2. The method requires the user to provide the derivatives of each function with respect to each variable. Therefore one must evaluate the n functions and the n^2 derivatives at each iteration. So solving systems of nonlinear equations is a difficult task. For systems of nonlinear equations that have analytical partial derivatives, Newton's method can be used; otherwise, multi-dimensional minimization techniques should be used.

Procedure 2.5 (Newton's Method for Two Nonlinear Equations)

1. *Choose the initial guess for the roots of the system, so that the determinant of the Jacobian matrix is not zero.*

2. *Establish Tolerance* $\epsilon (> 0)$.

3. *Evaluate the Jacobian at initial approximations and then find inverse of Jacobian.*

4. *Compute new approximation to the roots by using iterative formula (2.51).*

5. *Check tolerance limit. If* $\|(x_n, y_n) - (x_{n-1}, y_{n-1})\| \leq \epsilon$, *for* $n \geq 0$, *then end; otherwise, go back to step 3, and repeat the process.*

2.9 Exercises

1. Find the root of $f(x) = e^x - 2 - x$ in the interval $[-2.4, -1.6]$ accurate to 10^{-4} using bisection method.

2. Use bisection method to find solutions accurate to within 10^{-4} on the interval $[-5, 5]$ of the following functions:
(a) $f(x) = x^5 - 10x^3 - 4$, *(b)* $f(x) = 2x^2 + \ln(x) - 3$, *(c)* $f(x) = \ln(x) + 30e^{-x} - 3$.

3. The following equations have a root in the interval $[0, 1.6]$. Determine these with an error less than 10^{-4} using bisection method.
(a) $2x - e^{-x} = 0$; (b) $e^{-3x} + 2x - 2 = 0$.

4. Estimate the number of iterations needed to achieve an approximation with accuracy 10^{-4} to the solution of $f(x) = x^3 + 4x^2 + 4x - 4$ lying in the interval $[0, 1]$ using bisection method.

5. Use the bisection method for $f(x) = x^3 - 3x + 1$ in $[1, 3]$ to find:
(a) The first eight approximation to the root of the given equation.
(b) Find an error estimate $|\alpha - x_8|$.

6. The cubic equation $x^3 - 3x - 20 = 0$ can be written as

$$(a)\ x = \frac{(x^3 - 20)}{3}, \quad (b)\ x = \frac{3}{(x^3 - 3)}, \quad (c)\quad x = (3x + 20)^{1/3}.$$

Choose the form which satisfies the condition $|g'(x)| < 1$ on $[3, 4]$ and then find third approximation x_3 when $x_0 = 3.5$.

7. Consider the nonlinear equation $g(x) = \frac{1}{2}e^{0.5x}$ defined on the interval $[0, 1]$. Then
   (a) Show that there exists a unique fixed-point for g in $[0, 1]$.
   (b) Use fixed-point iterative method to compute x_3, set $x_0 = 0$.
   (c) Compute an error bound for your approximation in part *(b)*.

8. An equation $x^3 - 2 = 0$ can be written in form $x = g(x)$ in two ways:
   (a) $x = g_1(x) = x^3 + x - 2$, *(b)* $x = g_2(x) = \frac{(2 + 5x - x^3)}{5}$
   Generate first four approximations from $x_{n+1} = g_i(x_n)$, $i = 1, 2$ by using $x_0 = 1.2$. Show which sequence converge to $2^{1/3}$ and why ?

9. Find value of k such that the iterative scheme $x_{n+1} = \frac{x_n^2 - 4kx_n + 7}{4}$, $n \geq 0$ converges to 1. Also, find the rate of convergence of the iterative scheme.

10. Write the equation $x^2 - 6x + 5 = 0$ in the form $x = g(x)$, where $x \in [0, 2]$, so that the iteration $x_{n+1} = g(x_n)$ will converge to the root of the given equation for any initial approximation $x_0 \in [0, 2]$.

11. Which of the following iterations
    (a) $x_{n+1} = \frac{1}{4}\left(x_n^2 + \frac{6}{x_n}\right)$, *(b)* $x_{n+1} = \left(4 - \frac{6}{x_n^2}\right)$

    is suitable to find a root of the equation $x^3 = 4x^2 - 6$ in the interval $[3, 4]$? Estimate the number of iterations required to achieve 10^{-3} accuracy, starting from $x_0 = 3$.

12. An equation $e^x = 4x^2$ has a root in $[4, 5]$. Show that we cannot find that root using $x = g(x) = \frac{1}{2}e^{x/2}$ for the fixed-point iteration method. Can you find another iterative formula which will locate that root ? If yes, then find third iterations with $x_0 = 4.5$. Also find the error bound.

13. Let $f(x) = e^x + 3x^2$. Find Newton's formula $g(x_k)$. Start with $x_0 = 4$ and $x_0 = -0.5$, compute x_4.

14. Use Newton's formula for the reciprocal of square root of a number 15 and then find the 3rd approximation of number, with $x_0 = 0.05$.

15. Use Newton's method to find solution accurate to within 10^{-4} of the equation $\tan(x) - 7x = 0$, with initial approximation $x_0 = 4$.

16. Find Newton's formula for $f(x) = x^3 - 3x + 1$ in $[1, 3]$ to calculate x_3, if $x_0 = 1.5$. Also, find the rate of convergence of the method.

17. Rewrite the nonlinear equation $g(x) = \frac{1}{2}e^{0.5x}$ which defined in the interval $[0, 1]$ in the equivalent form $f(x) = 0$ and then use the Newton's method with $x_0 = 0.5$ to find third approximation x_3.

18. Given the iterative scheme $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, $n \geq 0$ with $f(\alpha) = f'(\alpha) = 0$ and $f''(\alpha) \neq 0$. Find the rate of convergence for this scheme.

19. Find x_4 for $x^3 - 2x - 5 = 0$ by secant method using $x_0 = 2$ and $x_1 = 3$.

20. Solve the equation $e^{-x} - x = 0$ by secant method,using $x_0 = 0$ and $x_1 = 1$, accurate to 10^{-4}.

21. Use secant method to find a solution accurate to within 10^{-4} for $\ln(x) + x - 5 = 0$ on $[3, 4]$.

22. Find the root of multiplicity of the function $f(x) = (x-1)^2 \ln(x)$ at $\alpha = 1$.

23. Show that if $f(x)$ has a root of multiplicity m at $x = \alpha$, then

$$f^{(n)}(x) = 0, \qquad n = 1, 2, \ldots, m-1.$$

24. Show that the root of multiplicity of the function $f(x) = x^4 - x^3 - 3x^2 + 5x - 2$ is 3 at $\alpha = 1$. Estimate the number of iterations required to solve the problem with accuracy 10^{-4}, start with the starting value $x_0 = 0.5$ by using:
(a) Newton's method; *(b)* First modified Newton's method; *(c)* Second modified Newton's method

25. If $f(x), f'(x)$ and $f''(x)$ are continuous and bounded on a certain interval containing $x = \alpha$ and if both $f(\alpha) = 0$ and $f'(\alpha) = 0$ but $f''(\alpha) \neq 0$, show that

$$x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n)}$$

will converge quadratically if x_n is in the interval.

26. Show that iterative scheme $x_{n+1} = 1 + x_n - \dfrac{x_n^2}{2}$, $n \geq 0$ converges to $\sqrt{2}$. Find the rate of convergence of the sequence.

27. Let α be the exact solution of the function $f(x) = 0$ such that $f'(\alpha) \neq 0$, $f''(\alpha) \neq 0$, then find the conditions on the constant K under which the rate of convergence of the sequence $x_{n+1} = x_n^2 - Kf(x_n)$, $n = 0, 1, 2, \ldots$ is quadratic.

28. Solve the following system using the Newton's method:

$$\begin{array}{rcl} 4x^3 + y & = & 6 \\ x^2 y & = & 1 \end{array}$$

Start with initial approximation $x_0 = y_0 = 1$. Stop when successive iterates differ by less than 10^{-7}.

29. Solve the following system using the Newton's method:

$$\begin{array}{rcl} x + e^y & = & 68.1 \\ \sin x - y & = & -3.6 \end{array}$$

Start with initial approximation $x_0 = 2.5$, $y_0 = 4$, compute the first three approximations.