$$

As the sequence $(v_n)_n$ defined by $v_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$ is divergent, then the integral of f is not absolutely convergent.

An other proof: we remark that $|\sin x| \geq \sin^2 x = \frac{1-\cos 2x}{2}$. As the integral $\int_1^{+\infty} \frac{\cos 2x}{2x} dx$ is convergent, the integral $\int_1^{+\infty} \frac{|\sin x|}{x} dx$ is divergent.

Proposition

Let $f\colon [a,b[\longrightarrow \mathbb{R}$ and $g\colon [a,b[\longrightarrow \mathbb{R}^+$ be two locally Riemann integrable functions. We assume that there exists $\ell \in \mathbb{R}\setminus\{0\}$ such that $f \approx \ell g$ (when t tends to b^-). Then $\displaystyle\int_a^b f(x)dx$ converges if and only if $\displaystyle\int_a^b g(x)dx$ converges.

Proof

If $f \approx \ell g$ (when t tends to b^-), then there exists a function h such that $f(t) = \ell h(t)g(t)$ and $\lim_{t \to b^-} h(t) = 1$. Thus $f(t) - \ell g(t) = (h(t)-1)\ell g(t)$ and, thus there exists c such that $\forall t \in]c, b[$, $|f(t) - \ell g(t)| \leq g(t)$, let $|f(t)| \leq (1 + |\ell|)g(t)$.

If the integral $\int_a^b g(x)dx$ converges, then the integral $\int_a^b f(x)dx$ converges absolutely.

If the integral $\int_a^b f(x)dx$ converges, as $\ell \neq 0$, there exists c such that $\forall t \in]c,b[\ ; |f(t) - \ell g(t)| \leq \frac{|\ell|}{2} g(t)$. If $x < y \in]c, b[$, we have $\Big| \int_x^y f(t) - \ell g(t) dt \Big| \leq \frac{|\ell|}{2} \int_x^y g(t)dt$, thus $\frac{|\ell|}{2} \int_x^y g(t)dt \leq \Big| \int_x^y f(t)dt \Big| \underset{x,y \to b}{\longrightarrow} 0$.

Remark

If g change of sign the previous result is not true. It suffices to take the function $f(t) = \dfrac{|\sin t|}{t} + \dfrac{\sin t}{\sqrt{t}}$ and $g(t) = \dfrac{\sin t}{\sqrt{t}}$, for $t \in [1, +\infty[$. The integral of the function g is convergent on $[1, +\infty[$, it suffices to use the Cauchy test and the second Mean Value Formula. The integral of the function f is divergent.

Proposition

Let $f\colon [1,+\infty[\longrightarrow \mathbb{R}^+$ be a piecewise continuous function.

1. If there exists $\alpha > 1$ such that $\lim\limits_{x\to+\infty} x^\alpha f(x) = 0$, then the integral of f is convergent on $[1,+\infty[$.
2. If there exists $\alpha < 1$ such that $\lim\limits_{x\to+\infty} x^\alpha f(x) = +\infty$, then the integral of f is not convergent on $[1,+\infty[$.

Proposition

Let $a, b \in \mathbb{R}$ and $f :]a, b] \longrightarrow \mathbb{R}+$ be a locally Riemann integrable function.

1. If there exists $\alpha < 1$ such that $\lim_{x \to a^+} (x-a)^\alpha f(x) = 0$, then the integral of f is convergent on $]a, b]$.
2. If there exists $\alpha > 1$ such that $\lim_{x \to +\infty} (x-a)^\alpha f(x) = +\infty$, then the integral of f is not convergent on $]a, b]$.

The Abel's Test

Theorem

[Abel's Theorem]
Let $a \in \mathbb{R}$ and $b \in \mathbb{R} \cup \{+\infty\}$, and f and g be two continuous functions on the interval $[a, b[$. Assume that
i) there exists $M \geq 0$ such that $\left|\int_x^y f(t)dt\right| \leq M$ for any x, y in $[a, b[$.
ii) g is monotonic on $[a, b[$ and $\lim_{t \to b} g(t) = 0$.
Then $\int_a^b f(x)g(x)dx$ converges.

Proof

We can assume that g is decreasing. By second mean value formula, theorem (24), for any $x < y$ in $[a, b[$,

$$\begin{aligned} \left| \int_x^y f(t)g(t)dt \right| &= g(x) \left| \int_x^c f(t)dt \right| \\ &\leq Mg(x) \underset{x \to b^-}{\longrightarrow} 0. \end{aligned}$$

□