

King Saud University  
Department of Mathematics  
M-203[Final Examination]  
(Differential and Integral Calculus)  
(Second-Semester 1434/35)

Max.Marks:40

Time:3hrs

**Marking Scheme:** Q.No:1[2+3+4+3]; Q.No:2[3+3+3+3]; Q.No:3[4+4+4+4]

- Q.No: 1 (a) Test the convergence or divergence of the series  $\sum_{n=1}^{\infty} \frac{3^n}{10^2 + 2^n}$ .
- (b) Determine whether the series  $\sum_{n=1}^{\infty} ne^{-n^3}$  converges or diverges.
- (c) Find the interval of convergence and radius of convergence of the power Series  $\sum_{n=0}^{\infty} \frac{e^n}{n+1} (x-1)^n$ .
- (d) Use the Maclaurin series for the function  $f(x) = \sin x$  and use its first three non-zero terms to approximate the integral  $\int_0^{0.1} \frac{\sin x^2}{x^2} dx$ .
- Q.No: 2 (a) Evaluate the integral  $\int_0^{\pi/4} \int_y^{\pi/4} \frac{x \sin x}{x^2 + y^2} dx dy$ .
- (b) Find the area of the surface  $x^2 + y^2 + z^2 = 25$  that lies above the plane  $z = 4$ .
- (c) Set up an iterated triple integral that can be used to find the mass of the solid region bounded by the paraboloid  $z = x^2 + y^2$  and the plane  $z = 4$  with density  $\delta(x, y, z) = x^2 + y^2$ .
- (d) Let Q be the solid region that lies inside the cone  $z = \sqrt{3x^2 + 3y^2}$  and hemi-sphere  $z = \sqrt{4 - x^2 - y^2}$ . Find the volume of Q.
- Q.No: 3 (a) Show that the line integral  $\int_{(-1,2)}^{(2,3)} (2xy - 3)dx + (x^2 + 4y^3 + 5)dy$  is independent of path, and find its Value.
- (b) (a) Use Green's theorem to evaluate the line integral  $\oint_C (e^x + 6xy)dx + (8x^2 + \sin y^2)dy$  where C is positively oriented boundary of the region bounded by the circle of radii 1 and 3 centered at the origin and lying in the first quadrant.
- (c) If S is the surface of the region bounded by the cone  $z = \sqrt{x^2 + y^2}$  and the planes  $z = 0$  and  $z = 2$   $\vec{F}(x, y, z) = (xy)\vec{i} + (yz)\vec{j} + (zx)\vec{k}$ . Use the Divergence theorem to find  $\iiint_S \vec{F} \cdot \vec{n} dS$ .
- (d) Use the Stoke's theorem to evaluate  $\iint_S (\text{Curl } \vec{F}) \cdot \vec{n} dS$  where S is the hemisphere  $z = \sqrt{1 - x^2 - y^2}$  and the vector field is given by  $\vec{F}(x, y, z) = (y)\vec{i} - (x)\vec{j} + (z)\vec{k}$

①

Q: 1 (a)  $\sum_{n=1}^{\infty} \frac{3^n}{10^2 + 2^n}$  [Mark: 2]

we can use limit comp. test with  $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$

Applying nth term test  $\lim_{n \rightarrow \infty} \frac{3^n}{10^2 + 2^n} = \frac{\infty}{\infty}$

$= \lim_{x \rightarrow \infty} \frac{3^x \ln 3}{2^x \ln 2} = \lim_{x \rightarrow \infty} \left(\frac{3}{2}\right)^x \frac{\ln 3}{\ln 2} = \infty$ , d'gt

(b)  $\sum_{n=1}^{\infty} n e^{-n^2}$  applying integral test

① [Mark: 3]

$f(x) = \frac{x}{e^{x^2}}$

(i)  $f'(x) = e^{-x^2} + x e^{-x^2} (-2x)$

$= \frac{1 - 2x^2}{e^{x^2}} < 0, x > 1 \Rightarrow$  DEC

①

(ii)  $\int_1^{\infty} x e^{-x^2} dx = \lim_{t \rightarrow \infty} \int_1^t x e^{-x^2} dx = \lim_{t \rightarrow \infty} \left[ -\frac{1}{2} e^{-x^2} \right]_1^t$   
 $= \lim_{t \rightarrow \infty} \left[ -\frac{1}{2} e^{-t^2} + \frac{1}{2} e^{-1} \right] = \frac{1}{2} e^{-1}$  c'gt

①

①

②

$\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{e^{n+1} (x-1)^{n+1}}{n+2} \times \frac{n+1}{e^n (x-1)^n} \right| = \frac{e(n+1)}{(n+2)} |x-1|$  [Mark: 4]

$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = e|x-1| \Rightarrow$  c'gt if  $e|x-1| < 1$

$\Rightarrow |x-1| < \frac{1}{e} \Rightarrow -\frac{1}{e} < x-1 < \frac{1}{e} \Rightarrow \boxed{1 - \frac{1}{e} < x < 1 + \frac{1}{e}}$

Convergence at  $x = 1 - \frac{1}{e}$

$\sum_{n=1}^{\infty} \frac{e^n (1 - \frac{1}{e} - 1)^n}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$

It is c'gt ①

Convergence at  $x = 1 + \frac{1}{e}$

$\sum_{n=1}^{\infty} \frac{e^n (1 + \frac{1}{e} - 1)^n}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n+1}$

It is d'gt ①

$\Rightarrow$  Interval of c'gt:  $\boxed{1 - \frac{1}{e} < x < 1 + \frac{1}{e}}$  Radius =  $\rho = \frac{1}{e}$  ①

(d)  $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$  (1) [Mark: 3]

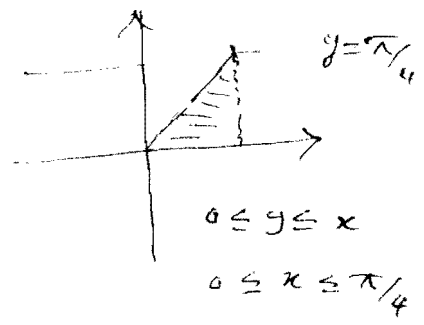
$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \dots$

$\frac{\sin(x^2)}{x^2} = 1 - \frac{x^4}{3!} + \frac{x^8}{5!} - \dots$  (1)

$\int_0^{0.1} \frac{\sin(x^2)}{x^2} dx = \int_0^{0.1} [1 - \frac{x^4}{6} + \frac{x^8}{(120)8} - \dots] dx$   
 $= [x - \frac{x^5}{30} + \frac{x^8}{(120)(64)} - \dots]_0^{0.1}$  (1)

Q. 2 (a)  $\int_0^{\pi/4} \int_y^{\pi/4} \frac{x \sin x}{x^2 + y^2} dx dy$  (1)  $0 \leq y \leq \pi/4$  [Mark: 3]  
 $y \leq x \leq \pi/4$

$= \int_0^{\pi/4} \int_0^x x \sin x \left[ \frac{1}{x^2 + y^2} \right] dy dx$



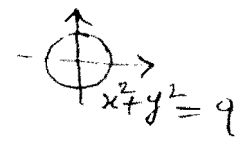
$= \int_0^{\pi/4} x \sin x \left[ \frac{1}{x} \tan^{-1}(y/x) \right]_0^x dx$

$= \int_0^{\pi/4} x \sin x \left[ \frac{1}{x} \tan^{-1}(1) - \frac{1}{x} \tan^{-1}(0) \right] dx = \frac{\pi}{4} \int_0^{\pi/4} \sin x dx$

$= -\cos x \Big|_0^{\pi/4} = \frac{\pi}{4} \left( \frac{1}{\sqrt{2}} + 1 \right)$  (1)

(b)  $z = \sqrt{25 - x^2 - y^2} = f(x, y) \Rightarrow f_x = \frac{-x}{\sqrt{25 - x^2 - y^2}}, f_y = \frac{-y}{\sqrt{25 - x^2 - y^2}}$  [Mark: 3]

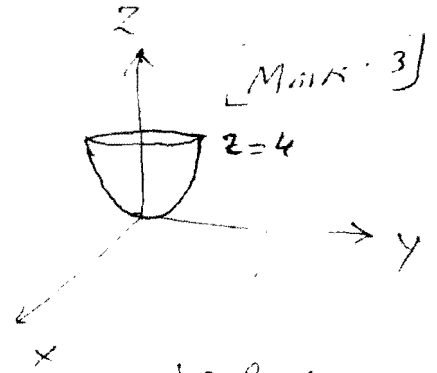
S.A. =  $\iint_{R_{xy}} \sqrt{1 + (f_x)^2 + (f_y)^2} dA = \iint_{R_{xy}} \frac{5}{\sqrt{25 - x^2 - y^2}} dA$  (1)



$= \int_0^{2\pi} \int_0^3 \frac{5}{\sqrt{25 - r^2}} r dr d\theta = -\frac{5}{2} \int_0^{2\pi} \int_0^3 (25 - r^2)^{-1/2} (-2r) dr d\theta$   
 $= -\frac{5}{2} \int_0^{2\pi} \left[ \frac{(25 - r^2)^{1/2}}{1/2} \right]_0^3 d\theta = -5 \int_0^{2\pi} (-1) d\theta = 10\pi$

(3)

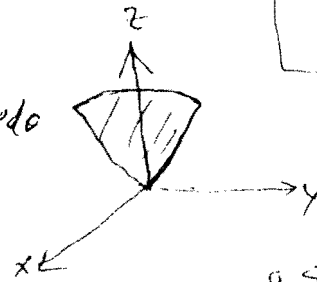
$$\begin{aligned}
 \textcircled{c} \quad m &= \iiint_Q \delta \, dV = \iiint_Q (x^2 + y^2) \, dV \\
 &= \int_0^{2\pi} \int_0^2 \int_0^4 (x^2 + y^2) r \, dy \, dr \, d\theta
 \end{aligned}$$



cylindrical shell

$$\begin{aligned}
 r^2 &\leq \leq 4 \\
 0 &\leq r \leq 2 \\
 0 &\leq \theta \leq 2\pi
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{d} \quad V &= \iiint_Q dV = \int_0^{2\pi} \int_0^{\pi/6} \int_0^2 \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta \\
 &= \int_0^{2\pi} \int_0^{\pi/6} \left[ \frac{\rho^3}{3} \right]_0^2 \sin \varphi \, d\varphi \, d\theta
 \end{aligned}$$



[Mark: 3]

$$\begin{aligned}
 0 &\leq \rho \leq 2 \\
 0 &\leq \varphi \leq \pi/6 \\
 0 &\leq \theta \leq 2\pi
 \end{aligned}$$

$$= \frac{8}{3} \int_0^{2\pi} \left[ -\cos \varphi \right]_0^{\pi/6} d\theta = \frac{8}{3} \left[ -\frac{\sqrt{3}}{2} + 1 \right] 2\pi = \frac{16\pi}{3} \left( 1 - \frac{\sqrt{3}}{2} \right)$$

Cylindrical shell

(1)

$$\sqrt{3}r \leq z \leq \sqrt{4-r^2}$$

$$V = \int_0^{2\pi} \int_0^1 \int_{\sqrt{3}r}^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^1 r \left[ \sqrt{4-r^2} - \sqrt{3}r \right] dr \, d\theta$$

$0 \leq r \leq 1$   
 $0 \leq \theta \leq 2\pi$

$$= \int_0^{2\pi} \int_0^1 \left[ \sqrt{4-r^2} (r) - \sqrt{3} r^2 \right] dr \, d\theta$$

$$= \int_0^{2\pi} \left[ -\frac{1}{2} \frac{(4-r^2)^{3/2}}{3/2} - \sqrt{3} \frac{r^3}{3} \right]_0^1 d\theta = \int_0^{2\pi} \left[ -\frac{(4-r^2)^{3/2}}{3} - \frac{r^3}{\sqrt{3}} \right]_0^1 d\theta$$

$$= \left( \frac{8}{3} - \frac{4}{\sqrt{3}} \right) 2\pi = \frac{16}{3}\pi - \frac{8\pi}{\sqrt{3}} \quad \#$$

(1)

Q. 3 (a)  $\int_{(-1,2)}^{(2,3)} (2xy-3) dx + (x^2+4y^3+5) dy$  [Mark: 4]

$$\left. \begin{aligned} f_x = M = 2xy - 3 &\Rightarrow \frac{\partial M}{\partial y} = 2x \\ f_y = N = x^2 + 4y^3 + 5 &\Rightarrow \frac{\partial N}{\partial x} = 2x \end{aligned} \right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$f_x = 2xy - 3 \Rightarrow f(x,y) = x^2y - 3x + C_1 \quad (2)$$

$$f_y = x^2 + 4y^3 + 5 \Rightarrow f(x,y) = x^2y + y^4 + 5y + C_2$$

$$\Rightarrow \left[ f(x,y) \right]_{(-1,2)}^{(2,3)} = \left[ x^2y + y^4 - 3x + 5y + C_2 \right]_{(-1,2)}^{(2,3)}$$

$$= \left[ (4)(3) + (3)^4 - 6 + 15 \right] - \left[ 2 + 16 + 3 + 10 \right] = 61$$

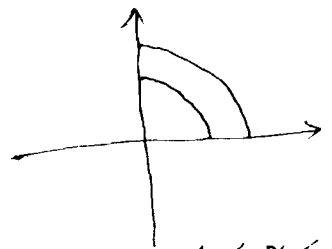
(2)

(b)  $\oint_C (e^x + 6xy) dx + (8x^2 + \sin y^2) dy$  [Mark: 4]



$$= \iint_R (16x - 6x) dA$$

(1)  $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$



$1 \leq r \leq 3$   
 $0 \leq \theta \leq \pi/2$

$$= \int_0^{\pi/2} \int_1^3 (10r \cos \theta) r dr d\theta$$

$$= 10 \int_0^{\pi/2} \cos \theta \left[ \frac{r^3}{3} \right]_1^3 d\theta =$$

$$= 10 \left( \frac{26}{3} \right) [\sin \theta]_0^{\pi/2} = \frac{260}{3} \quad \#$$

(1)

(5)

$$\textcircled{c} \quad M = xy, \quad N = yz, \quad P = zx$$

[Mark: 4]

$$\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = y + z + x$$

$$= \iiint (y + z + x) \, dV$$

$$= \int_0^{2\pi} \int_0^2 \int_0^2 (y \sin \theta + z + r \cos \theta) r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 \int_0^2 (r^2 \sin \theta + zr + r^2 \cos \theta) \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 \left[ r^2 z \sin \theta + \frac{z^2}{2} r + r^2 z \cos \theta \right]_0^2 \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 [2r^2 \sin \theta + 2r + 2r^2 \cos \theta] \, dr \, d\theta$$

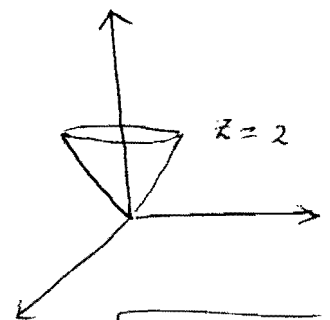
$$= 2 \int_0^{2\pi} \left[ \frac{r^3}{3} \sin \theta + \frac{r^2}{2} + \frac{r^3}{3} \cos \theta \right]_0^2 \, d\theta$$

$$= 2 \int_0^{2\pi} \left[ \frac{8}{3} \sin \theta + 2 + \frac{8}{3} \cos \theta \right] \, d\theta$$

$$= 2 \left[ -\frac{8}{3} \cos \theta \right]_0^{2\pi} + 2(2\pi) + \left[ \frac{8}{3} \sin \theta \right]_0^{2\pi}$$

$$= -\frac{16}{3} [1-1] + 4\pi + 0 = 4\pi$$

(1)



$$\begin{cases} 0 \leq z \leq 2 \\ 0 \leq r \leq 2 \\ 0 \leq \theta \leq 2\pi \end{cases}$$

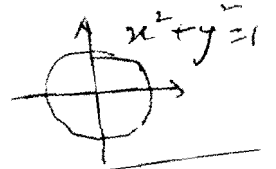
(6)

$$\textcircled{1} \oint_C \vec{F} \cdot d\vec{v}$$

$$\vec{F} = \langle y, -x, z \rangle$$
$$d\vec{v} = \langle dx, dy, dz \rangle$$

[Mark: 4]

$$= \int_C y dx - x dy + z dz$$



$$= \int_0^{2\pi} (\sin t)(-\sin t dt) - (\cos t)(\cos t dt) + 0$$

$$= - \int_0^{2\pi} (\sin^2 t + \cos^2 t) dt$$

$$= - \int_0^{2\pi} dt = -2\pi$$

$$C: x = \cos t, y = \sin t$$
$$z = 0, 0 \leq t \leq 2\pi$$

$$dx = -\sin t dt \quad \textcircled{2}$$

$$dy = \cos t dt$$

$$dz = 0$$

Q1

(a) Test the convergence or divergence of the series  $\sum_{n=1}^{\infty} \frac{3^n}{10^2 + 2^n}$

**solution:**

Using  $n^{\text{th}}$  term test

$$\lim_{n \rightarrow \infty} \frac{3^n}{10^2 + 2^n} \text{ is } \frac{\infty}{\infty}$$

By L'Hopital rule  $= \lim_{n \rightarrow \infty} \frac{3^n \cdot \ln 3}{2^n \cdot \ln 2}$

$$= \lim_{n \rightarrow \infty} \left(\frac{3}{2}\right)^n \cdot \frac{\ln 3}{\ln 2} = \infty$$

The series is divergent

(b) Determine whether the series  $\sum_{n=1}^{\infty} ne^{-n^2}$  converges or diverges

**solution:**

\* Applying integral test  $f(x) = xe^{-x^2}$

1.  $f(n) = a_n$  for  $n = 1, 2, 3, \dots$

2.  $f(x) = xe^{-x^2}$  is continuous on  $[1, \infty]$

3.  $f'(x) = (1)e^{-x^2} + x \cdot -2xe^{-x^2} = e^{-x^2}(1 - 2x^2) < 0$  for  $x \geq 1$

( $f(x)$  is decreasing)

\* integral test

$$\int_1^{\infty} xe^{-x^2} dx = \lim_{t \rightarrow \infty} \frac{1}{-2} \int_1^t (-2)x e^{-x^2} dx$$

$$= \lim_{t \rightarrow \infty} \frac{-1}{2} \left[ e^{-x^2} \right]_1^t$$

$$= \lim_{t \rightarrow \infty} \frac{-1}{2} \left[ e^{-t^2} - e^{-1} \right] = \frac{-1}{2} \left[ 0 - \frac{1}{e} \right] = \frac{1}{2e} \text{ convergent}$$

then the series  $\sum_{n=1}^{\infty} ne^{-n^2}$  is convergent



(d) Use the Maclaurin series for the function  $f(x) = \sin x$  and use its first three non-zero terms to approximate the integral  $\int_0^{0.1} \frac{\sin x^2}{x^2} dx$

**solution:**

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \end{aligned}$$

Maclaurin series  $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots + \frac{x^n}{n!}f^{(n)}(0) + \dots$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{x^{2n+1}}{(2n+1)!} + \dots$$

Replace  $x$  by  $x^2$   $\sin x^2 \approx x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!}$

Divide by  $x^2$   $\frac{\sin x^2}{x^2} \approx 1 - \frac{x^4}{3!} + \frac{x^8}{5!}$

$$\begin{aligned} \int_0^{0.1} \frac{\sin x^2}{x^2} dx &\simeq \int_0^{0.1} \left( 1 - \frac{x^4}{3!} + \frac{x^8}{5!} \right) dx \\ &\approx \left[ x - \frac{x^5}{(3!)(5)} + \frac{x^9}{(5!)(9)} \right]_0^{0.1} = \end{aligned}$$

(c) Find the interval of convergence and radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{e^n}{n+1} (x-1)^n$$

**solution:** Using absolute ratio test

$$u_n = \frac{e^n}{n+1} (x-1)^n, \quad u_{n+1} = \frac{e^{n+1}}{n+2} (x-1)^{n+1} = \frac{e \cdot e^n}{n+2} (x-1)(x-1)^n$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{e \cdot e^n}{n+2} (x-1)(x-1)^n \cdot \frac{n+1}{e^n (x-1)^n} \right| \\ &= e |x-1| \lim_{n \rightarrow \infty} \frac{n+1}{n+2} = e |x-1| \end{aligned}$$

\* The power series is convergent if  $e |x-1| < 1$

$$|x-1| < \frac{1}{e}, \quad \frac{-1}{e} < x-1 < \frac{1}{e}, \quad 1 - \frac{1}{e} < x < 1 + \frac{1}{e}$$

\* Checking convergent at  $x = 1 - \frac{1}{e}$

$$\sum_{n=0}^{\infty} \frac{e^n}{n+1} (x-1)^n = \sum_{n=0}^{\infty} \frac{e^n}{n+1} \left(1 - \frac{1}{e} - 1\right)^n = \sum_{n=0}^{\infty} \frac{e^n}{n+1} \left(-\frac{1}{e}\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1}$$

By Alternating series test

$$1. \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0$$

$$2. f(x) = \frac{1}{x+1}, \quad f'(x) = \frac{-1}{(x+1)^2} < 0 \text{ decreasing}$$

The series is convergent at  $x = 1 - \frac{1}{e}$

\* Checking convergent at  $x = 1 + \frac{1}{e}$

$$\sum_{n=0}^{\infty} \frac{e^n}{n+1} (x-1)^n = \sum_{n=0}^{\infty} \frac{e^n}{n+1} \left(1 + \frac{1}{e} - 1\right)^n = \sum_{n=0}^{\infty} \frac{e^n}{n+1} \left(\frac{1}{e}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n+1}$$

is divergent ( p-series )

The series is divergent at  $x = 1 + \frac{1}{e}$

$\Rightarrow$  The power series is convergent on  $\left[1 - \frac{1}{e}, 1 + \frac{1}{e}\right)$ , Radius =  $\frac{\left(1 + \frac{1}{e}\right) - \left(1 - \frac{1}{e}\right)}{2} = \frac{1}{e}$

Q2:

(a) Evaluate the integral  $\int_0^{\pi/4} \int_y^{\pi/4} \frac{x \sin x}{x^2 + y^2} dx dy$

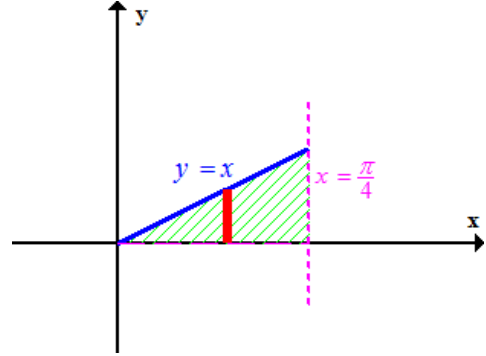
**solution:**

$$y \leq x \leq \frac{\pi}{4}, \quad 0 \leq y \leq \frac{\pi}{4}$$

changing to

$$0 \leq y \leq x, \quad 0 \leq x \leq \frac{\pi}{4}$$

$$\begin{aligned} \int_0^{\pi/4} \int_y^{\pi/4} \frac{x \sin x}{x^2 + y^2} dx dy &= \int_0^{\pi/4} \int_0^x \frac{x \sin x}{x^2 + y^2} dy dx \\ &= \int_0^{\pi/4} \int_0^x x \sin x \left[ \frac{1}{x^2 + y^2} \right] dy dx \\ &= \int_0^{\pi/4} x \sin x \left[ \frac{1}{x} \tan^{-1} \left( \frac{y}{x} \right) \right]_0^x dx \\ &= \int_0^{\pi/4} \cancel{x} \sin x \cdot \frac{1}{\cancel{x}} \left[ \tan^{-1} \left( \frac{x}{x} \right) - \tan^{-1}(0) \right] dx \\ &= \int_0^{\pi/4} \sin x \left[ \frac{\pi}{4} - 0 \right] dx \\ &= \frac{\pi}{4} [-\cos x]_0^{\pi/4} = \frac{\pi}{4} \left[ -\frac{\sqrt{2}}{2} + 1 \right] \end{aligned}$$



(b) Find the area of the surface  $x^2 + y^2 + z^2 = 25$  that lies above  $z = 4$

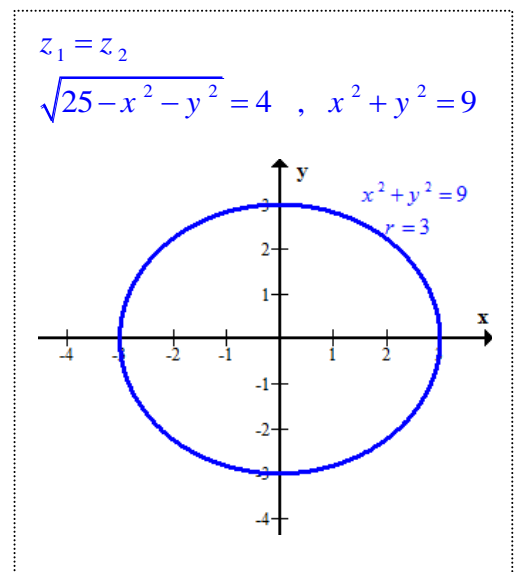
**solution:**

$$x^2 + y^2 + z^2 = 25, \quad z^2 = 25 - x^2 - y^2$$

$$z = \sqrt{25 - x^2 - y^2}, \quad f(x, y) = \sqrt{25 - x^2 - y^2}$$

$$f_x = \frac{-x}{\sqrt{25 - x^2 - y^2}}, \quad f_y = \frac{-y}{\sqrt{25 - x^2 - y^2}}$$

$$\begin{aligned} S.A &= \iint_R \sqrt{1 + [f_x]^2 + [f_y]^2} dA \\ &= \iint_R \sqrt{1 + \frac{x^2}{25 - x^2 - y^2} + \frac{y^2}{25 - x^2 - y^2}} dA \\ &= \iint_R \sqrt{\frac{25}{25 - x^2 - y^2}} dA = \iint_R \frac{5}{\sqrt{25 - x^2 - y^2}} dA \\ &= \int_0^{2\pi} \int_0^3 \frac{5}{\sqrt{25 - r^2}} r dr d\theta = 5 \cdot \frac{1}{-2} \int_0^{2\pi} \int_0^3 (25 - r^2)^{-\frac{1}{2}} (-2) r dr d\theta \\ &= \frac{-5}{2} \int_0^{2\pi} \left[ 2(25 - r^2)^{\frac{1}{2}} \right]_0^3 d\theta = -5[4 - 5][\theta]_0^{2\pi} = 10\pi \end{aligned}$$



(c) Set up the iterated triple integral that can be used to find the mass of the solid region bounded by the paraboloid  $z = x^2 + y^2$  and the plane  $z = 4$  with density  $\delta(x,y,z) = x^2 + y^2$

**solution**

$$x^2 + y^2 \leq z \leq 4, \quad r^2 \leq z \leq 4$$

Using cylindrical shell

$$r^2 \leq z \leq 4$$

$$0 \leq r \leq 2$$

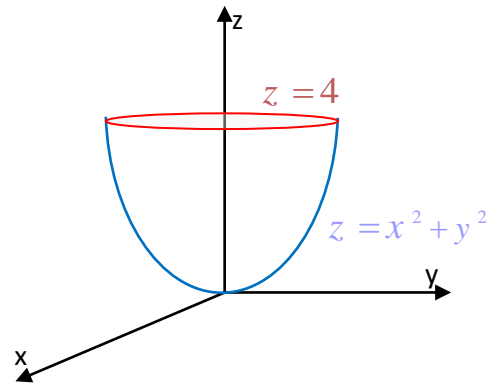
$$0 \leq \theta \leq 2\pi$$

$$\text{mass} = \iiint_Q \delta(x, y, z) dv$$

$$= \iiint_Q (x^2 + y^2) dv$$

$$= \int_0^{2\pi} \int_0^2 \int_{r^2}^4 r^2 \cdot r dz dr d\theta$$

$$= \int_0^{2\pi} \int_0^2 \int_{r^2}^4 r^3 dz dr d\theta$$



(d) Let Q be the solid region that lies inside the cone  $z = \sqrt{3x^2 + 3y^2}$  and hemisphere  $z = \sqrt{4 - x^2 - y^2}$ . Find the volume of Q

**solution:**

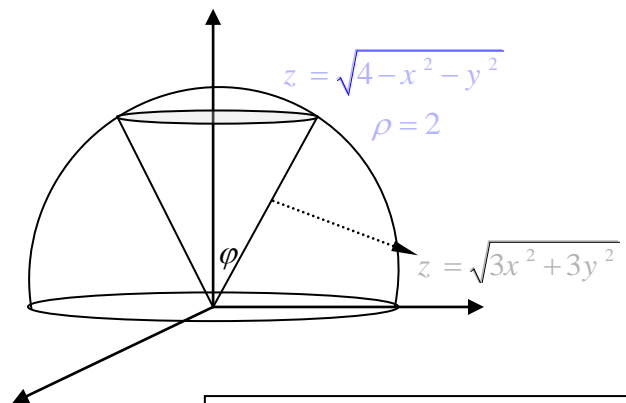
Using spherical coordinates

$$\sqrt{3x^2 + 3y^2} \leq z \leq \sqrt{4 - x^2 - y^2}$$

So  $0 \leq \rho \leq 2$

$$0 \leq \varphi \leq \frac{\pi}{6} \quad \text{?????}$$

$$0 \leq \theta \leq 2\pi$$



$$\text{Volume} = \iiint_Q dv$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{6}} \int_0^2 \rho^2 \sin \varphi d\rho d\varphi d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{6}} \left[ \frac{1}{3} \rho^3 \right]_0^2 \sin \varphi d\varphi d\theta$$

$$= \frac{8}{3} \int_0^{2\pi} [-\cos \varphi]_0^{\frac{\pi}{6}} d\theta$$

$$= \frac{8}{3} \left[ -\left(\frac{\sqrt{3}}{2} - 1\right) \right] [\theta]_0^{2\pi} = \frac{16\pi}{3} \left(1 - \frac{\sqrt{3}}{2}\right)$$

\* To find  $\varphi$

$$z = \sqrt{3x^2 + 3y^2}$$

$$\rho \cos \varphi = \sqrt{3(x^2 + y^2)}$$

$$\rho \cos \varphi = \sqrt{3\rho^2 \sin^2 \varphi}$$

$$\rho \cos \varphi = \sqrt{3} \rho \sin \varphi$$

$$\frac{\sin \varphi}{\cos \varphi} = \frac{1}{\sqrt{3}}$$

$$\tan \varphi = \frac{1}{\sqrt{3}}, \quad \varphi = \frac{\pi}{6}$$

Q3:

(a) Show that the line integral  $\int_{(-1,2)}^{(2,3)} (2xy - 3)dx + (x^2 + 4y^3 + 5)dy$  is independent of path and find its value .

solution:

$$M = 2xy - 3 \quad \longrightarrow \quad \frac{\partial M}{\partial y} = 2x$$

$$N = x^2 + 4y^3 + 5 \quad \longrightarrow \quad \frac{\partial N}{\partial x} = 2x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{the integral is independent of path}$$

$$f(x, y) = \int M dx = \int (2xy - 3) dx = x^2 y - 3x + g(y)$$

$$f(x, y) = \int N dy = \int (x^2 + 4y^3 + 5) dy = x^2 y + y^4 + 5y + h(x)$$

$$f(x, y) = x^2 y - 3x + y^4 + 5y + c$$

$$\int_{(-1,2)}^{(2,3)} (2xy - 3) dx + (x^2 + 4y^3 + 5) dy = \left[ x^2 y - 3x + y^4 + 5y \right]_{(-1,2)}^{(2,3)}$$

$$= [(4)(3) - 3(2) + 81 + 15] - [(1)(2) + 3 + 16 + 10]$$

$$= [102] - [31] = 71$$

(b) Use Green's theorem to evaluate the line integral  $\oint_C (e^x + 6xy) dx + (8x^2 + \sin y^2) dy$  where

C is positively oriented boundary of the region bounded by the circle of radii 1 and 3 centered at the origin and lying in the first quadrant.

solution:

$$\oint_C (e^x + 6xy) dx + (8x^2 + \sin y^2) dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

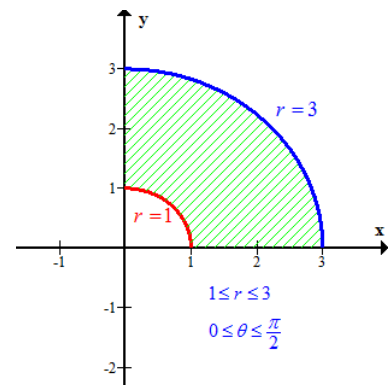
$$= \iint_R (16x - 6x) dA$$

$$= \iint_R (10x) dA$$

$$= \int_0^{\pi/2} \int_1^3 10r \cos \theta \cdot r dr d\theta$$

$$= 10 \int_0^{\pi/2} \left[ \frac{1}{3} r^3 \right]_1^3 \cos \theta d\theta$$

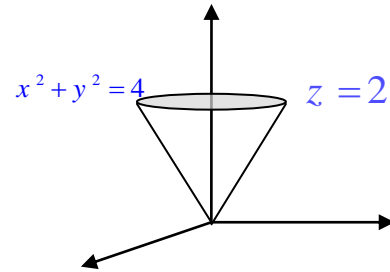
$$= 10 \cdot \frac{26}{3} [\sin \theta]_0^{\pi/2} = \frac{260}{3}$$



(c) If  $S$  is the surface of the region bounded by the cone  $z = \sqrt{x^2 + y^2}$  and the planes  $z = 0$  and  $z = 2$ .  $\vec{F}(x, y, z) = (xy)\vec{i} + ((yz))\vec{j} + (zx)\vec{k}$ . use the Divergence theorem to find  $\iint_S \vec{F} \cdot \vec{n} \, dS$

**solution:**

$$\begin{aligned}
 \iint_S \vec{F} \cdot \vec{n} \, dS &= \iiint_Q \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \right) dv \\
 &= \iiint_Q (y + z + x) \, dv \\
 &= \int_0^{2\pi} \int_0^2 \int_0^2 (r \sin \theta + z + r \cos \theta) r \, dz \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^2 \int_0^2 (r^2 \sin \theta + zr + r^2 \cos \theta) \, dz \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^2 \left( r^2 \sin \theta \cdot z + \frac{1}{2} z^2 r + r^2 \cos \theta \cdot z \right) \Big|_0^2 \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^2 (2r^2 \sin \theta + 2r + 2r^2 \cos \theta) \, dr \, d\theta \\
 &= \int_0^{2\pi} \left[ \frac{2}{3} r^3 \sin \theta + r^2 + \frac{2}{3} r^3 \cos \theta \right]_0^2 \, d\theta \\
 &= \int_0^{2\pi} \left( \frac{16}{3} \sin \theta + 4 + \frac{16}{3} \cos \theta \right) \, d\theta \\
 &= \left[ -\frac{16}{3} \cos \theta + 4\theta + \frac{16}{3} \sin \theta \right]_0^{2\pi} \\
 &= \left[ -\frac{16}{3} + 8\pi + 0 \right] - \left[ -\frac{16}{3} + 0 + 0 \right] = 8\pi
 \end{aligned}$$



$  \begin{aligned}  0 &\leq z \leq 2 \\  0 &\leq r \leq 2 \\  0 &\leq \theta \leq 2\pi  \end{aligned}  $
--

(d) Use the Stoke's theorem to evaluate  $\iint_S (\text{Curl } \vec{F}) \cdot \vec{n} dS$  where S is the hemi-sphere

$$z = \sqrt{1-x^2-y^2} \text{ and the vector field is given by } \vec{F} = (y)\vec{i} - (x)\vec{j} + (z)\vec{k}$$

**solution:**

$$C: x^2 + y^2 = 1$$

$$x = \cos t \quad , \quad y = \sin t \quad , \quad z = 0$$

$$y = -\sin t dt \quad , \quad dy = \cos t dt \quad , \quad dz = 0$$

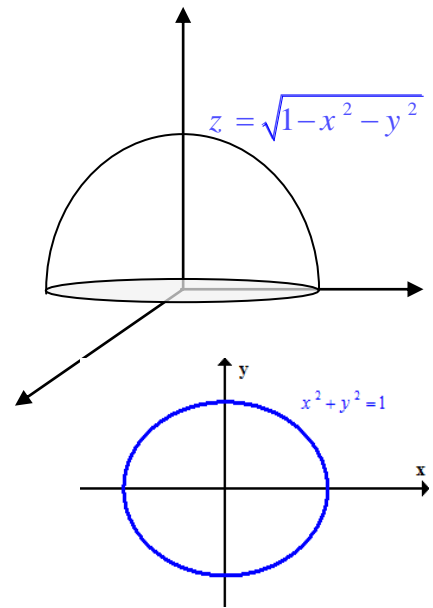
$$\iint_S (\text{Curl } \vec{F}) \cdot \vec{n} dS = \oint_C \vec{F} \cdot d\vec{r} \quad (\text{line integral})$$

$$= \oint_C y dx - x dy + z dz$$

$$= \int_0^{2\pi} (\sin t)(-\sin t dt) - (\cos t)(\cos t dt) + 0$$

$$= \int_0^{2\pi} -(\sin^2 t + \cos^2 t) dt$$

$$= \int_0^{2\pi} -dt = -[t]_0^{2\pi} = -2\pi$$



**Another solution**

(d) Use the Stoke's theorem to evaluate  $\iint_S (\text{Curl } \vec{F}) \cdot \vec{n} dS$  where S is the hemi-sphere

$$z = \sqrt{1-x^2-y^2} \text{ and the vector field is given by } \vec{F} = (y)\vec{i} - (x)\vec{j} + (z)\vec{k}$$

**solution:**

$$\vec{F} \cdot d\vec{r} = y dx - x dy + z dz$$

$$z = 0 \quad , \quad dz = 0$$

$$M = y \quad , \quad N = -x$$

$$\iint_S (\text{Curl } \vec{F}) \cdot \vec{n} dS = \oint_C \vec{F} \cdot d\vec{r} \quad (\text{Green's theorem})$$

$$= \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

$$= \iint_R (-1-1) dA$$

$$= \int_0^{2\pi} \int_0^1 -2r dr d\theta$$

$$= \int_0^{2\pi} [-r^2]_0^1 d\theta$$

$$= -[\theta]_0^{2\pi} = -2\pi$$

