King Saud University,
College of Sciences
Department of Mathematics

Mid-Term 2/S1/2013
Full Mark:25. Time 1H30mn
25/1/1435

Question 1[4,4]. **a)** Consider the initial value problem

$$\begin{cases} \sqrt{3-x}\, y'' + \frac{\ln x}{x-2}\, y' + xy = 0 \\ y(1) = 2,\ y'(1) = 3. \end{cases}$$

Find the largest interval for which the given initial value problem has a unique solution.

b) Show that $y_1 = \sin(\sin x)$ and $y_2 = \cos(\sin x)$ are linearly independent solutions on $(-\pi/2, \pi/2)$ of the differential equation

$$y'' + (\tan x)y' + y\cos^2 x = 0$$

Question 2[4,4]. **a)** If $y_1 = \frac{1}{\sqrt{x}\csc x}$ is a solution of the differential equation

$$4x^2y'' + 4xy' + \left(4x^2 - 1\right)y = 0, \qquad 0 < x < \pi.$$

Then find its general solution.

b) By using the undetermined coefficients method, give only the form of the particular solution y_p of the differential equation

$$y^{(3)} - 2y'' - 5y' + 6y = x^2e^x - x^3e^{-2x} + 7x^4e^{3x}$$

Question 3[4]. Solve the initial value problem

$$\begin{cases} x^2y'' - 3xy' + 4y = 0, \qquad x > 0 \\ y(1) = 5, \quad y'(1) = 3 \end{cases}$$

Question 4 [5] By sing the method of variation of parameters, solve the differential equation

$$y'' + y = \tan x, \quad 0 < x < \frac{\pi}{2}$$

Complete Solution of Mid-Term, 204 M
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Question ① ⓐ $\begin{cases} \sqrt{3-x}\, y'' + \dfrac{\ln x}{x-2} y' + x y = 0 \\ y(1) = 2, \quad y'(1) = 3 \end{cases}$

$a_2(x) = \sqrt{3-x}$ is continuous for $x \le 3$ 2

$a_1(x) = \dfrac{\ln x}{x-2}$ is continuous for $x > 0$ and $x \ne 2$

$a_0(x) = x$ is continuous on $\mathbb{R}$, $a_2(x) \ne 0$ on $(0,2)$ and $(2,3)$

As $1 \in (0,2)$, then the largest interval for which the IVP is $I = (0,2)$

0 2 3

Question ① ⓑ $W(y_1, y_2; x) = \begin{vmatrix} \sin(\sin x) & \cos(\sin x) \\ \cos x \cos(\sin x) & -\cos x \sin(\sin x) \end{vmatrix} = -\cos x \ne 0$ 4

for all $-\frac{\pi}{2} < x < \frac{\pi}{2}$

Then $y_1 = \sin(\sin x)$ and $y_2 = \cos(\sin x)$ are linearly independent on $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Question ② ⓐ $y'' + \frac{1}{x} y' + \frac{1}{x^2}(x^2 - 4) y = 0$; $0 < x < \pi$, $y_1 = \dfrac{\sin x}{\sqrt{x}}$, $P = \frac{1}{x}$

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{(y_1)^2} dx, \quad e^{-\int \frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

$$y_2 = \frac{\sin x}{\sqrt{x}} \int \frac{\frac{1}{x}}{\frac{\sin^2 x}{x}} dx = \frac{\sin x}{\sqrt{x}} \int \csc^2 x\, dx = \frac{\sin x}{\sqrt{x}} \cdot -\frac{\cos x}{\sin x}$$

$$y_2 = \frac{-\cos x}{\sqrt{x}} \quad \text{or} \quad y_2 = \frac{\cos x}{\sqrt{x}}$$

So the G-solution of the D.E is

$$y = c_1 y_1 + c_2 y_2 = c_1\left(\frac{\sin x}{\sqrt{x}}\right) - c_2\left(\frac{\cos x}{\sqrt{x}}\right)$$

$$y = c_1\left(\frac{\sin x}{\sqrt{x}}\right) + c_3\left(\frac{\cos x}{\sqrt{x}}\right)$$

Question (2) (b) $y''' - 2y'' - 5y' + 6y = x^2 e^x - x^3 e^{-2x} + 7x^4 e^{3x}$

$m^3 - 2m^2 - 5m + 6 = (m-1)(m^2 - m - 6) = 0$ - - - - (1)

$(m-1)(m+2)(m-3) = 0 \Rightarrow m = 1, -2, 3$ 1

$f_1(x) = x^2 e^{(1)x}$, $s=1$ is a simple root of (1) $\Rightarrow y_{p,1} = (A_0 + A_1 x + A_2 x^2) x e^x$ 1

$f_2(x) = -x^3 e^{(-2)x}$, $s=-2$ is a simple root of (1) $\Rightarrow y_{p,2} = (B_0 + B_1 x + B_2 x^2 + B_3 x^3) x e^{-2x}$ 1

$f_3(x) = 7x^4 e^{3x}$, $s=3$ is a simple root of (1) $\Rightarrow y_{p,3} = (C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4) x e^{3x}$ 1

So the form of the particular solution of the D.E is

$$y_p = (A_0 x + A_1 x^2 + A_2 x^3) e^x + (B_0 x + B_1 x^2 + B_2 x^3 + B_3 x^4) e^{-2x} +$$
$$+ (C_0 x + C_1 x^2 + C_2 x^3 + C_3 x^4 + C_4 x^5) e^{3x}$$

Question (3) $\begin{cases} x^2 y'' - 3xy' + 4y = 0 ; \quad x > 0 \\ y(1) = 5, \quad y'(1) = 3 \end{cases}$

$y = x^m$, $m(m-1) - 3m + 4 = m^2 - 4m + 4 = (m-2)^2 = 0$, $m = 2, 2$

The general solution of the D.E is 1

$$y = c_1 x^2 + c_2 x^2 \ln x$$ 1

But $y' = 2c_1 x + 2c_2 x \ln x + c_2 x$,

$y(1) = 5 \Rightarrow c_1 + 0 = 5 \Rightarrow c_1 = 5$

$y'(1) = 2c_1 + 0 + c_2 = 3 \Rightarrow 10 + c_2 = 3 \Rightarrow c_2 = -7$ 2

So the solution the IVP is $\boxed{y = 5x^2 - 7x^2 \ln x}$

Question (4) $y'' + y = \tan x$; $0 < x < \frac{\pi}{2}$

1) $y'' + y = 0 \Rightarrow m^2 + 1 = 0$, $m = \mp i$

then the G. solution of $y'' + y = 0$ is $y_c = c_1 \cos x + c_2 \sin x$, 1

2) We put $y_1 = \cos x$, $y_2 = \sin x$.

$y_p = u_1 y_1 + u_2 y_2$, where $\begin{cases} u_1' \cos x + u_2' \sin x = 0 \\ u_1'(-\sin x) + u_2' \cos x = \tan x \end{cases}$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1 , \quad W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \tan x \end{vmatrix} = \sin x ,$$

1

$$W_1 = \begin{vmatrix} 0 & \sin x \\ \tan x & \cos x \end{vmatrix} = \frac{-\sin^2 x}{\cos x} , \quad u_2' = \frac{W_2}{W} \Rightarrow u_2 = \int \sin x \, dx = \boxed{-\cos x}$$

1

$$u_1' = \frac{W_1}{W} \Rightarrow u_1 = -\int \frac{1-\cos^2 x}{\cos x} dx$$

$$u_1 = -\int \sec x \, dx + \int \cos x \, dx = \boxed{-\ln|\sec x + \tan x| + \sin x}$$

$$y_p = u_1 y_1 + u_2 y_2 = -\cos x \ln(\sec x + \tan x) + \cos x . \sin x - \sin x \cos x$$

2

$$\boxed{y_p = -\cos x \ln(\sec x + \tan x)}$$

Then the G. solution of the D. E

$$\boxed{y = y_c + y_p = c_1 \cos x + c_2 \sin x - \cos x \ln(\sec x + \tan x)}$$