Applications of Integration:

تطبيقات حساب التكامل

Let $f : [a, b] \longrightarrow \mathbb{R}$ $(a < b)$ be a positive function (nice function). The area of the region bounded by the curve of f, the x-axis, and the axes $x = a$; $x = b$

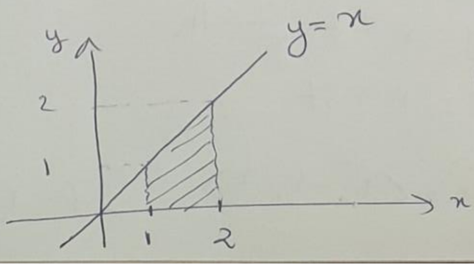
$f(x) \geq 0$

is equal to : $\int_a^b f(x)\,dx = F(b) - F(a)$

where $F(x)$ is an antiderivative (primitive) of $f(x)$.

$$\int f(x)\,dx = F(x) + C$$

Example!



The area is $F(2) - F(1)$

$\int x\,dx = \frac{x^2}{2} + C$

Area $= \frac{2^2}{2} - \frac{1^2}{2}$
$= \frac{3}{2}$

Remarks: ① If $F_1(x)$ and $F_2(x)$ are two primitives of $f(x)$, then $F_2(x) = F_1(x) + c$

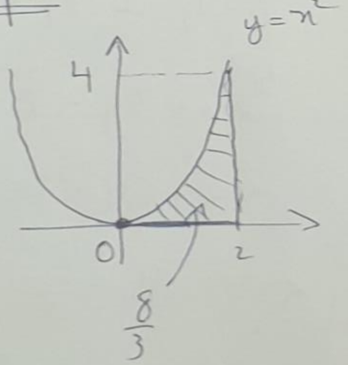
and
$$F_2(b) - F_2(a) = \big(F_1(b) + c\big) - \big(F_1(a) + c\big)$$
$$= F_1(b) - F_1(a)$$

The area does not depend on the choice of the primitive.

② We denote
$$F(b) - F(a) = \int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b$$

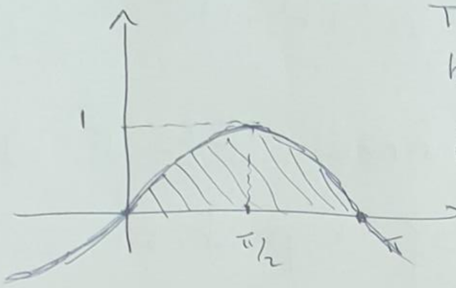
when $\int f(x)\,dx = F(x) + C$

Example:



The area is given by
$$\int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3} - \frac{0}{3} = \frac{8}{3}$$

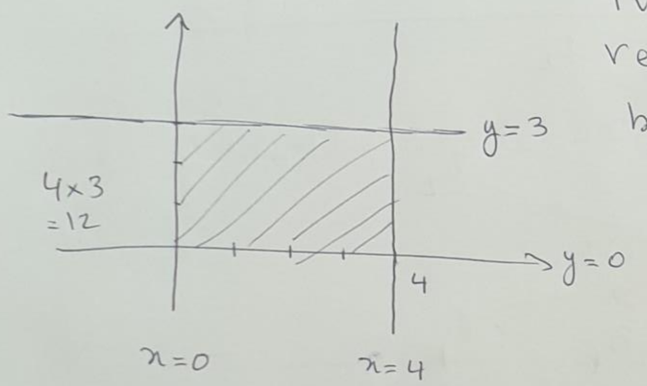
Example: $f(x) = \sin x$



The area of the region bounded by $y = \sin x$, $y = 0$, $x = 0$; $x = \pi$ is given by

$$\text{Area} = \int_0^{\pi} \sin x \, dx = \Big[-\cos x\Big]_0^{\pi} = \big(-(-1)-(-1)\big) = 2$$

Example:



The area of the rectangle is given by

$$\int_0^4 3\, dx = \Big[3x\Big]_0^4 = 12 - 0 = 12.$$

Properties:

① $$\int_a^b \big(f(x) \pm g(x)\big)dx = \Big[F(x) \pm G(x)\Big]_a^b = \Big[F(x)\Big]_a^b \pm \Big[G(x)\Big]_a^b$$

$$= \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

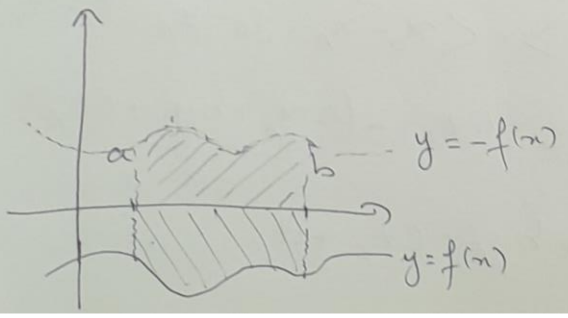
② $$\int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx$$

③ $$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \left(F(c) - F(a)\right) + \left(F(b) - F(c)\right)$$
$$= F(b) - F(a)$$
$$= \int_a^b f(x)\,dx$$

④ $$\int_a^a f(x)\,dx = F(a) - F(a) = 0$$

⑤ $$\int_b^a f(x)\,dx = F(a) - F(b) = -\left(F(b) - F(a)\right)$$
$$= -\int_a^b f(x)\,dx$$

If $f(x) \le 0$ on $[a,b]$ then $-f(x) \ge 0$ on $[a,b]$



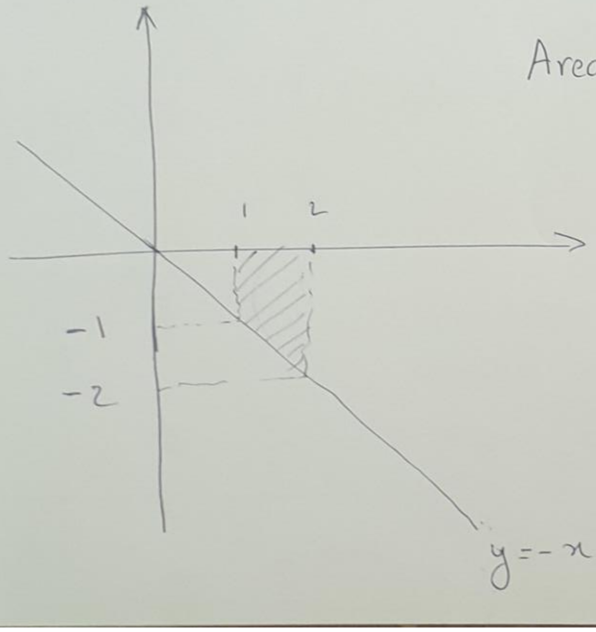
Area between $y = f(x)$ and the x-axis is equal to the area between $y = -f(x)$ and the x-axis

Therefore:

$$\text{The area} = \int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\boxed{\begin{array}{l} \text{Then} \quad \displaystyle\int_a^b f(x)\,dx = -\text{Area} \\ \text{If} \quad f(x) \le 0 \quad \text{in} \quad [a;b] \end{array}}$$

Example: Compute the area between $y = -x$; $y = 0$; $x = 1$; $x = 2$



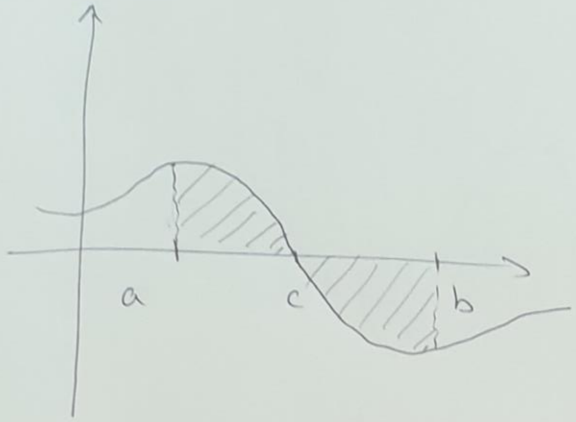
$$\text{Area} = -\int_1^2 (-x)\,dx$$

$$= -\left[-\frac{x^2}{2}\right]_1^2$$

$$= \left[\frac{x^2}{2}\right]_1^2 = 2 - \frac{1}{2} = \frac{3}{2}$$

What happens if $f(x) \ge 0$ on $[a, c]$

and $f(x) \le 0$ on $[c, b]$



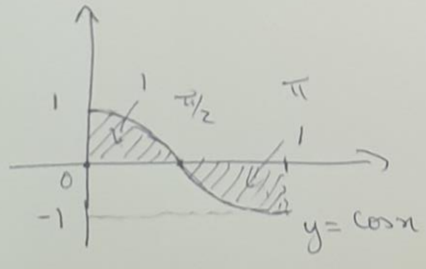
The area is given by:

$$\text{Area} = \int_a^c f(x)\,dx - \int_c^b f(x)\,dx$$

$\neq$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

<u>Example:</u>



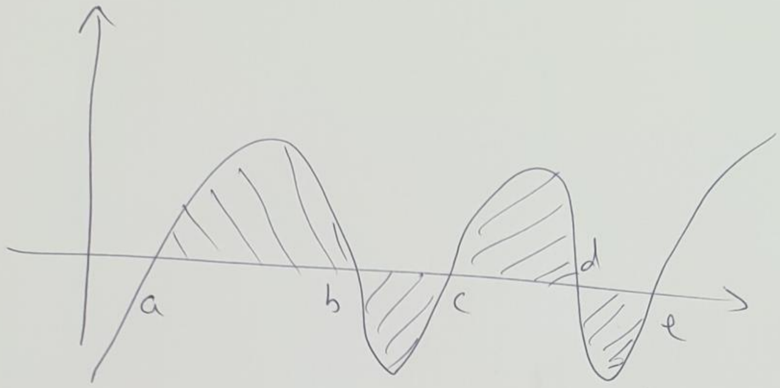
The area of the region showed in the drawing:

$$\text{Area} = \int_0^{\pi/2} \cos x\,dx - \int_{\pi/2}^{\pi} \cos x\,dx$$

$$= \Big[\sin x\Big]_0^{\pi/2} - \Big[\sin x\Big]_{\pi/2}^{\pi} = (1-0)-(0-1)$$

$$= 2$$

$$\int_0^{\pi} \cos x \, dx = \left[\sin x\right]_0^{\pi} = 0 - 0 = 0$$



$$\text{Area} = \int_a^b f(x)\,dx - \int_b^c f(x)\,dx + \int_c^d f(x)\,dx - \int_d^e f(x)\,dx$$

$\nparallel$

$$\int_a^e f(x)\,dx = \underbrace{\int_a^b f(x)\,dx}_{>0} + \underbrace{\int_b^c f(x)\,dx}_{<0} + \underbrace{\int_c^d f(x)\,dx}_{>0} + \underbrace{\int_d^e f(x)\,dx}_{<0}$$