King Saud University	College of Sciences	Department of Mathematics
Final Examination	Math 132	Trimester 2(1444) Time: 3H

Calculators are not allowed

The Examination contains 2 pages

Question 1: (12 Marks)

1. Without using truth tables, prove the following statement: **(3 Marks)**

$$[(p \wedge \neg q) \to (\neg p \vee q)] \equiv \neg p \vee q$$

2. Using the first question, prove that the following statement is contradiction: **(2 Marks)**

$$[(p \wedge \neg q) \to (\neg p \vee q)] \wedge (p \wedge \neg q)$$

3. Let a and b be real numbers. Use a proof by contraposition to show that if $2a + b > 11$ then $a > 4$ or $b > 3$. **(2 Marks)**

4. Assume that $\sqrt{7}$ is irrational. Use a proof by contradiction to show that $\dfrac{2+\sqrt{7}}{3}$ is irrational. **(2 Marks)**

5. Let $\{a_n\}_{n \geq 0}$ be a sequence defined as $\begin{cases} a_0 = 2, a_1 = 4 \\ a_{n+1} = 4a_n - 3a_{n-1}, & \text{for } n \geq 1 \end{cases}$.

   Show that $a_n = 1 + 3^n$, $\forall n \geq 0$. **(3 Marks)**

Question 2: (13 Marks)

1. Let E be the relation defined on the set $\mathbb{Z}$ by:

$$a, b \in \mathbb{Z}, (aEb) \iff 5|a + 4b.$$

   (a) Prove that E is an equivalence relation on $\mathbb{Z}$. **(3 Marks)**

   (b) Show that $[1] \neq [2]$. **(1 Mark)**

2. Let P be the relation defined on the set $C := \{a, b, c, d, e\}$ by :

   $P = \{(a,a), (a,b), (a,d), (a,e), (b,b), (b,e), (c,c), (c,d), (c,e), (d,d), (d,e), (e,e)\}$

   (a) Find P^{-1} and $\overline{P}$. **(2 Marks)**

   (b) Find P^2. **(2 Marks)**

   (c) Prove that P is partial ordering relation on C. **(3 Marks)**

   (d) Draw the Hasse diagram of P. **(1 Mark)**

   (e) Is P a total ordering? (Justify your answer) **(1 Mark)**

Question 3: (15 Marks)

1. Consider the following three sets $X := \{1, 2, 3, 4\}, Y := \{1, 2, 3\}$, and $Z := \{(x, y)|x \in X, y \in Y, x \leq y\}$. Find the following sets:
   (a) Z. **(1 Mark)**
   (b) $(X \times Y) - Z$. **(1 Mark)**
   (c) $(Y - X) \times X$. **(1 Mark)**
   (d) $Z - (X \times Y)$. **(1 Mark)**

2. Consider the sets $A := \{1, 2, 3, 4\}$ and $B := \{x, y, z\}$, and the function $f : A \longrightarrow B$ defined by: $f(1) = f(4) = y$, $f(2) = x$ and $f(3) = z$.
   (a) Find the image of each of the sets $\{1, 2\}, \{1, 4\}$, and $\{2, 3, 4\}$. **(1.5 Marks)**
   (b) Find the inverse image of each of the sets $\{x, y\}, \{z\}$, and $\{y, z\}$. **(1.5 Marks)**
   (c) For the function f, determine whether it is one-to-one, and whether it is onto B. (Justify your answer). **(2 Marks)**

3. Let g and h be the functions from the set of integers to the set of integers defined by $g(x) = 2 - x$ and $h(x) = -2x + 2$.
   (a) Prove that g is a one to one correspondence. **(2 Marks)**
   (b) Find the g^{-1} the inverse function of g. **(2 Marks)**
   (c) Find $g \circ h$ and $h \circ g$. **(2 Marks)**

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Q1)

1. $(p \wedge \neg q) \rightarrow (\neg p \vee q) \equiv \neg(p \wedge \neg q) \vee (\neg p \vee q)$

$\equiv (\neg p \vee q) \vee (\neg p \vee q)$

$\equiv (\neg p \vee \neg p) \vee (q \vee q)$

$\equiv \neg p \vee q$

$\frac{3}{3}$

2. $[(p \wedge \neg q) \rightarrow (\neg p \vee q)] \wedge (p \wedge \neg q)$

$\equiv [\neg p \vee q] \wedge (p \wedge \neg q)$ (from Q1)

$\frac{1.5}{2}$

p	q	$\neg p$	$\neg q$	$\neg p \vee q$	$p \wedge \neg q$	$(\neg p \vee q) \wedge (p \wedge \neg q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	F
F	T	T	F	T	F	F
F	F	T	T	T	F	F

Q1

$\frac{11.5}{12}$

So, it is contradiction.

3. If $2a+b \geq 11$ then $a \geq 4$ or $b \geq 3$.

The contrapositive is: If $a < 4$ and $b < 3$, then $2a+b < 11$

$a < 4 \overset{\times 2}{\Rightarrow} 2a < 8$ ①

$b < 3$ ②

From ① and ② $2a+b < 8+3 = 11$ $\frac{2}{2}$

$\therefore \neg q \rightarrow \neg p$ is true

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4. Assume $\sqrt{7}$ irrational

Assume by contradiction $\frac{2+\sqrt{7}}{3}$ is rational

$\Rightarrow \frac{2+\sqrt{7}}{3} = \frac{a}{b}$ $(b \neq 0,\ a,b \in \mathbb{Z})$

$\frac{2}{2}$

$\Rightarrow (2+\sqrt{7})b = 3a$

$\Rightarrow 2b + \sqrt{7}b = 3a$

$\Rightarrow \sqrt{7}b = 3a - 2b$

$\Rightarrow \sqrt{7} = \frac{3a-2b}{b}$, So $\sqrt{7}$ is rational

$\left(\frac{3a-2b}{b}\right.$ rational $b \neq 0$ $\left. 3a-2b,\ b \in \mathbb{Z}\right)$

By contradiction, $\frac{2+\sqrt{7}}{3}$ is irrational

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5. $\{a_n\}$ $\begin{cases} a_0 = 2 \quad a_1 = 4 \\ \cancel{a_{n+1}} = 4a_n - 3a_{n-1} \end{cases}$ $n \geq 1$

Let $P(n)$: "$a_n = 1 + 3^n \quad \forall n \geq 0$"

Basis step: $n=0$:

$a_0 = 1 + 3^0 = 1 + 1 = 2$ is true

$n = 1$:

$a_1 = 1 + 3^1 = 1 + 3 = 4$ is true

So $p(0)$ and $p(1)$ is true

Inductive step: Suppose $p(2) \wedge \dots \wedge p(k)$ is true

we want to show $p(k+1)$ is true $\forall n \geq 0$ $(a_{k+1} = 1 + 3^{k+1})$

$a_{k+1} = 4a_k - 3a_{k-1}$

$p(k)$: $a_k = 1 + 3^k$

$p(k-1)$: $a_{k-1} = 1 + 3^{k-1}$

$a_{k+1} = 4(1 + 3^k) - 3(1 + 3^{k-1})$

$= 4 + 4(3^k) - 3 - 3(3)^{k-1}$ $\cancel{4}\ \frac{3}{3}$

$= 4 - 3 + 4(3^k) - 3^k$

$= 1 + 3^k(4 - 1)$

$= 1 + 3^k(3)$

$= 1 + 3^{k+1}$

So, $p(k+1)$ is true

Q2. 1. $a, b \in \mathbb{Z}$ $\quad aEb \Leftrightarrow 5 \mid a + 4b$

(a) ① $\forall a \in \mathbb{Z}$ $\quad 5 \mid a + 4a = 5 \mid 5a$

aEa, E is reflexive +1

② let $a, b \in \mathbb{Z}$

$5 \mid a + 4b \Rightarrow a + 4b = 5h_1$ - $\quad aEb \quad h_1 \in \mathbb{Z}$

$5 \mid b + 4a \Rightarrow b + 4a = 5h_2$ $\quad bEa \quad h_2 \in \mathbb{Z}$

Assume aEb so $a + 4b = 5h_1$, we want to prove bEa

$a = 5h_1 - 4b$ $\quad h_1 \in \mathbb{Z}$

$\Rightarrow 4a = 20h_1 - 16b$

$\Rightarrow 4a + b = 20h_1 - 16b + b$

$4a + b = 20h_1 - 15b$ +1

$= 5(4h_1 - 3b)$

$= 5h$ $\quad h = 4h_1 - 3b \in \mathbb{Z}$

So bEa

$\therefore$ E is symmric

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③ $a,b,c \in \mathbb{Z}$

$aEb \wedge bEc \overset{?}{\Rightarrow} aEc$

$5 \mid a+4b \Rightarrow a+4b = 5m_1$ $\quad m_1 \in \mathbb{Z}$

$5 \mid b+4c \Rightarrow b+4c = 5m_2$ $\quad m_2 \in \mathbb{Z}$

$a+4b+b+4c = a+5b+4c = 5m_1+5m_2$

$\Rightarrow a+4c = 5m_1+5m_2-5b$

$a+4c = 5(m_1+m_2-b) = 5m$ $\quad m = m_1+m_2-b \in \mathbb{Z}$.

+1

So aEc

$\therefore E$ is transitive.

Q2 From ① → ③ E is equivalance relation on $\mathbb{Z}$. $\frac{4}{4}$

$\frac{13}{13}$

(b) $5 \mid 1+4(2)$

$\Rightarrow 5 \nmid 9$ +1

So $[1] \neq [2]$

2. (a) $P^{-1} = \{(a,a),(b,a),(d,a),(e,a),(b,b),(e,b),(c,c),(d,c),$
$(e,c),(d,d),(e,d),(e,e)\}$ +1

$\bar{P} = \{(a,c),(b,a),(b,e),(b,d),(c,a),(c,b),(d,a)$
$(d,b),(d,c),(e,a),(e,b),(e,c),(e,d)\}$ +1

(b) $P^2 = P \circ P = \{(a,a),(a,b),(a,d),(a,e),(b,e),(c,d),(c,e),(d,e)$
$(e,e),(b,b),(c,c),(d,d)\}$ +2

(c) ① Since $aPa \wedge bPb \wedge cPc \wedge dPd \wedge ePe$ $\frac{a}{a}$
P is reflexive +1

Complete

② aPb but $b \not P a$ $\quad [\text{if } aPb \wedge bPa \Rightarrow a=b]$ →
P is antisymmtric +1

③ $aPa \wedge aPb \Rightarrow aPb$ ✓ +1 $\quad bPb \wedge bPe \Rightarrow bPe$ ✓
$aPb \wedge bPb \Rightarrow aPb$ ✓ $\quad bPe \wedge ePe \Rightarrow bPe$ ✓
$aPb \wedge bPe \Rightarrow aPe$ ✓ $\quad cPc \wedge cPd \Rightarrow cPd$ ✓
$aPd \wedge dPd \Rightarrow aPd$ ✓ $\quad cPc \wedge cPe \Rightarrow cPe$ ✓
$aPd \wedge dPe \Rightarrow aPe$ ✓ $\quad cPd \wedge dPd \Rightarrow cPd$ ✓
$aPe \wedge ePe \Rightarrow aPe$ ✓ $\quad cPd \wedge dPe \Rightarrow cPe$ ✓
$dPd \wedge dPe \Rightarrow dPe$ ✓ $\quad cPe \wedge ePe \Rightarrow cPe$ ✓
$dPe \wedge ePe \Rightarrow dPe$ ✓

Complete →

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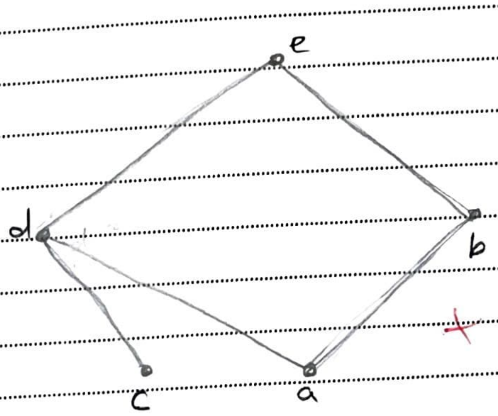
So, P is transitive
From ① → ③ P is partial ordering relation on C

(d) Hass diagram



+1

(e) No
Q3
Because c not related with a +1

$\frac{15}{15}$

Q3:
(a) $Z = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$ +1

(b) $X \times Y = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)$
$(4,1), (4,2), (4,3)\}$
$X \times Y - Z = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$ +1

(c) $Y - X = \varphi$
$(Y - X) \times X = \varphi$ +1

(d) $Z - (X \times Y) = \varphi$ +1 $\frac{4}{4}$

2.

(a) $f(\{1,2\}) = \{y, x\}$

$f(\{1,4\}) = \{y\}$ $+1.5$

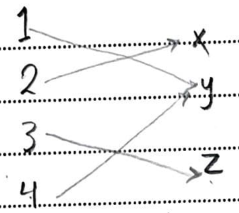
$f(\{2,3,4\}) = \{x, y, z\}$

(b) $f^{-1}\{x,y\} = \{2,1,4\}$

$f^{-1}\{z\} = \{3\}$ $+1.5$

$f^{-1}\{y,z\} = \{1,4,3\}$

(c)



$\frac{5}{5}$

① Since $f(1) = f(4) = y$, then f is not one to one

② Onto, because every element in the set B have pre image related with it. (and from graph.) $+2$

3.

$f : \mathbb{Z} \to \mathbb{Z}$

(a) ① $g(x) = g(y)$

$\Rightarrow 2 - x = 2 - y$

$\Rightarrow 2 - x - 2 = 2 - y - 2$

$\Rightarrow -x = -y$

$\Rightarrow x = y$

g is one to one

$g(1) = 1$

$g(-1) = 3$

$g(-1) \neq g(1)$

مع مثال للتأكد

② let $y \in \mathbb{Z}$ $g(x) = y$

$\Rightarrow 2 - x = y$

$\Rightarrow -x = y - 2$

$\Rightarrow x = 2 - y$

$g(x) = g(2-y) = 2 - (2-y) = y$ $+2$

g is onto

From ① & ② g is one to one correspondence

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(b) $g \circ g^{-1}(x) = g(2-x)$ ← صحح للتأكيد

$= 2-(2-x)$

$= 2-2+x$

$= x$

(b) $2-x = y \Rightarrow -2+x = -y$

$\Rightarrow x = 2-y$

$\therefore g^{-1}(x) = 2-x$ +2

(c) $g \circ h(x) = g(h(x))$

$= g(-2x+2) = 2-x$

$= g(-2x+2)$

$= 2-(-2x+2)$

$= \cancel{2}+2x-2$

$= 2x$

+2

$\frac{6}{6}$

$h \circ g(x) = h(g(x))$

$= h(2-x) \quad = -2x+2$

$= h(2-x)$

$= -2(2-x)+2$

$= -4+2x+2$

$= 2x-2$