King Saud University
Department of Mathematics
M-203
(Differential & Integral Calculus)
First Mid-Term Examination
(I-Semester 1437/38)

Max. Marks: 25 Time: 90 minutes

Marks: Q.1(4); Q.2(4); Q.3(6); Q.4(6); Q.5(5)

Q.No: 1 Determine whether or not the sequence $\left\{\sqrt{n^2+n}-n\right\}_{n=1}^{\infty}$ converges, and if it converges, find its limit.

Q. No: 2 Test the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{3^n+4^n}{5^n+7^n}$

Q. No: 3 Determine whether the series $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n(\ln n)}$ is absolutely convergent, conditionally convergent or divergent.

Q.No: 4 Find the interval of convergence and the radius of convergence of the power series: $\sum_{n=1}^{\infty} (-2)^n \frac{(x+3)^n}{n}$.

Q. N0: 5 Find the first four non-zero terms of the Taylor series for the function $f(x)=\ln(x)$ in powers of $(x-2)$.

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I Mid-term (I semester 1437/1438)

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Q#1) Determine whether or not the sequence $\{\sqrt{n^2+n}-n\}_{n=1}^{\infty}$ converges, and if it converges, fin[d] its limit [Marks: 4]

Soln. $$\lim_{n\to\infty} \frac{(\sqrt{n^2+n}-n)(\sqrt{n^2+n}+n)}{\sqrt{n^2+n}+n}$$

(2)

$$= \lim_{n\to\infty} \frac{\cancel{n^2}+n-\cancel{n^2}}{\sqrt{n^2+n}+n} = \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}\left(\sqrt{1+\frac{1}{n}}+1\right)}$$

$= \frac{1}{2}$ (1) ; Conv. (1)

Q#2) Test the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{3^n+4^n}{5^n+7^n}$. [Marks: 4]

Soln. we apply Limit Comparison test (LCT)

Choose $\sum b_n = \sum_{n=1}^{\infty} \left(\frac{4}{7}\right)^n$ which is a Conv. Geom. series.

(2)

$$\lim_{n\to\infty} \frac{3^n+4^n}{5^n+7^n} \times \left(\frac{7}{4}\right)^n$$

$$= \lim_{n\to\infty} \frac{\cancel{4^n}\left[\left(\frac{3}{4}\right)^n+1\right]}{\cancel{7^n}\left[\left(\frac{5}{7}\right)^n+1\right]} \times \cancel{\left(\frac{7}{4}\right)^n} = 1 \neq 0$$

(1)

Hence by LCT, $\sum_{n=1}^{\infty} \frac{3^n+4^n}{5^n+7^n}$ is also Conv.

(1)

②

Q#3) Determine whether the series $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n(\ln n)}$ is absolutely convergent, conditionally convergent or divergent. [Marks: 6]

Soln. $\sum_{n=2}^{\infty} \left| (-1)^n \frac{1}{n(\ln n)} \right| = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$ which is divergent series by Integral test ③

But $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n(\ln n)}$ is conv. by AST ②

Hence the series $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n(\ln n)}$ is conditionally conv. ①

Q#4) Find the interval of convergence and the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-2)^n (x+3)^n}{n}$. [Marks: 6]

Soln. We apply absolute Ratio test

$$\lim_{n \to \infty} \left| \frac{(-2)^{n+1} (x+3)^{n+1}}{(n+1)} \times \frac{n}{(-2)^n (x+3)^n} \right|$$

Please make corresponding change

$$= 2|x+3|.$$

For absolute conv $|x+3| < 1 \iff -1 < x+3 < 1$

$\iff -4 < x < -2$ ②

At $x = -4$, we have $\sum_{n=1}^{\infty} \frac{(-2)^n (-1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n} 2^n}{n} = \sum_{n=1}^{\infty} \frac{2^n}{n}$

Which is a div. series by nth term test ①

At $x = -2$, we have $\sum_{n=1}^{\infty} \frac{(-2)^n (1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n}$ ①

which is div. by AST, Hence the interval of conv: $(-4, -2)$ ①

Radius: $r = \frac{-2-(-4)}{2} = 1$ ①

Q#5) Find the first four non-zero terms of the Taylor Series for the function $f(x) = \ln(x)$ in powers of $(x-2)$. [Marks: 5]

Soln. We know the Taylor Series as

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2) + \cdots \quad (1)$$

$$f(x) = \ln(x) \Rightarrow f(2) = \ln 2$$

$$f'(x) = \frac{1}{x} \Rightarrow f'(2) = \frac{1}{2}$$

$$f''(x) = -\frac{1}{x^2} \Rightarrow f''(2) = -\frac{1}{4}$$

$$f'''(x) = \frac{2}{x^3} \Rightarrow f'''(2) = \frac{2}{8} = \frac{1}{4} \quad (2)$$

Hence, the first four non-zero terms of the Taylor Series

$$f(x) = \ln(x) = \ln(2) + \frac{1}{2}(x-2) - \frac{1}{2!(4)}(x-2)^2 + \frac{1}{3!(4)}(x-2)^3 - \cdots$$

$$\therefore f(x) = \ln x = \ln(2) + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3 - \cdots$$

(2)