Chapter 7
Simple linear regression and correlation

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Plan

1. Correlation
2. Simple linear regression

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2. Simple linear regression

Definition

The measure of linear association ρ between two variables X and Y is estimated by the sample correlation coefficient r, where

$$r = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}}$$

with $S_{xy} = \sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})$, $S_{xx} = \sum_{i=1}^{n}(x_i - \overline{x})^2$ and

$$S_{yy} = \sum_{i=1}^{n}(y_i - \overline{y})^2.$$

Example

Let consider the following grades of 6 students selected at random

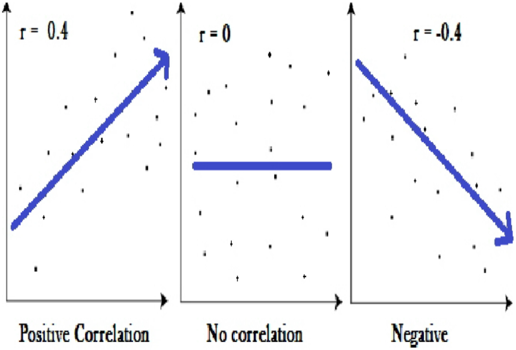
Mathematics grade	70	92	80	74	65	83
English grade	74	84	63	87	78	90

We have

$$n = 6, \quad S_{xy} = 115.33, \quad S_{xx} = 471.33, \quad and \quad S_{yy} = 491.33.$$

Hence

$$r = \frac{115.33}{\sqrt{(471.33)(491.33)}} = 0.24.$$



Properties of r

1. $r = 1$ iff all (x_i, y_i) pairs lie on straight line with positive slope,
2. $r = -1$ iff all (x_i, y_i) pairs lie on a straight line with negative slope.

Plan

The form of a relationship between the response Y (the dependent or the response variable) and the regressor X (the independent variable) is in mathematically the linear relationship

$$Y = \beta_0 + \beta_1 X + \varepsilon_i$$

where, β_0 is the intercept, β_1 the slope and ε_i, the error term in the model, is a random variable with mean 0 and constant variance. An important aspect of regression analysis is to estimate the parameters β_0 and β_1 (i.e., estimate the so-called regression coefficients). The method of estimation will be discussed in the next section. Suppose we denote the estimates b_0 for β_0 and b_1 for β_1. Then the estimated or fitted regression line is given by

$$\hat{Y} = b_0 + b_1 x$$

where $\hat{Y}$ is the predicted or fitted value.

Least Squares and the Fitted Model

Definition

Given a set of regression data $\{(x_i, y_i); i = 1, 2, ..., n\}$ and a fitted model, $\hat{y}_i = b_0 + b_1 x_i$, the i^{th} residual e_i is given by

$$e_i = y_i - \hat{y}_i,\ i = 1, 2, ..., n.$$

We shall find b_0 and b_1, the estimates of β_0 and β_1, so that the sum of the squares of the residuals is a minimum. This minimization procedure for estimating the parameters is called the method of least squares. Hence, we shall find b_0 and b_1 so as to minimize

$$SSE = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 = \sum_{i=1}^{n} (y_i - b_0 - b_1 x_i)^2$$

SSE is called the error sum of squares.

Theorem

Given the sample $\{(x_i, y_i); i = 1, 2, ..., n\}$, the least squares estimates b_0 and b_1 of the regression coefficients β_0 and β_1 are computed from the formulas

$$b_1 = \frac{\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})}{\sum_{i=1}^{n}(x_i - \overline{x})^2} = \frac{\sum_{i=1}^{n} x_i y_i - n\overline{x}\ \overline{y}}{\sum_{i=1}^{n} x_i^2 - n\overline{x}^2}$$

$$b_0 = \overline{y} - b_1\overline{x}$$

Example

Consider the experimental data in Table, which were obtained from 33 samples of chemically treated waste in a study conducted at Virginia Tech. Readings on x, the percent reduction in total solids, and y, the percent reduction in chemical oxygen demand, were recorded. We denote by
x: Solids Reduction
y: Oxygen Demand

x (%),	y(%)	x (%),	y (%)
3	5	36	34
7	11	37	36
11	21	38	38
15	16	39	37
18	16	39	36
27	28	39	45
29	27	40	39
30	25	41	41
30	35	42	40
31	30	42	44
31	40	43	37
32	32	44	44
33	34	45	46
33	32	46	46
34	34	47	49
36	37	50	51
36	38		

The estimated regression line is given by

$$\hat{y} = 3.8296 + 0.9036x.$$

Using the regression line, we would predict a 31% reduction in the chemical oxygen demand when the reduction in the total solids is 30%. The 31% reduction in the chemical oxygen demand may be interpreted as an estimate of the population mean $\mu_{Y|30}$ or as an estimate of a new observation when the reduction in total solids is 30%.

Properties of the Least Squares Estimators

Theorem

We have

1. $E(b_0) = \beta_0$, $E(b_1) = \beta_1$,
2. $V(b_1) = \dfrac{\sigma^2}{\sum_{i=1}^{n}(x_i - \overline{x})^2} = \dfrac{\sigma^2}{S_{xx}}$.

Theorem

An unbiased estimate of σ^2, named the mean squared error, is

$$\widehat{\sigma}^2 = \frac{SSE}{n-2} = \frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{n-2}$$

Theorem

Assume now that the errors ε_i are normally distributed. A $100(1-\alpha)\%$ confidence interval for the parameter β_1 in the regression line

$$b_1 - t_{\alpha/2}\frac{\widehat{\sigma}}{\sqrt{S_{xx}}} < \beta_1 < b_1 + t_{\alpha/2}\frac{\widehat{\sigma}}{\sqrt{S_{xx}}}$$

where $t_{\alpha/2}$ is a value of the t-distribution with $n-2$ degrees of freedom.

Example

Find a 95% confidence interval for β_1 in the regression line, based on the pollution data of Example 10.

Solution

We show that

$$\widehat{\sigma}^2 = \frac{SSE}{n-2} = \frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{n-2} = 0.4299.$$

Therefore, taking the square root, we obtain $\widehat{\sigma} = 3.2295$. Also,

$$S_{xx} = \sum_{i=1}^{n}(x_i - \overline{x})^2 = 4152.18.$$

Using Table of the t-distribution, we find that $t_{0.025} \approx 2.045$ for 31 degrees of freedom. Therefore, a 95% confidence interval for β_1 is

$$0.903643 - (2.045)\frac{3.2295}{\sqrt{4152.18}} < \beta_1 < 0.903643 + (2.045)\frac{3.2295}{\sqrt{4152.18}}$$

which simplifies to

$$0.8012 < \beta_1 < 1.0061.$$

Hypothesis Testing on the Slope

To test the null hypothesis H_0 that $\beta_1 = \beta_{10}$, we again use the t-distribution with $n - 2$ degrees of freedom to establish a critical region and then base our decision on the value of

$$t = \frac{b_1 - \beta_{10}}{\widehat{\sigma}/\sqrt{S_{xx}}}$$

which is t-distribution with $n - 2$ degrees of freedom.

Example

Using the estimated value $b_1 = 0.903643$ of Example 10, test the hypothesis that $\beta_1 = 1$ against the alternative that $\beta_1 < 1$.

Solution

The hypotheses are $H_0 : \beta_1 = 1$ and $H_1 : \beta_1 < 1$. So

$$t = \frac{0.903643 - 1}{3.2295/\sqrt{4152.18}} = -1.92,$$

with $n - 2 = 31$ degrees of freedom ($P \approx 0.03$).
Decision: P-value < 0.05, suggesting strong evidence that $\beta_1 < 1$

One important t-test on the slope is the test of the hypothesis $H_0 : \beta_1 = 0$ versus $H_1 : \beta_1 \neq 0$. When the null hypothesis is not rejected, the conclusion is that there is no significant linear relationship between $E(y)$ and the independent variable x. Rejection of H_0 above implies that a significant linear regression exists.

Measuring Goodness-of-Fit: the Coefficient of Determination

A goodness-of-fit statistic is a quantity that measures how well a model explains a given set of data. A linear model fits well if there is a strong linear relationship between x and y.

Definition

The coefficient of determination, R^2, is given by

$$R^2 = 1 - \frac{SSE}{SST}$$

where $SSE = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2$ and $SST = \sum_{i=1}^{n}(y_i - \overline{y})^2$.

Note that if the fit is perfect, all residuals $y_i - \hat{y}_i$ are zero, and thus $R^2 = 1$. But if SSE is only slightly smaller than SST, $R^2 \approx 0$. In the example of table 10, the coefficient of determination $R^2 = 0.913$, suggests that the model fit to the data explains 91.3% of the variability observed in the response, the reduction in chemical oxygen demand.