Properties of operations on matrices

If the operations are defined:

① $$(A+B)+C = A+(B+C) = A+B+C$$ Associative

② $$(A+B)C = AC+BC$$

③ $$C(A+B) = CA+CB$$

④ $$(AB)C = A(BC) = ABC$$ Associative

Example: $A=\begin{pmatrix}1&1\\0&1\end{pmatrix}$; $B=\begin{pmatrix}1&0\\1&1\end{pmatrix}$; $C=\begin{pmatrix}1&2\\3&1\end{pmatrix}$

$$AB=\begin{pmatrix}2&1\\1&1\end{pmatrix}$$

$$(AB)C=\begin{pmatrix}2&1\\1&1\end{pmatrix}\begin{pmatrix}1&2\\3&1\end{pmatrix}=\begin{pmatrix}5&5\\4&3\end{pmatrix}$$

$$BC=\begin{pmatrix}1&2\\4&3\end{pmatrix} \qquad \| $$

$$A(BC)=\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&2\\4&3\end{pmatrix}=\begin{pmatrix}5&5\\4&3\end{pmatrix}$$

⑤ $AB \stackrel{?}{=} BA$

For $A=\begin{pmatrix}1&1\\0&1\end{pmatrix}$; $B=\begin{pmatrix}1&0\\1&1\end{pmatrix}$; $A=\begin{pmatrix}1&1\\0&1\end{pmatrix}$

$$AB=\begin{pmatrix}2&1\\1&1\end{pmatrix} \quad ; \quad BA=\begin{pmatrix}1&1\\1&2\end{pmatrix}$$

In this case $AB \neq BA$.

If $AB = BA$ then we say that the matrices A and B commute

Some matrices:

① The matrix zero

$$\mathbb{O}_{2\times 3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad ; \quad \mathbb{O}_{3\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathbb{O}_{1\times 2} = \begin{pmatrix} 0 & 0 \end{pmatrix} \qquad \mathbb{O}_{2\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

ⓐ $A + \mathbb{O} = \mathbb{O} + A = A$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \end{pmatrix}$$

ⓑ $A\mathbb{O} = \mathbb{O} \quad ; \quad \mathbb{O}A = \mathbb{O}$

$$\underset{2\times 3}{\begin{pmatrix} 3 & 1 & 2 \\ 4 & 2 & 1 \end{pmatrix}} \underset{3\times 3}{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} = \underset{2\times 3}{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

② The identity matrix:

Plays the role of 1 in the numbers.

we denote it I (Id)

$$\underset{m\times n}{A}\ \underset{n\times n}{I} = \underset{m\times n}{A} \quad ; \quad \underset{m\times m}{I}\ \underset{m\times n}{A} = \underset{m\times n}{A}$$

I. is a square matrix.

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \; ; \; I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\underset{2\times 3}{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & -1 \end{pmatrix}} \underset{3\times 3}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & -1 \end{pmatrix}$$

$$\underset{2\times 2}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \underset{2\times 3}{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & -1 \end{pmatrix}} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}$$

The transpose of a matrix
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If A is a matrix of any size $m \times n$ the transpose of A is the matrix denoted A^t where the rows of A become columns in A^t

Example:

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 2 & -1 \end{pmatrix}_{2\times 3} \quad ; \quad A^t = \begin{pmatrix} 3 & 1 \\ 2 & 2 \\ 1 & -1 \end{pmatrix}_{3\times 2}$$

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad ; \quad A^t = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

Theorem:

1. $(A^t)^t = A$
2. $(A+B)^t = A^t + B^t$
3. $(kA)^t = kA^t$
4. $(AB)^t = B^t A^t$

Example:

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$$

We have $AB = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$; $(AB)^t = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$;

$$A^t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad B^t = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$$

We have $A^t B^t = \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix} \neq (AB)^t$

$$B^t A^t = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} = (AB)^t$$