Set Theory

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Elementarily Operations on Sets

In what follows, X is a nonempty set. We denote by $\mathscr{P}(X)$ the collection of subsets of X, ($\mathscr{P}(X)$ called also the power set, $\mathscr{P}(X) = \{A : A \subset X\}$. If A and B are in $\mathscr{P}(X)$, we put $A \setminus B := \{x \in A \text{ and } x \notin B\} = A \cap B^c$. $A \Delta B = (A \setminus B) \bigcup (B \setminus A)$ called symmetric difference of B from A, and if $A = X$, $X \setminus B = B^c$

Definition

[Characteristic functions of sets]
For any subset $A \in \mathscr{P}(X)$, we define the characteristic function χ_A (or the indicator function) of A by $\chi_A(x) = 1; \ \forall\ x \in A$ and $\chi_A(x) = 0; \ \forall\ x \notin A$.

Properties

All the operations on sets can be translated easily in term of characteristic functions of sets by the correspondence: $A \longrightarrow \chi_A$ when $A \in \mathscr{P}(X)$. We have the following relations:

1. $A \subset B \iff \chi_A \leq \chi_B$.
2. $\chi_{A\cap B} = \chi_A . \chi_B$.
3. $\chi_{A^c} = 1 - \chi_A$.
4. $\chi_{A\cup B} = \chi_A + \chi_B - \chi_A . \chi_B$.
5. $\chi_{A\setminus B} = \chi_A(1 - \chi_B)$.
6. $\chi_{A\Delta B} =\mid \chi_A - \chi_B \mid$.

Properties

7. If $(A_n)_{n\in\mathbb{N}}$ is a sequence of subsets of X, then

$$\chi_{\bigcap_n A_n} = \inf_n \chi_{\{\bigcap_{p\leq n} A_p\}} = \lim_{n\to+\infty} \prod_{k=1}^{n} \chi_{A_k}.$$

$$\chi_{\bigcup_n A_n} = \sup_n \chi_{\{\bigcup_{p\leq n} A_p\}} = \lim_{n\to+\infty} \chi_{\{\bigcup_{p\leq n} A_p\}}.$$

8. If $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ are two sequences of subsets of X, then

$$(\bigcup_{n=1}^{+\infty} A_n)\Delta(\bigcup_{n=1}^{+\infty} B_n) \subset \bigcup_{n=1}^{+\infty}(A_n\Delta B_n).$$

Definition

1. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real functions on X. We define

$$(\lim\ \sup)_{n\to+\infty}\ f_n\ =\overline{\lim}_{n\to+\infty}f_n\ =\ \inf_n\ \sup\ \{f_m; m\geq n\}$$

and

$$(\lim\ \inf)_{n\to+\infty}\ f_n\ =\underline{\lim}_{n\to+\infty}f_n\ =\ \sup_n\ \inf\ \{f_m; m\geq n\}.$$

These two limits are always exist and can take the values $\pm\infty$.

Definition

2. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of subsets of X. We define

$$\overline{\lim}_{n\to+\infty}A_n=\bigcap_{n=1}^{+\infty}\bigcup_{m=n}^{+\infty}A_m \text{ and } \underline{\lim}_{n\to+\infty}A_n=\bigcup_{n=1}^{+\infty}\bigcap_{m=n}^{+\infty}A_m.$$

$\overline{\lim}_{n\to+\infty}A_n$ (or $\limsup_{n\to+\infty} A_n$) is called the limit superior and $\underline{\lim}_{n\to+\infty}A_n$ (or $\liminf_{n\to+\infty} A_n$) is called the limit inferior.

Definition

Note that $(\bigcup_{m=n}^{+\infty} A_m)_n$ is a decreasing sequence of subsets of X

and t follows that $\lim_{n\to+\infty} \bigcup_{m=n}^{+\infty} A_m = \bigcap_{n=1}^{+\infty} \bigcup_{m=n}^{+\infty} A_m$ exists.

Similarly $(\bigcap_{m=n}^{+\infty} A_m)_n$ is an increasing sequence of subsets of X

and this implies that $\lim_{n\to+\infty} \bigcap_{m=n}^{+\infty} A_m = \bigcup_{n=1}^{+\infty} \bigcap_{m=n}^{+\infty} A_m$ exists.

The interpretation is that $\limsup_n A_n$ contains those elements of X that occur "infinitely often" in the sets A_n, and $\liminf_n A_n$ contains those elements that occur in all except finitely many of the sets A_n.

Remarks

1. If the sequence $(f_n)_{n\in\mathbb{N}}$ converges to the function f; then $\overline{\lim}_{n\to+\infty} f_n = \underline{\lim}_{n\to+\infty} f_n = f$.
2. $\overline{\lim}_{n\to+\infty} A_n$ is the set of the elements of X which are in an infinite sets of A_n. Thus

$$\overline{\lim}_{n\to+\infty} A_n \;=\; \{x\in X : \sum_{n=1}^{\infty} \chi_{A_n}(x) \;= +\infty\}.$$

3. $\underline{\lim}_{n\to+\infty}A_n$ is the set of elements of X which are in all the A_n except a finite number and thus

$$\underline{\lim}_{n\to+\infty}A_n = \{x \in X : \sum_{n=1}^{\infty} \chi_{A_n^c}(x) \ < +\infty\}.$$

4. $\underline{\lim}_{n\to+\infty}A_n \subset \overline{\lim}_{n\to+\infty}A_n$.
5. $\chi_{\overline{\lim}_{n\to+\infty}A_n} = \overline{\lim}_{n\to+\infty}\chi_{A_n}$.
6. $\chi_{\underline{\lim}_{n\to+\infty}A_n} = \underline{\lim}_{n\to+\infty}\chi_{A_n}$.

Example

Let $X = \mathbb{R}$ and let a sequence $(A_n)_n$ of subsets of $\mathbb{R}$ be defined by $A_{2n+1} = [0, \frac{1}{2n+1}]$, and $A_{2n} = [0, 2n]$. Then

$$\underline{\lim}_{n\to+\infty} A_n = \{x \in X;\ x \in A_n \text{for all but finitely many } n \in \mathbb{N}\} = \{0\}$$

and

$$\overline{\lim}_{n\to+\infty} A_n = \{x \in X;\ x \in A_n \text{for infinitely many } n \in \mathbb{N}\} = [0, \infty[.$$

Algebras and σ-Algebras

General Properties of $\sigma-$Algebras

Definition

Let $\mathscr{A}$ be a collection of subsets of X. $\mathscr{A}$ is called an algebra or a field if

1. $X \in \mathscr{A}$;
2. (Closure under complement) if $A \in \mathscr{A}$, then $A^c \in \mathscr{A}$;
3. (Closure under finite intersection) if $A_1, \dots, A_n \in \mathscr{A}$, then $\bigcap_{j=1}^n A_j \in \mathscr{A}$.
   $\mathscr{A}$ is called a $\sigma-$algebra or a $\sigma-$field if in addition
4. (Closure under countable intersection) if $(A_j)_{j\in\mathbb{N}}$ are in $\mathscr{A}$, then $\bigcap_{j=1}^{+\infty} A_j \in \mathcal{A}$.

If $\mathscr{A}$ is a $\sigma-$algebra, the pair $(X, \mathscr{A})$ is called a **measurable space**, and the subsets in $\mathscr{A}$ are called the measurable sets.

Remarks

By complementarity

1. If $\mathscr{A}$ is an algebra, then $\emptyset \in \mathscr{A}$.
2. (Closure under finite union) If $\mathscr{A}$ is an algebra and $A_1, \dots, A_n \in \mathscr{A}$,, then $\displaystyle\bigcup_{j=1}^{n} A_j \in \mathscr{A}$.
3. (Closure under countable union) If $\mathscr{A}$ is a $\sigma-$algebra and $(A_j)_{j\in\mathbb{N}}$ in $\mathscr{A}$, then $\displaystyle\bigcup_{j=1}^{+\infty} A_j \in \mathscr{A}$.

Example

$\mathscr{A} = \{\emptyset, X\}$ is an algebra and a $\sigma-$algebra. This is the smallest $\sigma-$algebra in $\mathscr{P}(X)$.

Example

$\mathscr{A} = \mathscr{P}(X)$ is an algebra and a $\sigma-$algebra. This is the largest $\sigma-$algebra in $\mathscr{P}(X)$.

Example

Let $\mathscr{F} = \{A, B, C\}$ be a partition of X. The set

$$\mathscr{A} = \{\emptyset, X, A, B, C, A \cup B = C^c, A \cup C = B^c, B \cup C = A^c\}.$$

is a $\sigma-$algebra.

Example

1. Let $X = \mathbb{R}$ and $\mathscr{A}$ the collection of subsets A of X such that either A or A^c is countable or $\emptyset$. $\mathscr{A}$ is a $\sigma-$algebra. In fact let $(A_j)_{j\in\mathbb{N}}$ be a sequence of elements of $\mathscr{A}$.
   If there exists p such that A_p is countable, then $\cap_{j=1}^{+\infty} A_j \subset A_p$ is countable and $\cap_{j=1}^{+\infty} A_j \in \mathscr{A}$.
   If every A_j is not countable, then all A_k^c are countable, and then $\cup_{j=1}^{+\infty} A_j^c$ is a countable subset of $\mathbb{R}$ and then $\cap_{j=1}^{+\infty} A_j \in \mathscr{A}$.
2. Let X be an infinite set and let $\mathscr{A}$ the collection of subsets A of X such that either A or A^c is finite, then $\mathscr{A}$ is an algebra but it is not a σ-algebra.

$\sigma-$Algebra Generated by a Subset $P \subset \mathscr{P}(X)$

Theorem

Any intersection of algebras (resp $\sigma-$ algebra) is an algebra (resp $\sigma-$ algebra).

Example

Definition

Let X be a non empty set and $\mathcal{B} \subset \mathscr{P}(X)$. There exists a smallest algebra (resp $\sigma-$algebra) denoted by $\mathcal{A}(\mathcal{B})$, (resp $\sigma(\mathcal{B})$) that contains $\mathcal{B}$. This algebra (resp $\sigma-$algebra) is called the algebra (resp $\sigma-$algebra) generated by $\mathcal{B}$.
$\mathcal{A}(\mathcal{B})$ (resp $\sigma(\mathcal{B})$) is the intersection of all algebras on X (resp $\sigma-$algebra) containing $\mathcal{B}$. So this is the smallest algebra (resp $\sigma-$algebra) which contains $\mathcal{B}$.

Example

Let A be a subset of X with $A \neq \emptyset$ and $A \neq X$. The $\sigma-$algebra generated by $\{A\}$ is $\{\emptyset, X, A, A^c\}$.

Exercise

Let X be an arbitrary nonempty set, and let $\mathscr{A}$ be the family of all subsets $A \subset X$ such that either A or $X \setminus A$ is countable. Show that $\mathscr{A}$ is the σ-algebra generated by the singleton sets $S = \{\{x\}; x \in X\}$.

Example

Borelian $\sigma-$Algebra in $\mathbb{R}$

If $X = \mathbb{R}$ and $\mathscr{B}$ is the $\sigma-$algebra generated by the family $\{[a, b[; (a, b) \in \mathbb{R}^2\}$. This $\sigma-$algebra is denoted by $\mathscr{B}_{\mathbb{R}}$ and called the Borel $\sigma-$algebra on $\mathbb{R}$. ($\mathscr{B}_{\mathbb{R}}$ contains all open and closed subsets of $\mathbb{R}$.) Every element of $\mathscr{B}_{\mathbb{R}}$ is called a Borel subset of $\mathbb{R}$.

We can prove easily that

$\mathscr{B}_{\mathbb{R}}$ is generated by $\{[a, b[; (a, b) \in \mathbb{R}^2\}$,

$\mathscr{B}_{\mathbb{R}}$ is generated by the family of open subsets in $\mathbb{R}$,

$\mathscr{B}_{\mathbb{R}}$ is generated by the family of closed subsets in $\mathbb{R}$,

$\mathscr{B}_{\mathbb{R}}$ is generated by $\{]a, +\infty[;\ a \in \mathbb{R}\}$,

$\mathscr{B}_{\mathbb{R}}$ is generated by $\{] -\infty, a];\ a \in \mathbb{R}\}$,

Example

Borelian $\sigma-$Algebra in a Topological Space

Let X be a topological space and $\mathcal{A}$ be the family of the open subsets of X. Let $\mathscr{B}_X$ be the $\sigma-$algebra generated by the family $\mathcal{A}$. Then $\mathscr{B}_X$ is called the Borel $\sigma-$algebra on X. All open and closed subsets of X are Borel subsets.

The family of the closed subsets of X generates $\mathscr{B}_X$.

Example

Product of $\sigma-$Algebras

Definition

Let $(X_1, \mathscr{A}_1)$ and $(X_2, \mathscr{A}_2)$ be two measurable spaces. We denote by X the cartesian product $X_1 \times X_2$. A subset $R = A_1 \times A_2$ of $X_1 \times X_2$ is called a rectangle with $A_1 \in \mathscr{A}_1$ and $A_2 \in \mathscr{A}_2$. We denote by $\mathcal{R}$ the set of all rectangles in X. The product $\sigma-$algebra of $\mathscr{A}_1$ and $\mathscr{A}_2$ on X is the $\sigma-$algebra generated by $\mathcal{R}$ and will be denoted by $\mathscr{A}_1 \otimes \mathscr{A}_2$.

Remark

In the same way if $(X_j, \mathscr{A}_j)$, $j = 1, \dots, n$ are n measurable spaces, we define the $\sigma-$algebra $\otimes_{j=1}^{n} \mathscr{A}_j$ on the space $X = \prod_{j=1}^{n} X_j$, and for the remainder of this course, we provide the product space X with this $\sigma-$algebra.

Example

Pull back of a $\sigma-$Algebra

Let X and X' two non empty sets, and let $f : X \longrightarrow X'$ a mapping. Let $\mathscr{B}$ be a family of subsets of X'. We define

$$f^{-1}(\mathscr{B}) = \{f^{-1}(A);\ A \in \mathscr{B}\}$$

Proposition

If $\mathscr{B}$ is a $\sigma-$algebra on X', then $f^{-1}(\mathscr{B})$ is a $\sigma-$algebra on X called the pull back of $\mathscr{B}$ under f.

Proof

We have $f^{-1}(X') = X$ and $\bigcup_j f^{-1}(A_j) = f^{-1}(\bigcup_j A_j)$ and $(f^{-1}(A))^c = f^{-1}(A'^c)$.

□

If X is a subset of X' and $f\colon X \longrightarrow X'$ is an injection, then the pull back of a $\sigma-$algebra on X' is called the **trace** of this $\sigma-$algebra on X.

Proposition

Let X and X' be two non empty sets and $f : X \longrightarrow X'$ a mapping. Let $\mathcal{B}$ be a family of subsets of X' and $\mathscr{B} = \sigma(\mathcal{B})$ the $\sigma-$algebra generated by $\mathcal{B}$. Then $f^{-1}(\mathscr{B})$ is the $\sigma-$algebra generated by $f^{-1}(\mathcal{B})$. In other words $f^{-1}(\sigma(\mathcal{B})) = \sigma(f^{-1}(\mathcal{B}))$.

Proof

Since $f^{-1}(\mathcal{B}) \subset f^{-1}(\sigma(\mathcal{B}))$, then $\sigma(f^{-1}(\mathcal{B})) \subset f^{-1}(\sigma(\mathcal{B})) = f^{-1}(\mathscr{B})$.
We shall prove the reverse inclusion in the particular case when f is surjective (onto).
Let $\mathscr{A}$ be a $\sigma-$algebra on X such that $f^{-1}(\mathcal{B}) \subset \mathscr{A} \subset f^{-1}(\mathscr{B})$.
Let $\mathscr{B}_1 = f(\mathscr{A}) = \{f(A); A \in \mathscr{A}\}$. The family $\mathscr{B}_1$ is closed under countable union and since f is surjective (onto) and $X \in \mathscr{A}$, then $X' \in \mathscr{B}_1$.
Let proving now that $\mathscr{B}_1$ is closed under complementarity.

For $K \in \mathscr{B}_1$, there exists $H \in \mathscr{A}$ such that $K = f(H)$. Since $H \in f^{-1}(\mathscr{B})$, there exists $L \in \mathscr{B}$ such that $H = f^{-1}(L)$. Thus $K = f(f^{-1}(L))$ with $L \in \mathscr{B}$. We deduce that $K^c = f(f^{-1}(L^c))$ and since $f^{-1}(L^c) = (f^{-1}(L))^c = H^c \in \mathscr{A}$, we conclude that $K^c = f(Z)$, with $Z = H^c \in \mathscr{A}$.
It results that $\mathscr{B}_1$ is a $\sigma-$algebra. So $\mathcal{B} \subset \mathscr{B}_1 \subset \mathscr{B}$, and since $\mathscr{B}$ is the $\sigma-$algebra generated by $\mathcal{B}$, we deduce that $\mathscr{B}_1 = \mathscr{B}$.
(Let $Y \in \mathscr{B}$ then $Y \in \mathscr{B}_1$, there exists thus $Z \in \mathscr{A}$ such that $Z = f^{-1}(Y) \Rightarrow f^{-1}(Y) \in \mathscr{A}$, for any $Y \in \mathscr{B}$ where $f^{-1}(\mathscr{B}) \subset \mathscr{A}$.)
Assume now that f is injective.

We can identify X as a subset of X' and f is the canonical injection of X to X'. Let $\mathscr{A}$ be a $\sigma-$algebra such that $f^{-1}(\mathcal{B}) \subset \mathscr{A} \subset f^{-1}(\mathscr{B})$. We set

$$\mathscr{B}_1 = \{C \in \mathscr{P}(X'); C \cap X \in \mathscr{A}\}.$$

$\mathscr{B}_1$ is a $\sigma-$algebra and contains $\mathcal{B}$. So $\mathscr{B}_1 \supset \mathscr{B}$. Thus $f^{-1}(\mathscr{B}_1) \supset f^{-1}(\mathscr{B})$. The result is deduced easily.
In the general case we set $Y = f(X)$. Let $f_1 : X \longrightarrow Y$ be the mapping defined by f. Let f_2 be the canonical injection of Y into X'. $f = f_2 \circ f_1$ with f_1 surjective (onto) and f_2 injective. Let $A = f^{-1}(\mathcal{B})$ and $\mathscr{A} = f^{-1}(\mathscr{B})$. Thus $\mathscr{A} = f_1^{-1}(f_2^{-1}(\mathscr{B}))$.
From the previous result, $\sigma(f^{-1}(\mathcal{B})) = f_2^{-1}(\mathscr{B})$ is a $\sigma-$algebra generated by $f_2^{-1}(\mathcal{B})$ and $f_1^{-1}(\sigma(f^{-1}(\mathcal{B})))$ is generated by $f_1^{-1}(f_2^{-1}(\mathcal{B}))$.
□

Monotone Class and $\sigma-$Algebra

Definition

A collection of sets $\mathcal{M}$ is called a **monotone class** if for any monotone sequence $(A_n)_n$ of $\mathcal{M}$; $\lim_{n\to+\infty} A_n \in \mathcal{M}$.

Remarks

1. Any $\sigma-$algebra is a monotone class.
2. An arbitrary intersection of monotone classes is a monotone class.
3. If $A \subset X$, the intersection of all monotone classes that contain A is called the monotone class generated by A and denoted by $\mathscr{M}(A)$.

Theorem

Let $\mathcal{A}$ be an algebra of X. Then $\mathscr{M}(\mathcal{A}) = \sigma(\mathcal{A})$. ($\mathscr{M}$ ($\mathcal{A}$) is the monotone class generated by $\mathcal{A}$ and by $\sigma(\mathcal{A})$ is the $\sigma-$algebra generated by $\mathcal{A}$.)

Proof

It follows from the above remark that $\sigma(\mathcal{A})$ is a monotone class, as $\sigma(\mathcal{A})$ contains $\mathcal{A}$, then $\sigma(\mathcal{A})$ contains the smallest monotone class containing $\mathcal{A}$ thus $\sigma(\mathcal{A}) \supset \mathscr{M}(\mathcal{A})$.
To prove $\sigma(\mathcal{A}) \subset \mathscr{M}(\mathcal{A})$, we define for every subset S of X the set $\tilde{S}$ by

$$\tilde{S} = \{T \in \mathscr{P}(X);\ S \cup T, S \setminus T \text{ and } T \setminus S \in \mathscr{M}(\mathcal{A})\}.$$

This definition is symmetric with respect to S and T, then $S \in \tilde{T} \iff T \in \tilde{S}$. We intend to prove that $\tilde{S}$ is a monotone class if it exists.
If $(A_n)_n$ is an increasing sequence of $\tilde{S}$; $(S \cup A_n)_n$ is a increasing sequence of $\mathscr{M}(\mathcal{A})$, the same for the sequence $(A_n \setminus S)_n$, the sequence $(S \setminus A_n)_n$ is a decreasing sequence of $\mathscr{M}(\mathcal{A})$. Then the limits of the sequences are in $\mathscr{M}(\mathcal{A})$. Hence $\tilde{S}$ is a monotone class.
Since for all $A, B \in \mathcal{A}$, $B \in \tilde{A}$, then $\tilde{A}$ is a monotone class containing $\mathcal{A}$ and $\tilde{A} \supset \mathscr{M}(\mathcal{A})$. So $\forall$ $S \in \mathscr{M}(\mathcal{A})$, $S \in \tilde{A}$ for any $A \in \mathcal{A}$, and so $A \in \tilde{S}$, then $\mathcal{A} \subset \tilde{S}$; $\forall S \in \mathscr{M}(\mathcal{A})$. As $\tilde{S}$ is a monotone class then $\mathscr{M}(\mathcal{A}) \subset \tilde{S}$.

We prove then
$\forall\, S, S' \in \mathscr{M}(\mathcal{A})$, $S \setminus S'$, $S' \setminus S$, $S \cup S' \in \mathscr{M}(\mathcal{A})$. If we take $S' = X$, we find that $S^c \in \mathscr{M}(\mathcal{A})$, so $\mathscr{M}(\mathcal{A})$ is an algebra.
Let now $(A_n)_n$ a sequence of $\mathscr{M}$. Consider $B_n = \bigcup\limits_{1\leq j\leq n} A_j$, the sequence $(B_n)_n$ is increasing in $\mathscr{M}$ and $\cup_{n=1}^{\infty} A_n = \cup_{n=1}^{\infty} B_n \in \mathscr{M}$.
$\square$