The chain rule   قاعدة السلسلة

$$y = f(x) \quad ; \quad x = g(s)$$

Then $y = f(g(s))$

Examples:

① $y = \sin x \quad ; \quad x = s^2$

$$y = \sin(s^2)$$

$$\boxed{\frac{dy}{ds} = \frac{dx}{ds} \cdot \frac{dy}{dx}} = 2s\cos(x) = 2s\cos(s^2)$$

The chain rule

② $y = (x^2 - x + 1)^2 \quad ; \quad x = se^s$

Then $y = (s^2e^{2s} - se^s + 1)^2$

$$\frac{dy}{ds} = 2\left(2se^{2s} + 2s^2e^{2s} - e^s - se^s\right)\left(s^2e^{2s} - se^s + 1\right)$$

$$\frac{dy}{ds} = \frac{dy}{dx} \cdot \frac{dx}{ds} = 2(2x-1)(x^2-x+1)(e^s+se^s)$$

$$\frac{dy}{ds} = 2(2se^s - 1)(s^2e^{2s} - se^s + 1)(e^s + se^s)$$

For $z = f(x, y)$; $x = x(s)$; $y = y(s)$

Example:

$z = x^2 + y^2$; $x = \cos s$; $y = \sin s$

Find $\frac{dz}{ds}$

$\frac{\partial z}{\partial x}$; $\frac{\partial z}{\partial y}$

By direct substitution:

$z = x^2 + y^2 = \cos^2 s + \sin^2 s = 1$

$z = 1$

$\frac{dz}{ds} = 0$

The chain rule:

$$\boxed{\frac{dz}{ds} = \frac{dx}{ds} \cdot \frac{\partial z}{\partial x} + \frac{dy}{ds} \cdot \frac{\partial z}{\partial y}}$$

$\frac{dx}{ds} = -\sin s$ $\qquad$ $\frac{\partial z}{\partial x} = 2x = 2\cos s$

$\frac{dy}{ds} = \cos s$ $\qquad$ $\frac{\partial z}{\partial y} = 2y = 2\sin s$

Then $$\frac{dz}{ds} = \frac{dx}{ds}\cdot\frac{\partial z}{\partial x} + \frac{dy}{ds}\cdot\frac{\partial z}{\partial y}$$

$$= -\sin s \times 2\cos s + \cos s \times 2\sin s$$

$$= 0$$

Example: $z = xy$; $x = s^2$; $y = \frac{1}{s}$

Find $\frac{dz}{ds}$ by using the chain rule.

$\frac{dx}{ds} = 2s$; $\frac{\partial z}{\partial x} = y = \frac{1}{s}$

$\frac{dy}{ds} = -\frac{1}{s^2}$; $\frac{\partial z}{\partial y} = x = s^2$

Then $$\frac{dz}{ds} = \frac{dx}{ds}\cdot\frac{\partial z}{\partial x} + \frac{dy}{ds}\cdot\frac{\partial z}{\partial y}$$

$$= 2s\cdot\frac{1}{s} - \frac{1}{s^2}\cdot s^2 = 1$$

By direct substitution:

$$z = xy = s^2\cdot\frac{1}{s} = s$$

$$\frac{dz}{ds} = 1$$

Example: $w = 3xy^2$; $x = t^2+1$; $y = \frac{1}{t}$

Find $\frac{dw}{dt}$ by using the chain rule:

$$\frac{dx}{dt} = 2t \qquad \frac{\partial w}{\partial x} = 3y^2 = \frac{3}{t^2}$$

$$\frac{dy}{dt} = -\frac{1}{t^2} \qquad \frac{\partial w}{\partial y} = 6xy = 6(t^2+1)\frac{1}{t} = 6\left(t+\frac{1}{t}\right).$$

$$\boxed{\frac{dw}{dt} = \frac{dx}{dt}\cdot\frac{\partial w}{\partial x} + \frac{dy}{dt}\cdot\frac{\partial w}{\partial y}}$$

$$= 2t \times \frac{3}{t^2} - \frac{1}{t^2} \times 6\left(t+\frac{1}{t}\right)$$

$$= \frac{6}{t} - \frac{6}{t} - \frac{6}{t^3} = \boxed{-\frac{6}{t^3}}$$

By substitution

$$w = 3xy^2 = 3(t^2+1)\frac{1}{t^2} = 3 + \frac{3}{t^2}$$

$$\frac{dw}{dt} = 0 - \frac{6}{t^3} = \boxed{-\frac{6}{t^3}}$$

$w = f(x, y) \quad ; \quad x = g(u, v) \quad ; \quad y = h(u, v)$

$$\begin{tikzcd} & x \arrow[dr] & \\ u, v \arrow[ur] \arrow[dr] & & w \\ & y \arrow[ur] & \end{tikzcd}$$

$$\frac{\partial w}{\partial u} = \frac{\partial x}{\partial u} \cdot \frac{\partial w}{\partial x} + \frac{\partial y}{\partial u} \cdot \frac{\partial w}{\partial y}$$

$$\frac{\partial w}{\partial v} = \frac{\partial x}{\partial v} \cdot \frac{\partial w}{\partial x} + \frac{\partial y}{\partial v} \cdot \frac{\partial w}{\partial y}$$

Example: $w = x y^2 \qquad x = u + v \;;\; y = u - v$

Find $\frac{\partial w}{\partial u}$; $\frac{\partial w}{\partial v}$ by using the chain rule.

$\frac{\partial x}{\partial u} = 1 \qquad \frac{\partial x}{\partial v} = 1 \qquad \frac{\partial w}{\partial x} = y^2 = (u-v)^2$

$\frac{\partial y}{\partial u} = 1 \qquad \frac{\partial y}{\partial v} = -1 \qquad \frac{\partial w}{\partial y} = 2xy = 2(u^2 - v^2)$

Then $\boxed{\frac{\partial w}{\partial u}} = \frac{\partial x}{\partial u} \cdot \frac{\partial w}{\partial x} + \frac{\partial y}{\partial u} \cdot \frac{\partial w}{\partial y} = 1 \cdot (u-v)^2 + 1 \cdot 2(u^2 - v^2)$

$= u^2 + v^2 - 2uv + 2u^2 - 2v^2$

$= \boxed{3u^2 - v^2 - 2uv}$

$(u+v)(u-v) = u^2 - v^2$

$$\boxed{\frac{\partial w}{\partial v}} = \frac{\partial x}{\partial v} \cdot \frac{\partial w}{\partial x} + \frac{\partial y}{\partial v} \cdot \frac{\partial w}{\partial y}$$

$$= (u-v)^2 - 2(u^2-v^2)$$

$$= u^2 + v^2 - 2uv - 2u^2 + 2v^2$$

$$= \boxed{3v^2 - u^2 - 2uv}$$

By substitution:

$$w = xy^2 = (u+v)(u-v)^2$$

$$= (u+v)(u^2+v^2-2uv)$$

$$= u^3 + uv^2 - 2u^2v + vu^2 + v^3 - 2uv^2$$

$$= u^3 - uv^2 - u^2v + v^3$$

$$\frac{\partial w}{\partial u} = 3u^2 - v^2 - 2uv$$

$$\frac{\partial w}{\partial v} = -2uv - u^2 + 3v^2$$