KING SAUD UNIVERSITY
DEPARTMENT OF MATHEMATICS
(SEMESTER I, 1435-1436)

M - 107
FULL MARKS :80
FINAL
TIME: 3hours

Question: 1 . (a) Find the condition on a , b and c such that the following system is consistent

[5+ 5+5]

$$\begin{aligned} x + 2y - 2z - 2t &= a \\ -y + 3z + 2t &= b \\ -x + y + 4z + 3t &= c \\ 4y - z - t &= 2 \end{aligned}$$

(b) If $A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$ **and** $B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$ **find the inverse of the matrix** $(BA)^T$

(c) Use Cramer's rule to solve the system

$$2x + y + z = 1, \quad x - y + 4z = 0, \quad x + 2y - 2z = 3$$

Question: 2.(a) Find the parametric equations of the line passing through point A (2, 1, 3)

[5+ 5+6] **and perpendicular to the plane** $3x - 2y + z = 2$

(b) Determine if the three vectors

$a = <1,4,-7>, \; b = <2,-1,4>$ *and* $c = <0,-9,18>$ **lie in the same plane .**

(c) If given points $P(4,-3,-2), Q(2,3,5), R(1,0,5)$ *and* $S(-3,2,7)$ **. Find**

(i)The angle between $\vec{PQ}$ and $\vec{RS}$.

(ii)The component of $\vec{PS}$ along $\vec{QR}$.

Question: 3. (a) Consider the function

$u = f(x,y)$ where $x = r\cos\theta$ and $y = r\sin\theta$

[6+ 6+6]

prove that $\left(\dfrac{\partial u}{\partial r}\right)^2 + \dfrac{1}{r^2}\left(\dfrac{\partial u}{\partial \theta}\right)^2 = \left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2$

(b)Find the unit tangent vector for the curve C

$x = t^2 + 1, y = 4t - 3, z = 2t^2 - 6t$ **at** $t = 2$.

(c) Find the curvature, radius of curvature and center of curvature of

$y = \sin(3x)$ **at the point** $\mathbf{P}\left(\frac{\pi}{6}, 1\right)$.

Question: 4. (a) Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2 + y^4}$ **does not exist.**

[5+ 5+5] (b) Find $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z}$ if $u(x,y,z) = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

(c) Use differential to find approximate change in

$f(x,y) = x^2 + 2xy - 4x,$ if the point$(-1,2)$ change to $(-1.01,1.98)$

Question: 5. (a) Find the directional derivative of $f(x,y,z) = xy + yz + xz$ **at the point P** $(1,1,1)$

[8+8] **in the direction of the line** $x = -1 + 2t, \; y = 2 + t, \; z = 1 - t$,

- **In which direction** $f(x,y,z)$ **increases most rapidly at p?**
- **What is the maximum rate of increase of** $f(x,y,z)$ **at P?**

(b) Let $f(x,y,z) = x^2 + y^2 + 4z^2$**. Use Lagrange multipliers to find the point on the plane** $x + y + z = 1$ **at which** $f(x,y,z)$ **has minimum value. Find the value.**

Question: 1 . (a) Find the condition on a , b and c such that the following system is consistent

[5+ 5+5]

$$\begin{aligned} x + 2y - 2z - 2t &= a \\ -y + 3z + 2t &= b \\ -x + y + 4z + 3t &= c \\ 4y - z - t &= 2 \end{aligned}$$

(b) If $A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$ **and** $B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$ find the inverse of the matrix $(BA)^T$

(c) Use Cramer's rule to solve the system

$$2x + y + z = 1, \quad x - y + 4z = 0, \quad x + 2y - 2z = 3$$

Solution (a)

[5]

$$\left[\begin{array}{cccc|c} 1 & 2 & -2 & -2 & a \\ 0 & -1 & 3 & 2 & b \\ -1 & 1 & 4 & 3 & c \\ 0 & 4 & -1 & -1 & 2 \end{array}\right] \overset{1}{\equiv} \left[\begin{array}{cccc|c} 1 & 2 & -2 & -2 & a \\ 0 & -1 & 3 & 2 & b \\ 0 & 3 & 2 & 1 & a+c \\ 0 & 4 & -1 & -1 & 2 \end{array}\right]$$

$$\overset{2}{\equiv} \left[\begin{array}{cccc|c} 1 & 2 & -2 & -2 & a \\ 0 & -1 & 3 & 2 & b \\ 0 & 0 & 11 & 7 & a+c+3b \\ 0 & 0 & 11 & 7 & 2+4b \end{array}\right] \overset{1}{\equiv} \left[\begin{array}{cccc|c} 1 & 2 & -2 & -2 & a \\ 0 & -1 & 3 & 2 & b \\ 0 & 0 & 11 & 7 & a+c+3b \\ 0 & 0 & 0 & 0 & a+c+3b-2-4b \end{array}\right]$$

System is consistent if $a - b + c = 2$ 1

(b)

[5]

$$BA \overset{2}{=} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}, \quad \overset{1}{(BA)^T} = \begin{bmatrix} 2 & 2 \\ 4 & 3 \end{bmatrix}$$

2

$$\text{Inverse } (BA)^T = -\frac{1}{2} \begin{bmatrix} 3 & -2 \\ -4 & 2 \end{bmatrix}$$

[5] (c)

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 4 \\ 1 & 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

1 $\det A = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & 4 \\ 1 & 2 & -2 \end{vmatrix} = -3$, $\quad \det A_1 = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & 4 \\ 3 & 2 & -2 \end{vmatrix} = 9$ 1

1 $\det A_2 = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 0 & 4 \\ 1 & 3 & -2 \end{vmatrix} = -15$ $\quad \det A_3 = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 2 & 3 \end{vmatrix} = -6$ 1

$x = \frac{9}{-3} = 3$, $\quad y = \frac{-15}{-3} = 5$, $\quad z = \frac{-6}{-3} = 2$ 1

Question: 2.(a) **Find the parametric equations of the line passing through point A (2, 1, 3)**
[5+ 5+6] **and perpendicular to the plane** $3x-2y+z=2$

(b) Determine if the three vectors
$a=<1,4,-7>$, $b=<2,-1,4>$ *and* $c=<0,-9,18>$ **lie in the same plane.**

(c) If given points $P(4,-3,-2)$, $Q(2,3,5)$, $R(1,0,5)$ *and* $S(-3,2,7)$. **Find**

(i)The angle between $\bar{PQ}$ and $\bar{RS}$.

(ii)The component of $\bar{PS}$ along $\bar{QR}$.

[5] (a) vector parallel to line is $N = <3,-2,1>$
Line is passing through point $A(2,1,3)$ } 2

$x = 2+3t$, $y = 1-2t$, $z = 3+t$, $t \in \mathbb{R}$. 3

[5] 1(b) If three vectors lie in same plane, then volume of box will be zero

$$Vol = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix} = 18 - 144 + 126 = 0$$ 5

$\Rightarrow$ vectors lie in same plane.

[6] (c) (i) $\vec{PQ} = <-2,6,7>$, $\vec{RS} = <-4,2,2>$ $\boxed{0.5 + 0.5}$

$\vec{PQ}\cdot\vec{RS} = 8+12+14 = 34$

$\|\vec{PQ}\| = \sqrt{4+36+49} = \sqrt{89}$, $\|\vec{RS}\| = \sqrt{16+4+4} = \sqrt{24}$ } 1

$$\theta = \cos^{-1}\frac{\vec{PQ}\cdot\vec{RS}}{\|\vec{PQ}\|\|\vec{RS}\|} = \cos^{-1}\frac{34}{\sqrt{89}\sqrt{24}}$$ 1

(ii) 1 $$\text{Comp}_{\vec{QR}}\vec{PS} = \frac{\vec{PS}\cdot\vec{QR}}{\|\vec{QR}\|},\quad \vec{PS} = <-7,5,9>,\ \vec{QR} = <-1,-3,0>$$ 1

$$= \frac{7-15}{\sqrt{1+9}} = \frac{-8}{\sqrt{10}}$$ 1

Question: 3. (a) Consider the function

$u = f(x, y)$ where $x = r\cos\theta$ and $y = r\sin\theta$

[6+ 6+6] prove that $\left(\dfrac{\partial u}{\partial r}\right)^2 + \dfrac{1}{r^2}\left(\dfrac{\partial u}{\partial \theta}\right)^2 = \left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2$

(b)Find the unit tangent vector for the curve C

$x = t^2 + 1, y = 4t - 3, z = 2t^2 - 6t$ **at** $t = 2$.

(c) Find the curvature, radius of curvature and center of curvature of

$y = \sin(3x)$ **at the point** $P\left(\frac{\pi}{6}, 1\right)$.

(a) [6]

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cdot\cos\theta + \frac{\partial u}{\partial y}\sin\theta \quad 1$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta) \quad 1$$

$$1 \quad \left(\frac{\partial u}{\partial r}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2\cos^2\theta + \left(\frac{\partial u}{\partial y}\right)^2\sin^2\theta + 2\,\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}\cos\theta\sin\theta \quad \rightarrow 1$$

$$1 \quad \left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 r^2\sin^2\theta + \left(\frac{\partial u}{\partial y}\right)^2 r^2\cos^2\theta - 2r^2\,\frac{\partial u}{\partial x}\,\frac{\partial u}{\partial y}\cos\theta\sin\theta \quad \rightarrow 2$$

$$2 \quad \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

(b) [6]

$$r(t) = \langle t^2+1,\ 4t-3,\ 2t^2-6t\rangle$$

$$r'(t) = \langle 2t, 4, 4t-6\rangle \quad 1$$

$$r'(2) = \langle 4, 4, 2\rangle \quad 1$$

$$\|r'(2)\| = \sqrt{16+16+4} = \sqrt{36} = 6 \quad 1$$

$$T'(2) = \frac{r'(2)}{\|r'(2)\|} = \frac{1}{6}\langle 4, 4, 2\rangle = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right\rangle. \quad 3$$

(c) [6]

$y = \sin 3x \qquad y' = 3\cos 3x \qquad y'' = -9\sin 3x$

At $\frac{\pi}{6} \qquad y'(\frac{\pi}{6}) = 0; \qquad y''(\frac{\pi}{6}) = -9$ } 2

curvature $\quad K = \dfrac{|y''|}{(1+y'^2)} = \dfrac{|-9|}{(1+0)} = +9 \quad 1$

Radius of curvature $\quad \rho = \dfrac{1}{K} = \dfrac{1}{9} \quad 1$

centre of curvature $\quad h = x - \dfrac{y'(1+y'^2)}{y''} = \dfrac{\pi}{6} - 0 = \dfrac{\pi}{6} \quad 1$

$k = y + \dfrac{1+y'^2}{y''} = 1 - \dfrac{1}{9} = \dfrac{8}{9} \quad 1$

$(h, k) = \left(\dfrac{\pi}{6}, \dfrac{8}{9}\right)$

Question: 4. (a) Show that $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ does not exist.

[5+ 5+5] **(b)** Find $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}$ if $u(x,y,z)=\begin{vmatrix}1 & x & x^2\\ 1 & y & y^2\\ 1 & z & z^2\end{vmatrix}$

(c) Use differential to find approximate change in $f(x,y)=x^2+2xy-4x$, if the point$(-1,2)$ change to $(-1.01,1.98)$

(a)

[5]

(i) $y = x \Rightarrow \lim_{(x,x)\to(0,0)} \frac{x^3}{x^2+x^4} = \lim_{(x,x)\to(0,0)} \frac{x}{1+x^2} = \frac{0}{1} = 0$ [2]

(ii) $y=\sqrt{x}$ or $x = y^2$ $\lim_{(x,\sqrt{x})\to(0,0)} \frac{x^2}{x^2+x^2} = \lim_{(x,\sqrt{x})\to(0,0)} \frac{1}{2} = \frac{1}{2}$ [2]

$\Rightarrow$ Limit doesnot exist. [1]

(b)

[5]

$u(x,y,z) = yz^2 - zy^2 + zx^2 - xz^2 + xy^2 - yx^2$ [2]

$\frac{\partial u}{\partial x} = 2zx - z^2 + y^2 - 2xy$

$\frac{\partial u}{\partial y} = z^2 - 2zy + 2xy - x^2$

$\frac{\partial u}{\partial z} = 2yz - y^2 + x^2 - 2xz$ [2]

$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$ [1]

(c)

[5]

$f(x,y) = x^2 + 2xy - 4x$

$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$ [1]

$= (2x+2y-4)\,dx + (2x)\,dy$ [2]

$x=-1, \quad y=2, \quad dx = -.01, \quad dy = -.02$ [0.5+0.5]

$df = (-2+4-4)(-.01) + (2(-1))(-.02$

$= .02 + .04 = .06$ [1]

Question: 5. (a) Find the directional derivative of $f(x,y,z) = xy + yz + xz$ at the point P(1,1,1)
[8+8] **in the direction of the line $x = -1 + 2t,\ y = 2 + t,\ z = 1 - t$,**

- **In which direction it increases most rapidly?**
- **What is the maximum rate of increase of f at P?.**

(b) Let $f(x,y,z) = x^2 + y^2 + 4z^2$. Use Lagrange multipliers to find the point on the plane $x + y + z = 1$ at which $f(x,y,z)$ has minimum value. Find the value.

(a) [8]

$\nabla f = \langle y+z, x+z, y+x \rangle$ 1

$a = \langle 2, 1, -1 \rangle$, $\|a\| = \sqrt{6}$ 1

$u = \frac{1}{\sqrt{6}} \langle 2, 1, -1 \rangle$ 1

$D_u f(1,1,1) = \langle 2,2,2 \rangle \cdot \frac{1}{\sqrt{6}} \langle 2,1,-1 \rangle$

$= \frac{1}{\sqrt{6}} (4 + 2 - 2) = \frac{4}{\sqrt{6}}$ 1

- f increase most rapidly in direction of ∇f 2

  $\nabla f(1,1,1) = \langle 2,2,2 \rangle$

- Maximum rate of increase is $\|\nabla f\| = \sqrt{12}$ 2

(b) [8]

$f(x,y,z) = x^2 + y^2 + 4z^2$, condition is $g(x,y,z) = x + y + z - 1 = 0$

1 $\nabla f = \langle 2x, 2y, 8z \rangle$, $\nabla g = \langle 1, 1, 1 \rangle$ 1

2

$\nabla f = \lambda \nabla g$

$\langle 2x, 2y, 8z \rangle = \lambda \langle 1, 1, 1 \rangle$

$2x = \lambda$ $\qquad x = \frac{\lambda}{2}$

$2y = \lambda$ $\qquad y = \frac{\lambda}{2}$

$8z = \lambda$ $\qquad z = \frac{\lambda}{8}$

$x + y + z = 1$

$\frac{\lambda}{2} + \frac{\lambda}{2} + \frac{\lambda}{8} = 1$

$\frac{9\lambda}{8} = 1 \Rightarrow \lambda = \frac{8}{9} \Rightarrow$ $x = \frac{4}{9}$, $y = \frac{4}{9}$, $z = \frac{1}{9}$ (1 1 1)

Minimum value $f\left(\frac{4}{9}, \frac{4}{9}, \frac{1}{9}\right) = \frac{16}{81} + \frac{16}{81} + \frac{1}{81} = \frac{33}{81}$ 1