KING SAUD UNIVERSITY
DEPARTMENT OF MATHEMATICS
TIME: 1H 30 mn, FULL MARKS: 40, 07/06/1432
MATH 204

Question 1. a) [7] Show that $f(x) = \cosh 2x$, $g(x) = e^{-2x}$ are solutions of the differential equation: $y'' - 4y = 0$, and show that $\{\cosh 2x,\ e^{-2x}\}$ is a fundamental set of solutions of the differential equation: $y'' - 4y = 0$.

b) [8] If the function $y_1 = x$ is a solution of the differential equation

$$x^2 y'' - (x^2 + 2x)y' + (x+2)y = 0, \qquad x > 0.$$

Use the method of reduction of order to find the second solution y_2, and hence find the general solution.
(**Hint**: $\int \sec \alpha x dx = \frac{1}{\alpha} \ln |\sec \alpha x + \tan \alpha x|$)

Question 2. [8] Use the method of variation of parameters to find the general solution of the nonhomogeneous differential equation:

$$y'' - 2y' + 2y = e^{x} \tan\ x.$$

Question 3. [8] Determine only the form of the particular solution of the differential equation

$$y''' - 2y'' + y' = 7e^{-x} \sin 2x.$$

Question 4. [9] Solve the system of differential equations

$$\begin{cases} \frac{dx}{dt} + 2y = -x \\ 2\frac{dx}{dt} + \frac{dy}{dt} = -y. \end{cases}$$

Answer Sheet

Mid.Term 2

Q1 a) $y_1 = \cosh 2x$, $y_1' = 2\,\mathrm{sh}\,2x$, $y_1'' = 4\,\mathrm{ch}\,2x$

$\Rightarrow y_1'' - 4y_1 = 0$

$y_2 = e^{-2x}$, $y_2' = -2e^{-2x}$, $y_2'' = 4e^{-2x}$ (3)

$\Rightarrow y_2'' - 4y_2 = 0$

To show that $y_1 = \cosh 2$, $y_2 = \sinh 2x$ form a fundamental set of solutions for the DE, we have to show that $W[y_1, y_2](x) \neq 0$ (4)

where: $W[y_1, y_2](x) = \begin{vmatrix} \cosh 2x & e^{-2x} \\ 2\sinh 2x & -2e^{-2x} \end{vmatrix} = -2 \neq 0 \quad \forall x \in \mathbb{R}$

b) Let $y = xu$, $y' = xu' + u$, $y'' = xu'' + 2u'$

Thus $x^2[u''x + 2u'] - (x^2 + 2x)[xu' + u] + (x+2)xu = 0$ (3)

$\Rightarrow x^3 u'' - x^3 u' = 0 \Rightarrow u'' - u' = 0$

Let $u' = w$, then $w' - w = 0 \Rightarrow \frac{dw}{w} = dx$

$\Rightarrow w(x) = c_1 e^{x}$ (2)

$\Rightarrow u' = c_1 e^{x} \Rightarrow u = c_1 e^{x} + c_2$.

Hence $y = (c_1 e^{x} + c_2)x$, take $c_1 = 1$, $c_2 = 0$, $y_2 = xe^{x}$

A fundamental set of solution is $\{x, xe^{x}\}$ (2)

$y_g = c_1 x + c_2 x e^{x}$

$\underline{Q_2}$: $y'' - 2y' + 2y = e^x \tan \cancel{x}$

HE: $y'' - 2y' + 2y = 0$, charact Eq : $m^2 - 2m + 2 = 0$

$m_1 = 1 + i, \quad m_2 = 1 - i$ (2)

$y_{gh} = (C_1 \cos x + C_2 \sin x) e^x$

$y_p = C_1(x) e^x \cos x + C_2(x) e^x \sin x$

where $\begin{cases} C_1' e^x \cos x + C_2' e^x \sin x = 0 \\ C_1' e^x (\cos x - \sin x) + C_2' e^x (\sin x + \cos x) = e^x \tan \cancel{x} \end{cases}$

$$\Delta = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x (\cos x - \sin x) & e^x (\sin x + \cos x) \end{vmatrix} = e^{2x} \neq 0$$

$\forall x \in R$

$$C_1'(x) = \frac{\begin{vmatrix} 0 & e^x \sin x \\ e^x \tan 2x & e^x (\sin x + \cos 2x) \end{vmatrix}}{e^{2x}} = -\tan \cancel{x} \cdot \sin x$$ (2)

$$= -\frac{\sin^2 x}{\cos x} = -\frac{1 - \cos^2 x}{\cos x} = -\frac{1}{\cos x} + \cos x$$

$\Rightarrow C_1(x) = -\ln|\sec x + \tan x| + \sin x$

$$C_2'(x) = \frac{\begin{vmatrix} e^x \cos x & 0 \\ e^x (\cos x - \sin x & e^x \tan x \end{vmatrix}}{e^{2x}} = \tan x \cos x$$ (2)

$\Rightarrow C_2(x) = \int \sin x \, dx = -\cos x$

$y_a = y_{gh} + e^x \cos x (-\ln|\sec x + \tan x| + \sin x) \cancel{+} e^x \sin x \cos x$ (2)

$\underline{Q_3}$: $y''' - 2y'' + y' = 7e^{-x}\sin 2x$ 3

$m^3 - 2m^2 + m = 0 \Rightarrow m(m^2 - 2m + 1) = 0 \Rightarrow m_1 = 0,\ m_2 = 1$ (double)

$y_p = Ae^{-x}\sin 2x + Be^{-x}\cos 2x$ 5

$\underline{Q_4}$: $\begin{cases} \frac{dx}{dt} + 2y = -x \\ 2\frac{dx}{dt} + \frac{dy}{dt} = -y \end{cases}$

Operator form: $\begin{cases} (D+1)[x] + 2y = 0 \longrightarrow (1) \\ 2D[x] + (D+1)[y] = 0 \longrightarrow (2) \end{cases}$

Then to eliminate x, we apply $(D+1)$ to (2)

and apply $-2D$ to (1), we have

$-2D(D+1)[x] - 4D[y] = 0$ (3)

$2(D+1)D[x] + (D+1)(D+1)[y] = 0$

$(D+1)(D+1)[y] - 4D[y] = 0$

$\Rightarrow$ $y'' - 2y' + y = 0$

$\Rightarrow$ Charact eq: $m^2 - 2m + 1 = 0 \Leftrightarrow (m-1)^2 = 0 \Rightarrow m = 1$ (double root)

Thus $y_{gh} = (c_1 t + c_2)e^t$ (3)

From equation (2), $2x' = -(c_1 t + c_2)e^t$

$- e^t(c_1 t + c_2) - c_1 e^t$

$= -2e^t(c_1 t + c_2) - c_1 e^t$

$\Rightarrow 2x(t) = -c_1 e^t - 2c_2 e^t - 2c_1 \int t e^t \, dt$

$= -c_1 e^t - 2c_2 e^t - 2c_1 [te^t - e^t]$

$= -2c_1 te^t + c_1 e^t - 2c_2 e^t$

$= e^t(-2c_1 t + c_1 - 2c_2)$.

$\Longrightarrow x(t) = -e^t\left(c_1 t + \frac{c_1}{2} - c_2\right)$. (3)