King Saud University,
College of Sciences
Mathematical Department.

Mid-Term 2/S2/2014
Full Mark: 25. Time 1H30mn
12/2/1436

**Question 1[3,4].** **a)** Consider the initial value problem

$$\begin{cases} x^2y'' + \frac{x}{(4-x)^{\frac{1}{3}}}y' + \frac{2}{\sqrt{x-2}}y = 0 \\ y(3) = 1, \ y'(3) = -1. \end{cases} \qquad (*)$$

Find the largest interval for which the initial value problem (*) has a unique solution.

**b)** Find the general solution of the differential equation

$$x^2y'' + x^2y' - (x+2)y = 0, \quad x > 0,$$

if $y_1 = x^{-1}e^{-x}$ is one solution of the differential equation.

**Question 2 [4,4].** **a)** By using the undetermined coefficients method , **give only** the general form of the particular solution $y_p$ of the differential equation

$$y^{(4)} + y^{(3)} = 1 - e^{-x} + xe^{-x}\cos x,$$

**b)** Find the general solution of the differential equation

$$x^2y'' - 3xy' + 5y = 0 \ ; \ x > 0.$$

**Question 3[5]** Use the method of variation of parameters to find the general solution of the differential equation

$$y'' + 2y' + y = x^{-2}e^{-x}\ln x \ ; \ x > 0.$$

**Question 4 [5]** Solve the given linear system of differential equations ~~subject to the indicated initial conditions~~

$$\begin{cases} -x + \frac{dy}{dt} = t - 1 \\ \frac{dx}{dt} + \frac{dy}{dt} = 2x + 2y \end{cases}$$

**Question 1**

(a) $\begin{cases} x^2 y'' + \dfrac{x}{(4-x)^{1/3}} y' + \dfrac{2}{\sqrt{x-2}}\, y = 0 \\ y(3)=1, \quad y'(3)=-1 \end{cases}$

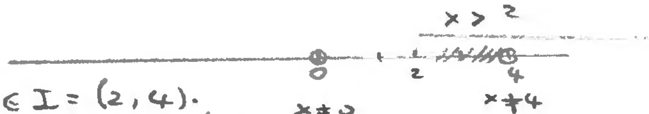
$a_2(x) = x^2$ is cont. on $\mathbb{R}$ and $a_2(x) \neq 0$ for $x \neq 0$

1

$a_1(x) = \dfrac{x}{\sqrt[3]{4-x}}$ is cont. on $\mathbb{R} \setminus \{4\}$

$a_0(x) = \dfrac{2}{\sqrt{x-2}}$ is cont for $x > 2$

1




But $x = 3 \in I = (2,4)$.

1

Then $(2,4)$ is the largest interval, for which the IVP has a unique Solution

(b) $x^2 y'' + x^2 y' - (x+2) y = 0; \quad x > 0, \quad y_1 = x^{-1} e^{-x}$

$y'' + y' - \dfrac{x+2}{x^2} y = 0, \qquad p(x) = 1$

1

$e^{-\int p(x)dx} = e^{-x}, \qquad y_2 = y_1 \displaystyle\int \frac{e^{-\int p(x)dx}}{(y_1)^2}\, dx$

$y_2 = e^{-x} x^{-1} \displaystyle\int \frac{e^{-x}}{x^{-2} e^{-2x}}\, dx = e^{-x} \int x^2 e^{x}\, dx$

2

$y_2 = x^{-1} e^{-x} [\, x^2 e^{x} - 2 \int x e^{x}\, dx \,]$

$y_2 = x^{-1} e^{-x} [\, x^2 e^{x} - 2x e^{x} + 2 e^{x} \,] = (x - 2 + \frac{2}{x})$

1

So the G. solution of the D. Eq. is $\boxed{y = c_1 x^{-1} e^{-x} + c_2 (x + \frac{2}{x} - 2)}$

**Question 2**

$y^{(4)} + y^{(3)} = 0 \Rightarrow m^3 (m+1) = 0 \Rightarrow m = 0, 0, 0$; $m = -1$

1

$1 = 1 e^{0x} \Rightarrow$ $r = 0$ is a root repeated 3 times $\Rightarrow$ $1 \to Ax^3$

2

$e^{-x} = e^{(-1)x}$, $r = -1$ is a simple root $\Rightarrow e^{-x} \to Bxe^{-x}$

$xe^{-x}\cos(x)$, $r = -1 \mp i$ is not a root $\Rightarrow$

1

$xe^{-x}\cos(x) \to (Cx+D)e^{-x}\cos x + (Ex+F)e^{-x}\sin x$

Then $\boxed{y_p = Ax^3 + Bxe^{-x} + (Cx+D)e^{-x}\cos x + (Ex+F)e^{-x}\sin x}$

(b) $y = x^m$, $m(m-1) - 3m + 5 = m^2 - 4m + 5 = 0$

$m = \frac{+4 \mp 2i}{2} = +2 \mp i$ ✓ (2)

$\boxed{y = c_1 x^2 \cos(\ln x) + c_2 x^2 \sin(\ln x)}$ is the G. solution of the D. E. (2)

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Question (3) $y'' + 2y' + y = x^{-2} e^{-x} \ln x$ ; $x > 0$

1) $y'' + 2y' + y = 0 \Rightarrow m^2 + 2m + 1 = (m+1)^2 = 0 \Rightarrow m = -1, -1$

$\boxed{y_c = c_1 e^{-x} + c_2 x e^{-x}}$ , $y_1 = e^{-x}$, $y_2 = x e^{-x}$ (1)

2) $y_p = u_1 y_1 + u_2 y_2$ s. t $u_1$ and $u_2$ satisfy:

$$\begin{cases} u_1'(e^{-x}) + u_2'(x e^{-x}) = 0 \\ u_1'(-e^{-x}) + u_2'(e^{-x} - x e^{-x}) = x^{-2} e^{-x} \ln x \end{cases}$$

or $u_1' + u_2'(x) = 0$
$-u_1' + u_2'(1-x) = x^{-2} \ln x$ , $W = \begin{vmatrix} 1 & x \\ -1 & 1-x \end{vmatrix} = 1$

$W_1 = \begin{vmatrix} 0 & x \\ x^{-2}\ln x & 1-x \end{vmatrix} = -x^{-1} \ln x$ , $u_1' = \frac{W_1}{W} = -x^{-1} \ln x$ , $u_1 = -\int \frac{\ln x}{x} dx$

$\boxed{u_1 = -\frac{1}{2}(\ln x)^2}$ (1)

$W_2 = \begin{vmatrix} 1 & 0 \\ -1 & x^{-2}\ln x \end{vmatrix} = x^{-2} \ln x$, $u_2' = \frac{W_2}{W} = \frac{\ln x}{x^2}$ ,

$u_2 = \int \frac{\ln x}{x^2} dx = [\frac{-1}{x} \ln x] + \int \frac{1}{x^2} dx$

$\boxed{u_2 = \left(\frac{-1}{x} \ln x - \frac{1}{x}\right)}$ (1)

$y_p = \frac{-1}{2} e^{-x} (\ln x)^2 + x e^{-x}\left(-\frac{1}{x} \ln x - \frac{1}{x}\right)$

$\boxed{y_p = e^{-x}\left[-\frac{1}{2}(\ln x)^2 - \ln x - 1\right)}$ (2)

So the G. solution is $\boxed{y = y_c + y_p = c_1 e^{-x} + c_2 x e^{-x} + e^{-x}\left[-\frac{1}{2}(\ln x)^2 - \ln x - 1\right)}$

**Question 4** $(D-2)(-x+Dy=t-1)$ $\Rightarrow$ $-(D-2)x+D(D-2)y=(D-2)(t-1)$

$(D-2)x+(D-2)y=0$ $\qquad$ $(D-2)x+(D-2)y=0$ $\quad +$

$$(D-2)y+(D^2-2D)y=D(t-1)-2t+2=-2t+3$$

$$y''-2y'+y'-2y=-2t+3 \qquad 1$$

$$\boxed{y''-y'-2y=-2t+3}$$

1) $y''-y'-2y=0 \Rightarrow m^2-m-2=(m-2)(m+1)=0$

$m=2,\quad m=-1$ $\qquad 1$

$\boxed{y_c=c_1e^{2t}+c_2e^{-t}}$, $\quad y_p=A+Bt$, $y_p'=B$, $y_p''=0$

$\Rightarrow -B-2A-2Bt=-2t+3 \Rightarrow B=1$, $-B-2A=3$ $\qquad 1$

$A=-2$

$\boxed{y_p=t-2}$, $\quad \boxed{y=c_1e^{2t}+c_2e^{-t}+t-2}$

But $x=y'-t+1 \Rightarrow x=2c_1e^{2t}-c_2e^{-t}+1-t+1$ $\qquad 2$

$$\boxed{x=2c_1e^{2t}-c_2e^{-t}-t+2}.$$

~~$x(0)=5 \Rightarrow 2c_1-c_2=3$ $\quad 3c_1=6 \Rightarrow c_1=2$~~

~~$y(0)=1 \Rightarrow c_1+c_2=3$ $\quad c_2=1$~~

~~Then the solution of the system is~~

~~$x(t)=4e^{2t}-e^{-t}-t+2$~~

~~$y(t)=2e^{2t}+e^{-t}+t-2$~~