# The implicit functions

Explicit functions:

$$y = f(x) \quad : \quad f(x) = x^2 + \cos x + 1$$

$$y = x^3 - e^x$$

Implicit functions:

The function $y = f(x)$ satisfies:

$$\begin{cases} y^2 - xy + e^y = 1 \\ y(2) = 0 \end{cases}$$

$(x, y) = (2, 0)$ is a solution.

There are some implicit functions that we cannot find explicitly

Can we compute $\frac{dy}{dx}(2)$ ?

Yes ; by using the chain rule

Consider $w = y^2 - xy + e^y - 1$

Our implicit function $y = y(x)$

satisfies $\quad w(x, y) = 0$

$$w(x; y(x)) = 0$$

$$\begin{tikzcd}
 & x \arrow[dr] & \\
x \arrow[ur] \arrow[dr] & & w(x, y(x)) = 0 \\
 & y(x) \arrow[ur] &
\end{tikzcd}$$

$$0 = \frac{dw}{dx} = \underbrace{\frac{dx}{dx}}_{1} \cdot \frac{\partial w}{\partial x} + \frac{dy}{dx} \cdot \frac{\partial w}{\partial y}$$

Therefore: $\quad \dfrac{\partial w}{\partial x} + \dfrac{dy}{dx} \cdot \dfrac{\partial w}{\partial y} = 0$

And $\quad \boxed{\dfrac{dy}{dx} = -\dfrac{\frac{\partial w}{\partial x}}{\frac{\partial w}{\partial y}}}$

Here: $\quad \dfrac{\partial w}{\partial x} = -y$

$$\frac{\partial w}{\partial y} = 2y - x + e^y$$

Therefore:

$$\frac{dy}{dx} = + \frac{y}{2y - x + e^y}$$

At $x = 2$ ; $y = 0$ : $\quad \frac{dy}{dx}(1) = \frac{0}{-2+1} = 0$

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<u>Example</u>: Let $y = f(x)$ be defined by

$$x^2 + 2y^2 x - 2x = 0$$

Compute $\frac{dy}{dx}$.

<u>Solution</u>: Let $w = x^2 + 2y^2 x - 2x$

So $y = f(x)$ is defined by $w(x,y) = 0$

We have :

$$\frac{\partial w}{\partial x} = 2x + 2y^2 - 2$$

$$\frac{\partial w}{\partial y} = 4xy$$

Therefore

$$\frac{dy}{dx} = - \frac{2x + 2y^2 - 2}{4xy}$$

$$\frac{dy}{dx} = -\frac{x + y^2 - 1}{2xy}$$

If $y(1) = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$ satisfies ~~$w(1, \frac{\sqrt{2}}{2})$~~

$w = 0$

$w(1, \frac{\sqrt{2}}{2}) = 0$

Then $y'(1) = -\frac{1 + \frac{1}{2} - 1}{\cancel{2}\frac{\sqrt{2}}{\cancel{2}}} = -\frac{\sqrt{2}}{4}$

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<u>Example</u>: Let $y = f(x)$ be a function defined by $x^3 - 3xy^2 + y^3 = \underline{1}$

Find $\frac{dy}{dx}$ with $y(1) = 3$ and $y'(1) = ?$

<u>Solution</u>: Let $w(x, y) = x^3 - 3xy^2 + y^3 - \underline{1}$

We have $\frac{\partial w}{\partial x} = 3x^2 - 3y^2$

$\frac{\partial w}{\partial y} = -6xy + 3y^2$

Therefore $\frac{dy}{dx} = -\frac{3x^2 - 3y^2}{-6xy + 3y^2} = \frac{y^2 - x^2}{y^2 - 2xy}$

And $y'(1) = \dfrac{9-1}{9-6} = \dfrac{8}{3}$.

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If $z = f(x,y)$ is defined implicitely by $w(x,y,z) = 0$

then $\dfrac{\partial z}{\partial x} = - \dfrac{\frac{\partial w}{\partial x}}{\frac{\partial w}{\partial z}}$ and $\dfrac{\partial z}{\partial y} = - \dfrac{\frac{\partial w}{\partial y}}{\frac{\partial w}{\partial z}}$

Example: Let $z = f(x,y)$ be defined implicitely by $z^3 - xy + yz + y^3 - 2 = 0$

Compute $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Solution: Let $w(x,y,z) = z^3 - xy + yz + y^3 - 2$

we have:

$$\frac{\partial w}{\partial x}(x,y,z) = -y$$

$$\frac{\partial w}{\partial y}(x,y,z) = -x + z + 3y^2$$

$$\frac{\partial w}{\partial z}(x,y,z) = 3z^2 + y$$

Therefore:

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial w}{\partial x}}{\frac{\partial w}{\partial z}} = -\frac{-y}{3z^2+y} = \frac{y}{3z^2+y}$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial w}{\partial y}}{\frac{\partial w}{\partial z}} = -\frac{-x+z+3y^2}{3z^2+y} = \frac{x-z-3y^2}{3z^2+y}$$

**Example:** Let $z = f(x,y)$ be defined implicitly by: $x \sin y + z^2 = 2xz$

with $z(1;0) = 2$

Find $\frac{\partial z}{\partial x}(1,0)$ ; $\frac{\partial z}{\partial y}(1,0)$

**We have:** Let $w(x,y,z) = x\sin y + z^2 - 2xz$

$$\frac{\partial w}{\partial x} = \sin y - 2z \ ; \ \frac{\partial w}{\partial x}(1,0,2) = -4$$

$$\frac{\partial w}{\partial y} = x\cos y \ ; \ \frac{\partial w}{\partial y}(1,0,2) = 1$$

$$\frac{\partial w}{\partial z} = 2z - 2x \ ; \ \frac{\partial w}{\partial z}(1,0,2) = 2$$

Then $\boxed{\frac{\partial z}{\partial x}(1,0) = -\frac{-4}{2} = 2 \ ; \ \frac{\partial z}{\partial y}(1,0) = -\frac{1}{2}}$