System of linear equations

Example:

Solve the system of linear equations

$$\begin{cases} 2x - 3y + z = 1 \\ x + 2y - z = 2 \\ 3x + y - 2z = 3. \end{cases}$$

We are asked to find all (x, y, z) which satisfy all 3 equations at the ~~Er~~ same time

Example: $(2, 1, 0)$ satisfies equation 1.
$\overset{\shortparallel}{x}\ \overset{\shortparallel}{y}\ \overset{\shortparallel}{z}$ but does not satisfy equation 2 nor equation 3.

Then $(2,1,0)$ is not a solution

Any system of linear equations can be written in a matrix form.

$$\underset{3\times 3}{\begin{pmatrix} 2 & -3 & 1 \\ 1 & 2 & -1 \\ 3 & 1 & -2 \end{pmatrix}} \underset{3\times 1}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \underset{3\times 1}{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}$$

Any linear system can be written in the form

$$AX = B$$

A is called the matrix of the coefficients of the system

$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is called the matrix of the unknowns

$B = \begin{pmatrix} \\ \end{pmatrix}$ is " " " of the constants.

<u>Example 2</u>: write the system

$$\begin{cases} x + z = 1 \\ y + w = 2 \\ 2x + y - z + w = 3 \\ x + w = 4 \end{cases}$$

in a matrix form:

Solution:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

About the matrix of coefficients A:

The number of rows: Is the number of equations

The number of columns: Is the number of unknowns

<u>Theorem</u>: If A is a square matrix then the system has a unique solution if and only if $\det A \neq 0$

We call this system (in this case) a Cramer' system.

We solve this system by using Cramer' method.

<u>Example</u> For the system:

$$\begin{cases} 2x - 3y + z = 1 \\ x + 2y - z = 2 \\ 3x + y - 2z = 3 \end{cases}$$

We write it:

$$\begin{pmatrix} 2 & -3 & 1 \\ 1 & 2 & -1 \\ 3 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}_{B}$$

$$\det A = -8 + 9 + 1 - 6 - (-2) - 6$$
$$= -8 \neq 0$$

The system is a Cramer system.

How to solve it:

<u>For x:</u>

$$\det \begin{pmatrix} 1 & -3 & 1 \\ 2 & 2 & -1 \\ 3 & 1 & -2 \end{pmatrix} = -4 + 9 + 2 - 6 - (-1) - 12$$
$$= -10$$

$\uparrow$ B

Then $x = \dfrac{-10}{-8} = \dfrac{5}{4}$.

<u>For y:</u>

$$\det \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 3 & 3 & -2 \end{pmatrix} = -8 - 3 + 3 - 6 - (-6) - (-2)$$
$$= -6$$

$\uparrow$ B

Then $y = \dfrac{-6}{-8} = \dfrac{3}{4}$

<u>For z:</u>

$$\det \begin{pmatrix} 2 & -3 & 1 \\ 1 & 2 & 2 \\ 3 & 1 & 3 \end{pmatrix} = 12 - 18 + 1 - 6 - 4 - (-9)$$
$$= -6$$

Then $z = \dfrac{-6}{-8} = \dfrac{3}{4}$

We have a unique solution:

$$\left(\frac{5}{4}, \frac{3}{4}, \frac{3}{4}\right)$$

Example: Solve the linear system

$$\begin{cases} 2x + 3y = 1 \\ x + 2y = 2 \end{cases}$$

Solution:

$$\underbrace{\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\det A = 4 - 3 = 1 \neq 0$$

$$\det A_x = \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} = 2 - 6 = -4$$

$$\Rightarrow x = \frac{-4}{1} = -4$$

$$\det A_y = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3$$

$$\Rightarrow y = \frac{3}{1} = 3$$

The unique solution is $(x, y) = (-4, 3)$