Integration by substitution

التكامل بالتعويض

If $f(x) = \sin(3x^2 + 6x + 1)$

Then $f'(x) = (6x + 6)\cos(3x^2 + 6x + 1)$

If $f(x) = (x^2 - 2x + \sin x)^4$

Then $f'(x) = 4(2x - 2 + \cos x)(x^2 - 2x + \sin x)^3$

If $g(x) = x^4$ then $g'(x) = 4x^3$

If $f(x) = e^{(5x^3 + x - 1)}$

then $f'(x) = (15x^2 + 1)e^{(5x^3 + x - 1)}$

If $g(x) = e^x$ then $g'(x) = e^x$

The chaine rule:

$$\boxed{\text{If } h(x) = f(g(x)) \text{ then } h'(x) = g'(x)\, f'(g(x)).}$$

Then

$$\int g'(x)\, f'(g(x))\, dx = f(g(x)) + C$$

Integration by substitution:

If we have $\int g'(x) f'(g(x))\, dx$

we put $u = g(x)$

then $du = g'(x)\, dx$

and

$$\int g'(x) f'(g(x))\, dx = \int f'(u)\, du$$

$$= f(u) + C = f(g(x)) + C$$

Example:

$$\int x \cos(x^2 - 3)\, dx$$

We know that $\int \cos x\, dx = \sin x + C$

Let $u = x^2 - 3$

then $du = 2x\, dx \Leftrightarrow x\, dx = \frac{1}{2} du$

Then $\int x \cos(x^2 - 3)\, dx = \int \frac{1}{2} \cos u\, du$

$$= \frac{1}{2} \sin u + C = \frac{1}{2} \sin(x^2 - 3) + C$$

Compute $\int (3x^2 - \cos x)(2x^3 - 2\sin x + 3)^4 \, dx$

Remark: We know that $\int x^4 \, dx = \frac{x^5}{5} + C$

Let $u = 2x^3 - 2\sin x + 3$

then $du = (6x^2 - 2\cos x)\, dx$

and $(3x^2 - \cos x)\, dx = \frac{1}{2} du$

~~Thef~~ Therefore:

$$\int (3x^2 - \cos x)(2x^3 - 2\sin x + 3)^4 \, dx$$

$$= \int \frac{1}{2} u^4 \, du = \frac{1}{2} \frac{u^5}{5} + C$$

$$= \frac{(2x^3 - 2\sin x + 3)^5}{10} + C$$

Example: $\int \frac{5x^4 + 2x - e^x}{x^5 + x^2 + 3 - e^x} \, dx$

We know that $\int \frac{1}{x} \, dx = \ln|x| + C$

Let $u = x^5 + x^2 + 3 - e^x$

Then $du = (5x^4 + 2x - e^x)\,dx$

Therefore:

$$\int \frac{5x^4 + 2x - e^x}{x^5 + x^2 + 3 - e^x}\,dx = \int \frac{1}{u}\,du$$

$$= \ln|u| + C = \ln\left|x^5 + x^2 + 3 - e^x\right| + C$$

In general:

$$\boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C}$$

Let $u = f(x)$. Then $du = f'(x)\,dx$

and $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \int \frac{1}{u}\,du = \ln|u| + C$

$= \ln|f(x)| + C$

Examples:

$$\int \frac{1}{3x-2}\,dx = \frac{1}{3}\int \frac{3}{3x-2}\,dx$$

and therefore $= \frac{1}{3}\ln|3x-2| + C$

In general

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \cos(2x+3)\,dx = \frac{1}{2}\sin(2x+3) + C$$

$$\int e^{(ax+b)}\,dx = \frac{1}{a}e^{ax+b} + C$$

$$\int \frac{1}{x-1}\,dx = \ln|x-1| + C$$

Example:

① $$\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx$$

We know that: $(\sin^{-1} x)' = \dfrac{1}{\sqrt{1-x^2}}$

Let $u = \sin^{-1} x$

then $du = \dfrac{1}{\sqrt{1-x^2}}\, dx$

and therefore

$$\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx = \int u\, du = \frac{u^2}{2} + C$$

$$= \frac{(\sin^{-1} x)^2}{2} + C$$

② $$\int 3(x^3-3)^9 x^2\, dx$$

Let $u = x^3 - 3$

then $du = 3x^2\, dx$

and $\displaystyle\int 3(x^3-3)^9 x^2\, dx = \int u^9\, du = \frac{u^{10}}{10} + C$

$$= \frac{(x^3-3)^{10}}{10} + C$$

③ $$\int \frac{\sec^2\left(\frac{1}{x}\right)}{x^2}\,dx$$

Let $u = \frac{1}{x} = x^{-1}$ then $du = -\frac{1}{x^2}\,dx$

Therefore $$\int \frac{\sec^2\left(\frac{1}{x}\right)}{x^2}\,dx = -\int \sec^2(u)\,du$$

$$= -\tan u + C = -\tan\left(\frac{1}{x}\right) + C$$

④ $$\int \frac{x^5}{\sqrt{x^3+1}}\,dx$$

Let $u = x^3 + 1$, then $du = 3x^2\,dx$ and $x^2\,dx = \frac{1}{3}\,du$

$$\frac{x^5 = \overset{= u-1}{x^3}\,x^4}{\sqrt{x^3+1}}$$

Therefore $$\int \frac{x^5}{\sqrt{x^3+1}}\,dx = \frac{1}{3}\int \frac{u-1}{\sqrt{u}}\,du$$

$$= \frac{1}{3}\int\left(\underset{u^{1/2}}{\sqrt{u}} - \underset{u^{-1/2}}{\frac{1}{\sqrt{u}}}\right)du = \frac{1}{3}\left(\frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2}\right) + C$$

$$= \frac{2u^{3/2}}{9} - \frac{2}{3}u^{1/2} + C$$

$$= \frac{2}{9}\left(x^3+1\right)^{3/2} - \frac{2}{3}\sqrt{x^3+1} + C$$

(5) $$\int \frac{10t}{e^{(5t^2-1)}}\,dt$$

Let $u = 5t^2 - 1$ Then $du = 10t\,dt$

Therefore $$\int \frac{10t}{e^{(5t^2-1)}}\,dt = \int \underbrace{\frac{1}{e^{u}}}_{e^{-u}}\,du$$

$$= -e^{-u} + C = \frac{-1}{e^{5t^2-1}} + C$$