Chapter 6: Residue Theory

Li, Yongzhao

State Key Laboratory of Integrated Services Networks, Xidian University

June 7, 2009

Introduction

- In the previous chapters, we have seen how the theory of contour integration lends great insight into the properties of analytic functions
- The goal this chapter is to explore another dividend of this theory, namely, its usefulness in evaluating certain real integrals
- We shall begin by presenting a technique for evaluating contour integrals that is known as **residue theory**
- Then we will introduce some application of the theory to the evaluating the real integrals

The Residue Theorem

- If $f(z)$ is analytic on and inside a simple closed positively oriented contour Γ except a single isolated singularity, z_0, lying interior to Γ, $f(z)$ has a Laurent series expansion
$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j$$
converging to some punctured neighborhood of z_0
- In particular, the above equation is valid for all z on the small positively oriented circle C continuously deformed from Γ (as shown in Fig. 6.1)

The Residue Theorem (Cont'd)

- According to the Continuous Deformation Invariance Theorem (page 231), we have

$$\int_\Gamma f(z)dz = \int_C f(z)dz$$

- The last integral can be computed by termwise integration of the series along C. For all $j \neq -1$ the integral is zero, and for $j = -1$ we obtain the value $2\pi i a_{-1}$
- Consequently we have

$$\int_\Gamma f(z)dz = 2\pi i a_{-1}$$

The Residue Theorem (Cont'd)

- Thus the constant a_{-1} plays an important role in contour integration. Accordingly, we adopt the following terminology

Definition

If f has an isolated singularity at the point z_0, then the coefficient a_{-1} of $(z - z_0)^{-1}$ in the Laurent expansion for f around z_0 is called the **residue** of f at z_0 and is denoted by

$$\mathrm{Res}(f;\ z_0) \text{ or } \mathrm{Res}(z_0)$$

How to Compute the Residue

- If f has a **removable singularity** at z_0, all the coefficients of the negative powers of $(z - z_0)$ in its Laurent expansion are zero, and so, in particular, the residue at z_0 is zero
- If f has an **essential singularity** at z_0, we have to use its Laurent expansion to find the residue at z_0 (See Example 1 on page 308)
- If f has **a pole of order** m at z_0, we have the following theorem to find the residue

Theorem

If f has a pole of order m at z_0, then

$$Res(f;\ z_0) = \lim_{z \to 0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z)\right]$$

How to Compute the Residue (Cont'd)

- Example 2 gives us another way to compute the residue when f is a rational polynomial
- Let $f(z) = P(z)/Q(z)$, where the functions $P(z)$ and $Q(z)$ are both analytic at z_0 and Q has a simple zero at z_0, while $P(z_0) \neq 0$. Then we have

$$\mathrm{Res}(f;\ z_0) = \frac{P(z_0)}{Q'(z_0)}$$

How to Compute the Residue (Cont'd)

- When there are a finite number of isolated singularities inside the simple closed positively oriented contour Γ, we have the following theorem

Theorem

If Γ is a simple closed positively oriented contour and f is analytic inside and on Γ except at the points $z_1, z_2, \cdots, z_n$ inside Γ, then

$$\int_\Gamma f(z)dz = 2\pi i \sum_{j=1}^{n} Res(z_j)$$

Trigonometric Integrals Over $[0,\ 2\pi]$

- Our goal of this section is to apply the residue theory to evaluate real integrals of the form

$$\int_0^{2\pi} U(\cos\theta, \sin\theta)d\theta \tag{1}$$

- We use $z = e^{i\theta}$ $(0 \le \theta \le 2\pi)$ to parameterize the closed positively oriented contour $|z| = 1$. Then a contour integral can be transformed into a real integral
- According to Euler's equation, we have

$$\cos\theta = \left(e^{i\theta} + e^{i\theta}\right)/2 = (z + z^{-1})/2$$
$$\sin\theta = \left(e^{i\theta} - e^{i\theta}\right)/2i = (z - z^{-1})/2i$$

Trigonometric Integrals Over $[0,\ 2\pi]$ (Cont'd)

- Also taking $dz = ie^{i\theta}d\theta = izd\theta$ into account, Eq. (1) can be transformed into a complex contour integration as

$$\int_0^{2\pi} U(\cos\theta, \sin\theta)d\theta = \oint_{|z|=1} F(z)dz$$

where the new integrand F is

$$F(z) := U\left[\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right] \cdot \frac{1}{iz}$$

Trigonometric Integrals Over $[0,\ 2\pi]$ (Cont'd)

- Of course, the function F must be a rational function of z
- Hence, it has only removable singularities (which can be ignored in evaluation integrals) or poles
- Consequently, by the residue theorem, our trigonometric integral equals $2\pi i$ time the sum of the residues at those poles of F that lie inside the unite circle

Improper Integrals of Certain Functions Over $(-\infty,\ \infty)$

- Given any function f continuous on $(-\infty, \infty)$, the limit
$$\lim_{\rho\to\infty}\int_{-\rho}^{\rho} f(x)dx$$
is called the Cauchy principal value of the integral of f over $(-\infty, \infty)$, and we write
$$\text{p.v.}\int_{-\infty}^{\infty} f(x)dx := \lim_{\rho\to\infty}\int_{-\rho}^{\rho} f(x)dx$$
- We shall now show how the theory of residue can be used to compute p.v. integrals for certain functions of f
- See Example 1 on page 319 to learn the basic idea of the algorithm

Improper Integrals of Certain Functions Over $(-\infty,\ \infty)$ (Cont'd)

Lemma

If $f(z) = P(z)/Q(z)$ is the quotient of two polynomials such that
$$\text{degree } Q \geq 2 + \text{degree } P$$
then
$$\lim_{\rho\to\infty}\int_{C_\rho^+} f(z)dz = 0$$
where C_ρ^+ is the upper half-circle of radius ρ defined in Eq. (4) on page 320 as shown in Figure 6.4

Improper Integrals of Certain Functions Over $(-\infty,\ \infty)$ (Cont'd)

- Then the improper integral $\int_{-\infty}^{\infty} f(x)dx$ can be computed as follows
$$\text{p.v.}\int_{-\infty}^{\infty} f(x)dx = \lim_{\rho\to\infty} 2\pi i \sum(\text{residues inside } \Gamma_\rho)$$

Improper Integrals Involving Trigonometric Functions

- The purpose of this section is to use residue theory to evaluate integrals of the general forms:
$$\text{p.v.}\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\cos mx\ dx, \quad \text{p.v.}\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\sin mx\ dx$$
- If we obtain the value of the integral
$$\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}e^{imx}\ dx$$
the above two integrals can be obtained by computing the real and imaginary parts

Improper Integrals Involving Trigonometric Functions (Cont'd)

Lemma

If $m > 0$ and P/Q is the quotient of two polynomials such that

$$degree\ Q \geq 1 + degree\ P$$

then

$$\lim_{\rho \to \infty} \int_{C_\rho^+} e^{imx} \frac{P(x)}{Q(x)} dz = 0$$

where C_ρ^+ is the upper half-circle of radius ρ

Improper Integrals Involving Trigonometric Functions (Cont'd)

- Then the improper integral $\int_{-\infty}^{\infty} f(x)dx$ can be computed as follows

$$\text{p.v.} \int_{-\infty}^{\infty} e^{imx} \frac{P(x)}{Q(x)} dx = \lim_{\rho \to \infty} 2\pi i \sum (\text{residues inside } \Gamma_\rho)$$

- Thus

$$\text{p.v.} \int_{-\infty}^{\infty} \cos mx \frac{P(x)}{Q(x)} dx = \Re \left\{ \text{p.v.} \int_{-\infty}^{\infty} e^{imx} \frac{P(x)}{Q(x)} dx \right\}$$

$$\text{p.v.} \int_{-\infty}^{\infty} \sin mx \frac{P(x)}{Q(x)} dx = \Im \left\{ \text{p.v.} \int_{-\infty}^{\infty} e^{imx} \frac{P(x)}{Q(x)} dx \right\}$$