Determinants المحددات

For any square matrix we associate a number called the determinant.

We denote: $\det A$; $|A|$

How to compute this number:

For a 2x2 matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad ; \quad \det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Examples:

$$\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 1\times1 - 3\times2 = -5$$

$$\begin{vmatrix} 4 & 3 \\ 3 & 2 \end{vmatrix} = 4\times2 - 3\times3 = -1$$

$$\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = 0\times0 - (-1)\times1 = 1$$

For a 3x3 matrix:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = aek + bfg + cdh - gec - hfa - kdb$$

مسودة :

$$\begin{array}{ccccc} & & - & - & - \\ a & b & c & a & b \\ d & e & f & d & e \\ g & h & k & g & h \\ & & + & + & + \end{array}$$

Example :

$$\begin{vmatrix} 4 & 3 & 2 \\ 1 & -1 & 2 \\ 2 & 0 & 5 \end{vmatrix} = -20 + 12 + 0 - (-4) - 0 - 15$$

$$= -19$$

في المسودة

$$\begin{array}{ccccc} 4 & 3 & 2 & 4 & 3 \\ 1 & -1 & 2 & 1 & -1 \\ 2 & 0 & 5 & 2 & 0 \end{array}$$

$$\begin{vmatrix} 2 & 1 & -1 \\ 4 & 3 & 2 \\ -2 & 2 & 1 \end{vmatrix} = 6 + (-4) + (-8) - 6 - 8 - 4$$

$$= -24$$

المسودة

$$\begin{matrix} 2 & 1 & -1 & 2 & 1 \\ 4 & 3 & 2 & 4 & 3 \\ -2 & 2 & 1 & -2 & 2 \end{matrix}$$

Another method:

We choose one row or one column
We put signs in a special way
and then we distribute with
respect to the row/column

$$\begin{vmatrix} a^+ & b^- & c^+ \\ d^- & e^+ & f^- \\ g^+ & h^- & k^+ \end{vmatrix} = +g\begin{vmatrix} b & c \\ e & f \end{vmatrix} - h\begin{vmatrix} a & c \\ d & f \end{vmatrix} + k\begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

$$= g(bf - ce) - h(af - cd) + k(ae - bd)$$

$$= gbf + cdh + aek - gce - haf - kbd$$

Example:

$$\begin{vmatrix} 1^+ & 2 & -1 \\ 2^- & 0^+ & 1^- \\ 4 & 3 & 1 \end{vmatrix} = -2\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & -1 \\ 4 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix}$$

$$= -2\times 5 + 0 - (-5) = -5$$

$$\begin{vmatrix}1^+ & 2^- & -1\\ 2 & 0^+ & 1\\ 4 & 3^- & 1\end{vmatrix} = -2\begin{vmatrix}2 & 1\\ 4 & 1\end{vmatrix} + 0\begin{vmatrix}1 & -1\\ 4 & 1\end{vmatrix} - 3\begin{vmatrix}1 & -1\\ 2 & 1\end{vmatrix}$$

$$= (-2)\times(-2) + 0 - 3\times 3$$

$$= 4 - 9 = -5$$

For a 4×4 matrix:

We cannot use the method of adding columns
The result will be wrong

We use the second method

Example:

$$\begin{vmatrix}1^+ & 2 & 1 & 3\\ 4^- & 0 & 5 & 3\\ 2^+ & -1^- & 0^+ & 0\\ 3 & 0 & 2 & 1\end{vmatrix} = +2\begin{vmatrix}2^+ & 1 & 3\\ 0^- & 5 & 3\\ 0^+ & 2 & 1\end{vmatrix} - (-1)\begin{vmatrix}1^+ & 1 & 3^+\\ 4 & 5 & 3\\ 3 & 2 & 1\end{vmatrix}$$

$$+ 0\begin{vmatrix}1 & 2 & 3\\ 4 & 0 & 3\\ 3 & 0 & 1\end{vmatrix} - 0\begin{vmatrix}1 & 2 & 1\\ 4 & 0 & 5\\ 3 & 0 & 2\end{vmatrix}$$

$$= 2\left(2\begin{vmatrix}5 & 3\\ 2 & 1\end{vmatrix} - 0 + 0\right) + \left(+\begin{vmatrix}5 & 3\\ 2 & 1\end{vmatrix} - \begin{vmatrix}4 & 3\\ 3 & 1\end{vmatrix} + 3\begin{vmatrix}4 & 5\\ 3 & 2\end{vmatrix}\right)$$

ساروس

$$\begin{matrix}1 & 1 & 3 & 1 & 1\\ 4 & 5 & 3 & 4 & 5\\ 3 & 2 & 1 & 3 & 2\end{matrix}$$

$$5 + 9 + 24 - 45 - 6 - 4 = -17$$

$$= -4 + (-1 + 5 - 21)$$

$$= \boxed{-21}$$

طريقة أخرى

$$\begin{vmatrix}1^{+} & 2^{-} & 1 & 3\\ 4 & 0^{+} & 5 & 3\\ 2 & -1^{-} & 0 & 0\\ 3 & 0^{+} & 2 & 1\end{vmatrix} = -2\begin{vmatrix}4^{+} & 5 & 3\\ 2^{-} & 0 & 0\\ 3 & 2 & 1\end{vmatrix} + 0 + 1\begin{vmatrix}1^{+} & 1 & 3\\ 4^{-} & 5^{+} & 3^{-}\\ 3 & 2 & 1\end{vmatrix}$$

$$+ 0$$

$$= -2\left(-2\begin{vmatrix}5 & 3\\ 2 & 1\end{vmatrix}\right) + \left(-4\begin{vmatrix}1 & 3\\ 2 & 1\end{vmatrix} + 5\begin{vmatrix}1 & 3\\ 3 & 1\end{vmatrix} - 3\begin{vmatrix}1 & 1\\ 3 & 2\end{vmatrix}\right)$$

$$= -4 + 20 - 40 + 3 = \boxed{-21}$$