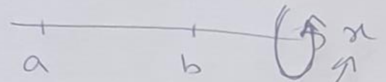
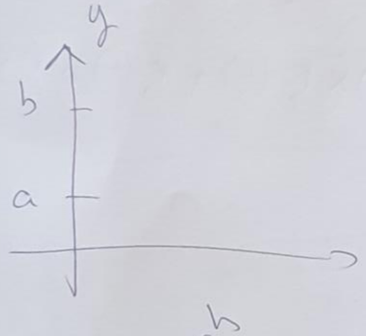
The Shell Method طريقة الشرائح الاسطوانية

The disk/washer method

We compute the integral with respect to the variable of the axis of revolution



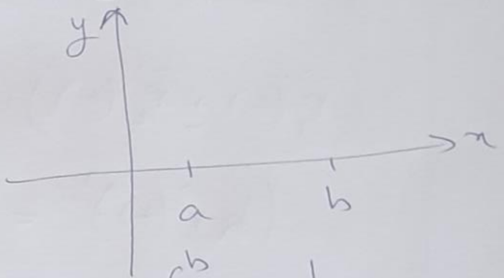
$$V = \_\int_a^b \_\, dx$$



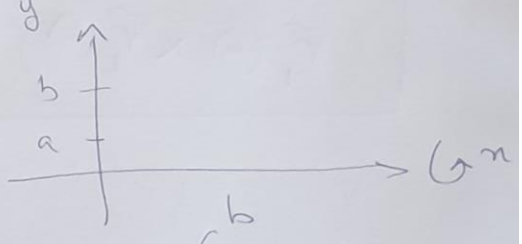
$$V = \_\int_a^b \_\, dy$$

The Shell Method

We compute the integral with respect to the variable diffrent from the axis of revolution



$$V = \_\int_a^b \_\, dx$$



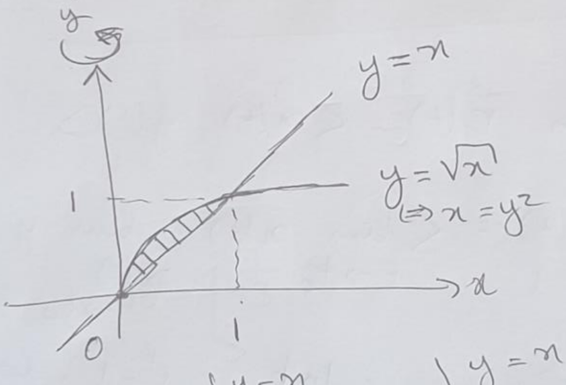
$$V = \_\int_a^b \_\, dy$$

The Formula

The volume of the region bounded by $y=f(x)$; $g=g(x)$; $x=a$; $x=b$ (with $f(x)\geq g(x)$ on $[a,b]$) rotated about the y-axis is given by

$$V = 2\pi \int_a^b x\big(f(x)-g(x)\big)\,dx$$

Example:



We solve $\begin{cases} y=x \\ y=\sqrt{x} \end{cases} \Leftrightarrow \begin{cases} y=x \\ x=\sqrt{x} \end{cases} \Leftrightarrow \begin{cases} y=x \\ \sqrt{x}(\sqrt{x}-1)=0 \end{cases} \Leftrightarrow \begin{cases} x=0, y=0 \\ \text{or} \\ x=1, y=1 \end{cases}$

The shell method to complite the volume

$$V = 2\pi \int_0^1 x(\sqrt{x} - x)\,dx = 2\pi \int_0^1 (x^{3/2} - x^2)\,dx$$

$$= 2\pi \left[\frac{2x^{5/2}}{5} - \frac{x^3}{3} \right]_0^1 = 2\pi \left(\frac{2}{5} - \frac{1}{3} - 0 \right)$$

$$= \boxed{\frac{2\pi}{15}}$$

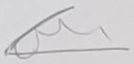
The washer method:

$$V = \pi \int_0^1 \left(y^2 - (y^2)^2\right) dy = \pi \left[\frac{y^3}{3} - \frac{y^5}{5} \right]_0^1$$

$$= \pi \left[\frac{1}{3} - \frac{1}{5} - 0 \right] = \boxed{\frac{2\pi}{15}}.$$

Example:

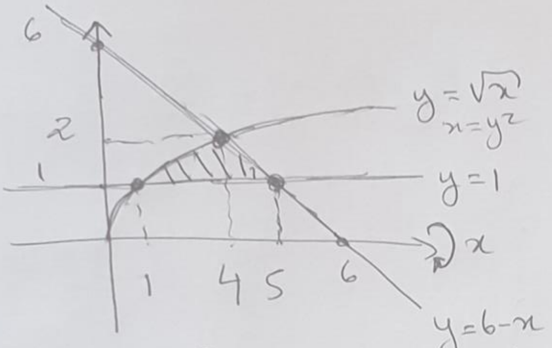
Find the volume of the region bounded by $y=\sqrt{x}$; $y=6-x$; $y=1$ rotated about the x-axis



(1) The intersection points:

$$\begin{cases} y=1 \\ y=\sqrt{x} \end{cases} \Leftrightarrow \begin{cases} y=1 \\ \sqrt{x}=1 \end{cases} \Rightarrow \begin{cases} y=1 \\ x=1 \end{cases}$$

$(1,1)$



$$\begin{cases} y=6-x \\ y=1 \end{cases} \Rightarrow \begin{cases} y=1 \\ x=5 \end{cases}$$

$(5,1)$

$$\begin{cases} y=6-x \\ y=\sqrt{x} \end{cases} \Leftrightarrow \begin{cases} \sqrt{x}=6-x \\ y=\sqrt{x} \end{cases}$$

$$\underset{6-x\geq 0}{\Leftrightarrow} \begin{cases} x=(6-x)^2 \\ y=\sqrt{x} \end{cases} \underset{6\geq x}{\Leftrightarrow} \begin{cases} x^2-13x+36=0 \\ y=\sqrt{x} \end{cases}$$

$$\underset{6\geq x}{\Leftrightarrow} \begin{cases} (x-4)(x-9)=0 \\ y=\sqrt{x} \end{cases} \Leftrightarrow \begin{cases} x=4 \\ y=2 \end{cases}$$

The washer method

$$V = \pi\int_1^4 \left((\sqrt{x})^2 - 1^2\right)dx + \pi\int_4^5 \left(\underbrace{(6-x)^2}_{(x-6)^2} - 1^2\right)dx$$

$$= \pi\left[\frac{x^2}{2} - x\right]_1^4 + \pi\left[\frac{(x-6)^3}{3} - x\right]_4^5$$

$$= \pi\left[(8-4) - \left(-\frac{1}{2}\right)\right] + \pi\left[\left(-\frac{1}{3} - 5\right) - \left(-\frac{8}{3} - 4\right)\right]$$

$$= \frac{9\pi}{2} + \frac{4\pi}{3} = \boxed{\frac{35\pi}{6}}$$

The shell method

$$V = 2\pi\int_1^2 y\left((6-y) - y^2\right)dy = 2\pi\int_1^2 (6y - y^2 - y^3)\,dy$$

$$= 2\pi\left[3y^2 - \frac{y^3}{3} - \frac{y^4}{4}\right]_1^2 = 2\pi\left(12 - \frac{8}{3} - 4 - 3 + \frac{1}{3} + \frac{1}{4}\right)$$

$$= 2\pi\left(5 - \frac{7}{3} + \frac{1}{4}\right) = 2\pi\left(\frac{60 - 28 + 3}{3\times 4}\right) = \boxed{\frac{35\pi}{6}}$$