Method of Gauss to solve a linear system

(1) If we have the system:

$$\begin{cases} x - 2y + 3z = 1 \\ y - z = 2 \\ z = 1 \end{cases}$$

To solve this system, we use back substitution:

$z = 1$: then $y - 1 = 2 \Rightarrow y = 3$

$y = 3$; $z = 1$: then $x - 6 + 3 = 1$
$\Rightarrow x = 4$

So $(x, y, z) = (4, 3, 1)$ is the unique solution

(2) Solve the system:

$$\begin{cases} x - 3y = 1 \\ 2x + y = -5 \end{cases}$$

Solution:

$$\begin{cases} x - 3y = 1 \\ 2x + y = -5 \end{cases} \underset{Eq_2 + 2Eq_1}{\overset{Eq_2 - 2Eq_1}{\Longleftrightarrow}} \begin{cases} x - 3y = 1 \\ 7y = -7 \end{cases}$$

$$\underset{7Eq_2}{\overset{\frac{1}{7}Eq_2}{\Longleftrightarrow}} \begin{cases} x - 3y = 1 \\ y = -1 \end{cases}$$

We have: $y = -1$. Then $x + 3 = 1 \Leftrightarrow x = -2$

The unique solution is

$$(x,y)=(-2,-1)$$

The augmented matrix

For the system $\begin{cases} x-3y=1 \\ 2x+y=-5 \end{cases}$

we associate the augmented matrix

$$\left(\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & 1 & -5 \end{array}\right)$$

To solve

$$\left(\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & 1 & -5 \end{array}\right) \equiv \left(\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 7 & -7 \end{array}\right) R_2-2R_1$$

$$\equiv \left(\begin{array}{cc|c} \boxed{1} & -3 & 1 \\ 0 & \boxed{1} & -1 \end{array}\right) \frac{1}{7}R_2$$

Row echelon form

$\boxed{1}$ Leading one

Example: Solve by using Gauss elimination method the system:

$$\begin{cases} 2x+z=2 \\ x+y+z=2 \\ x-y+2z=0 \end{cases}$$

The augmented matrix

$$\left(\begin{array}{ccc|c} 2 & 0 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & -1 & 2 & 0 \end{array}\right) \equiv \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 0 & 1 & 2 \\ 1 & -1 & 2 & 0 \end{array}\right) R_1 \rightleftarrows R_2$$

$$\equiv \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & -1 & -2 \\ 0 & -2 & 1 & -2 \end{array}\right) \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - R_1 \end{array} \equiv \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & -1 & -2 \\ 0 & 0 & 2 & 0 \end{array}\right) \begin{array}{l} \\ \\ R_3 - R_2 \end{array}$$

$$\equiv \left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 2 \\ 0 & \boxed{1} & \frac{1}{2} & 1 \\ 0 & 0 & \boxed{1} & 0 \end{array}\right) \begin{array}{l} \\ -\frac{1}{2} R_2 \\ \frac{1}{2} R_3 \end{array}$$

This is a row echelon form.

Back to the system:

$$\begin{cases} x + y + z = 2 \\ \quad y + \frac{1}{2} z = 1 \\ \qquad z = 0 \end{cases}$$

We apply back substitution:

$z = 0$ then $y + 0 = 1 \iff y = 1$

$y = 1;\ z = 0$ then $x + 1 + 0 = 2 \iff x = 1$

So the unique solution is

$$(x, y, z) = (1, 1, 0).$$

<u>The method of Gauss:</u>

$\{$System $\to$ $\begin{pmatrix}\text{augmented} \\ \text{matrix}\end{pmatrix} \xrightarrow[\text{operations}]{\text{Elementary}} \begin{pmatrix}\text{Row} \\ \text{echelon} \\ \text{form}\end{pmatrix} \to \{$Syst

back substitution

Elementary operations:

① Permutation between two rows

$$R_i \leftrightarrows R_j$$

② Adding / substracting a multiple of a row to another row:

$$R_i \leftarrow R_i \pm k R_j$$

③ Multiplying / Dividing a row by a number ($\neq 0$)

$$R_i \leftarrow \begin{matrix} kR_i \\ \frac{1}{k} R_i \end{matrix}$$

The method of Gauss - Jordan

Example:

Solve the system $\begin{cases} x + 2y = 1 \\ 2x + y = -1 \end{cases}$

The augmented matrix:

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 1 & -1 \end{array}\right) \equiv \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -3 & -3 \end{array}\right) R_2 - 2R_1$$

$$\equiv \left(\begin{array}{cc|c} \boxed{1} & 2 & 1 \\ 0 & \boxed{1} & 1 \end{array}\right) -\frac{1}{3} R_2 \equiv \left(\begin{array}{cc|c} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 1 \end{array}\right) R_1 - 2R_2$$

$$x = -1 \quad ; \quad y = 1$$

The unique solution is $(x, y) = (-1, 1)$

Example: Solve by using Gauss-Jordan method the system:

$$\begin{cases} 2x + z = 2 \\ x + y + z = 2 \\ x - y + 2z = 0 \end{cases}$$

We repeat the same ~~sp~~ steps until we get to

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 2 \\ 0 & \boxed{1} & \frac{1}{2} & 1 \\ 0 & 0 & \boxed{1} & 0 \end{array}\right) \equiv \left(\begin{array}{ccc|c} \boxed{1} & 1 & 0 & 2 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & 0 \end{array}\right) \begin{array}{l} R_1 - R_3 \\ R_2 - \frac{1}{2}R_3 \end{array}$$

$$\equiv \left(\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & 0 \end{array}\right) R_1 - R_2$$

Reduced row echelon form

The unique solution is

$$(x, y, z) = (1, 1, 0)$$

Example: Solve the system ~~by~~

$$\begin{cases} 3x + y = 5 \\ 2y - 3z = -5 \\ x + 2z = 7 \end{cases}$$

The augmented matrix:

$$\left(\begin{array}{ccc|c} 3 & 1 & 0 & 5 \\ 0 & 2 & -3 & -5 \\ 1 & 0 & 2 & 7 \end{array}\right) \underset{R_1 \leftrightarrows R_3}{\equiv} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 2 & -3 & -5 \\ 3 & 1 & 0 & 5 \end{array}\right)$$

$$\underset{R_3 - 3R_1}{\equiv} \left(\begin{array}{ccc|c} \boxed{1} & 0 & 2 & 7 \\ 0 & 2 & -3 & -5 \\ 0 & 1 & -6 & -16 \end{array}\right) \underset{R_2 \leftrightarrows R_3}{\equiv} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & -6 & -16 \\ 0 & 2 & -3 & -5 \end{array}\right)$$

$$\underset{R_3 - 2R_2}{\equiv} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & -6 & -16 \\ 0 & 0 & 9 & 27 \end{array}\right) \underset{\frac{1}{9}R_3}{\equiv} \left(\begin{array}{cccc} \boxed{1} & 0 & 2 & 7 \\ 0 & \boxed{1} & -6 & -16 \\ 0 & 0 & \boxed{1} & 3 \end{array}\right)$$

$$\equiv \left(\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & \boxed{1} & 3 \end{array}\right) \begin{array}{l} R_1 - 2R_3 \\ R_2 + 6R_3 \end{array}$$

The unique solution is

$$(x, y, z) = (1, 2, 3)$$