King Saud University
Department of Mathematics
M-203
(Differential and Integral Calculus)
Second Mid-Term Examination
(I-Semester 1436/1437)

Max. Marks: 25 | Time: 90 Minutes

Note: All questions carry equal marks

Q. No: 1 Evaluate the integral

$$\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \sin(x^2)\,dx\,dy .$$

Q. No: 2 Use **polar coordinates** to evaluate the double integral

$$\int_0^2 \int_x^{\sqrt{8-x^2}} xy\,dy\,dx .$$

Q. No: 3 Find **the surface area** of the portion of the graph of the equation $x+3y+z=6$ that lies in the first octant.

Q. No:4 Find the **centroid** of the solid bounded by the graphs of the equations $z=\sqrt{x^2+y^2}$ and $z=2-\sqrt{x^2+y^2}$.

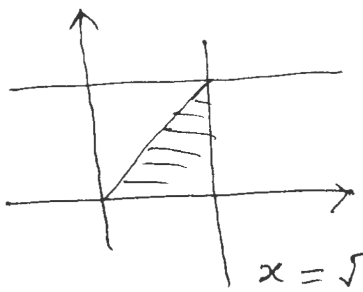
Q.No: 5 Evaluate the triple integral by changing to **spherical** coordinates:

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{\sqrt{x^2+y^2}} \left(x^2+y^2+z^2\right) dz\,dy\,dx .$$

Q:1 $\displaystyle\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \sin(x^2)\, dx\, dy$ [Marks: 5]

$\boxed{\begin{array}{l} 0 \le x \le \sqrt{\pi} \\ 0 \le y \le x \end{array}}$

$\boxed{\begin{array}{l} 0 \le y \le \sqrt{\pi} \\ y \le x \le \sqrt{\pi} \end{array}}$



$$= \int_0^{\sqrt{\pi}} \int_0^{x} \sin(x^2)\, dy\, dx$$

4

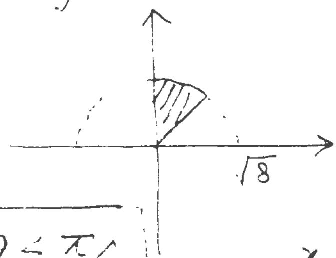
$$= \int_0^{\sqrt{\pi}} \sin(x^2)\, x\, dx$$

$$= \frac{1}{2} \int_0^{\sqrt{\pi}} \sin(x^2)(2x)\, dx = -\frac{1}{2} \cos(x^2) \Big]_0^{\sqrt{\pi}}$$

$$= -\frac{1}{2}\left[\cos(\pi) - \cos(0)\right] = -\frac{1}{2}\left[-1-1\right] = 1$$

1

Q.2 $\displaystyle\int_0^2\int_x^{\sqrt{8-x^2}} xy\,dy\,dx$ [Marks: 5]



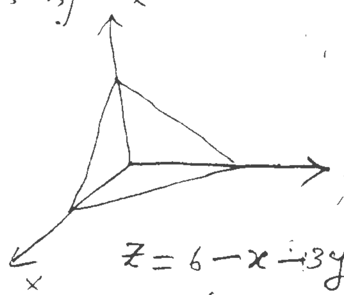
$$= \int_{\pi/4}^{\pi/2}\int_0^{\sqrt{8}} r^3 \sin\theta\cos\theta\,dr\,d\theta \quad (3) \qquad \boxed{\begin{array}{c}\pi/4 \le \theta \le \pi/2 \\ 0 \le r \le \sqrt{8}\end{array}}$$

$x \le y \le \sqrt{8-x^2}$
$0 \le x \le 2$

$$= \int_{\pi/4}^{\pi/2} \left[\frac{r^4}{4}\right]_0^{\sqrt{8}} \sin\theta\cos\theta\,d\theta = \frac{64}{4}\left[\frac{\sin^2\theta}{2}\right]_{\pi/4}^{\pi/2} = 8\left[(1) - (1/\sqrt{2})^2\right]$$

$$= 8(1/2) = 4 \quad (2)$$

Q.3 $\displaystyle S.A = \iint_{R_{xy}} \sqrt{1+(f_x)^2+(f_y)^2}\,dA$ (1) [Marks 5]



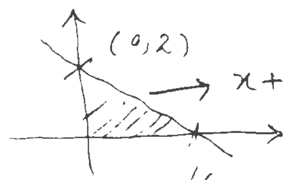
$z = 6 - x - 3y$
$= f(x,y)$

$$= \int_0^6\int_0^{\frac{1}{3}(6-x)} \sqrt{1+(-1)^2+(-3)^2}\,dy\,dx \quad (3)$$

$$= \sqrt{11}\int_0^6 \frac{1}{3}(6-x)\,dx$$

$$= \sqrt{11}\left[2x - \frac{x^2}{6}\right]_0^6$$

$$= \sqrt{11}\,[12-6] = 6\sqrt{11} \quad (1)$$

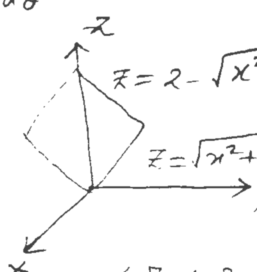


Q.4 [Marks: 5]

$$\text{mass} = \int_0^{2\pi}\int_0^1\int_r^{2-r} r\,dz\,dr\,d\theta \quad (3) = \int_0^{2\pi}\int_0^1 r\,[(2-r)-r]\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 [2r - 2r^2]\,dr\,d\theta = \frac{1}{3}\int_0^{2\pi} d\theta$$

$$= \frac{2\pi}{3} \quad (1)$$



$r \le z \le 2 - r$
$0 \le r \le 1$
$0 \le \theta \le 2\pi$

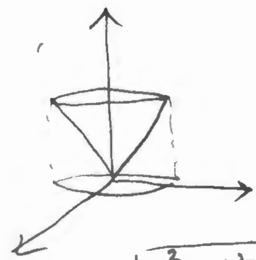
$$\int_0^{2\pi}\int_0^1\int_r^{2-r} z\,r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \left[\frac{z^2}{2}\right]_r^{2-r} r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 (2r - 2r^2)\,dr\,d\theta = 2\int_0^{2\pi}\left[\frac{r^2}{2} - \frac{r^3}{3}\right]_0^1 d\theta = \frac{2\pi}{3}$$

$$\bar{z} = \frac{2\pi/3}{2\pi/3} = 1, \quad \bar{x} = 0, \quad \bar{y} = 0 \qquad (0, 0, 1) \quad (1)$$

Q:5 $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{x^2+y^2}} (x^2+y^2+z^2)\, dz\, dy\, dx$ [Marks: 5]

$$=\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{\csc\varphi} \rho^2\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta \quad 3$$



$$=\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{\csc\varphi} \rho^4 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

$\rho^2\sin^2\varphi = 1$ $\boxed{x^2+y^2=1}$

$\rho = \csc\varphi$

$$=\int_0^{2\pi}\int_{\pi/4}^{\pi/2} \left.\frac{\rho^5}{5}\right]_0^{\csc\varphi} \sin\varphi\, d\varphi\, d\theta$$

$$\begin{cases} 0 \le \rho \le \csc\varphi \\ \pi/4 \le \varphi \le \pi/2 \\ 0 \le \theta \le 2\pi \end{cases}$$

$$=\frac{1}{5}\int_0^{2\pi}\int_{\pi/4}^{\pi/2} \csc^4\varphi\, d\varphi\, d\theta$$

$$=\frac{1}{5}\int_0^{2\pi}\int_{\pi/4}^{\pi/2} (1+\cot^2\varphi)\csc^2\varphi\, d\varphi\, d\theta$$

$$=\frac{1}{5}\int_0^{2\pi} -\left[\cot\varphi + \frac{\cot^3\varphi}{3}\right]_{\pi/4}^{\pi/2} d\theta$$

$u = \cot\varphi$

$du = -\csc^2\varphi\, d\varphi$

$-\int (1+u^2)\,du$

$-u - \frac{u^3}{3}$

$$= -\frac{1}{5}\left[(0+0) - (1+1/3)\right] 2\pi$$

$$= -1/5\,(-4/3)\, 2\pi = \frac{8}{15}\pi \quad 2$$