Partial Derivatives

الاشتقاق الجزئي

Usualy we study functions of one variable : $y = f(x)$

$$y = \sqrt{1 - x^2} \quad ; \quad \frac{dy}{dx} = y'$$

Here we study functions of more than one variable:

$$z = f(x, y) \quad ; \quad w = f(x, y, z)$$

Examples:

$$z = x^2 + y^2 - e^{xy}$$

$$z = \sqrt{1 - xy + y^2}$$

$$z = \cos(x + y^2)$$

$$w = x e^{yz} + y e^{z} + z$$

Partial derivatives:

For more than one variable we talk about partial derivatives and never about derivatives.

We compute the derivative with respect to one of the variables.

Example: $f(x,y) = x^2 + y^2 - e^{xy}$

The partial derivative with respect to x:

$$\frac{\partial f}{\partial x}(x,y) = f_x(x,y) = 2x + 0 - y e^{xy}$$

The partial derivative with respect to y:

$$\frac{\partial f}{\partial y}(x,y) = f_y(x,y) = 0 + 2y - x e^{xy}$$

When we compute the partial derivative with respect to a variable we deal with the other variable as constants

Examples: $f(x,y) = x^3 y - 2x^2 y^2 + x y^3 + 1$

Compute: $f(0,0)$; $f(0,1)$; $f(1,0)$; $f(1,1)$

$f(0,0) = 1$; $f(0,1) = 1$; $f(1,0) = 1$

$f(1,1) = 1$; $f(2,1) = 3$

Compute $f_x(x,y)$; $f_y(x,y)$

$$f_x(x,y) = 3x^2 y - 2y^2 \times 2x + y^3 = 3x^2y - 4xy^2 + y^3$$

$$f_y(x,y) = x^3 - 4x^2 y + 3xy^2$$

$f_x(0,1) = 1$; $f_y(0,1) = 0$

Compute $f_{xx}(x,y)$; $f_{xy}(x,y)$; $f_{yx}(x,y)$; $f_{yy}(x,y)$

We have: $f_x(x,y) = 3x^2 y - 4xy^2 + y^3$

$$\frac{\partial^2}{\partial x^2} f(x,y) = f_{xx}(x,y) = 6xy - 4y^2 + 0$$

$$\frac{\partial^2}{\partial y \partial x} f(x,y) = f_{xy}(x,y) = 3x^2 - 8xy + 3y^2$$

$$f_y(x,y) = x^3 - 4x^2 y + 3xy^2$$

$$\frac{\partial^2}{\partial x \partial y} f(x,y) = f_{yx}(x,y) = 3x^2 - 8xy + 3y^2$$

$$\frac{\partial^2}{\partial y^2} f(x,y) = f_{yy}(x,y) = 0 - 4x^3 + 6xy$$

Example: $f(x,y) = x e^{y} + y \cos x - \sin\left(\frac{x}{y}\right)$

$\left(\frac{1}{y}\right)' = \left(y^{-1}\right)' = -1 y^{-2} = -\frac{1}{y^2}$

We have:

$$f_x(x,y) = e^{y} - y \sin x - \frac{1}{y} \cos \frac{x}{y}$$

$$f_{xx}(x,y) = 0 - y \cos x + \frac{1}{y^2} \sin \frac{x}{y}$$

$$f_{xy}(x,y) = e^{y} - \sin x + \frac{1}{y^2} \cos \frac{x}{y} + \frac{1}{y}\left(-\frac{x}{y^2}\right) \sin \frac{x}{y}$$

$$= e^{y} - \sin x + \frac{1}{y^2} \cos \frac{x}{y} - \frac{x}{y^3} \sin \frac{x}{y}$$

$$f_y(x,y) = x e^{y} + \cos x + \frac{x}{y^2} \cos \frac{x}{y}$$

$$f_{yx}(x,y) = e^{y} - \sin x + \frac{1}{y^2} \cos \frac{x}{y} - \frac{x}{y^3} \sin \frac{x}{y}$$

$$f_{yx}(x,y) = f_{xy}(x,y)$$

$$f_{yy}(x,y) = x e^{y} + 0 + (-2)\frac{x}{y^3} \cos \frac{x}{y} - \frac{x^2}{y^4} \sin \frac{x}{y}$$

$$= x e^{y} - 2\frac{x}{y^3} \cos \frac{x}{y} - \frac{x^2}{y^4} \sin \frac{x}{y}.$$