King Saud University,
College of Sciences
Mathematical Department.

Mid-Term 2/S2/2012
Full Mark:40. Time 1H30mn
01/05/2012

Question 1[8]. Given the initial value problem

$$\left\{ \begin{array}{l} x(x^2-4)y'' + y = e^x \\ y(-1) = 1,\ y'(-1) = 0. \end{array} \right. \qquad (*)$$

Find the largest interval for which the initial value problem (*) has a unique solution.

Question 2[7,7]. a) Use reduction of order method to find a second solution of the differential equation

$$xy'' - (x+1)y' + y = 0,$$

given that $y_1 = e^x$ is a solution

b) By using the undetermined coefficients method, give only the form of the particular solution y_p of the differential equation

$$y^{(4)} - y = e^x - xe^{-x} + 2\sin x + x^3 \cos x$$

Question 3[8]. Solve the differential equation

$$x^2y'' - 2xy' + 2y = 2x^3 \ln x.$$

Question 4[10]. Solve the following linear system of differential equations

$$\left\{ \begin{array}{l} \frac{dx}{dt} = 2 + 4y \\ \frac{dy}{dt} = 1 + x. \end{array} \right.$$

Answer Sheet

Mid. Term 2 (S2)

Math 204.

Q1: $(*)\begin{cases} x(x^2-4)y''+y=e^x \\ y(-1)=1, \quad y'(-1)=0 \end{cases}$

The functions $a_2(x)=x(x^2-4)$, $a_1(x)=0$, $a_0(x)=1$, $f(x)=e^x$

are Continuous on $\mathbb{R}$. (4)

$a_2(x)=0$ if $x=0$, $x=\pm 2$

The interval $I=(-2,0)$ Contains $x_0=-1$

and $a_2(x)\neq 0$ on I. Thus the I.V.P $(*)$ (4)

admits a unique solution $I=(-2,0)$

Q2 a) $xy''-(x+1)y'+y=0$, $y_1=e^x$ (1)

Let $y_2=y_1u=e^xu$

$y_2'=e^x(u+u')$

$y_2''=e^x(u''+2u'+u)$

By substitution in the DE, we get. (2)

$xe^xu''+(x-1)e^xu'=0$

Let $u'=w$, Then we have the equation,

$xw'+(x-1)w=0 \Rightarrow \frac{dw}{w}=\frac{-x+1}{x}dx \Rightarrow$ (2)

$\ln\left|\frac{w}{c}\right|=-x+\ln|x| \Rightarrow W(x)=c\,x\,e^{-x}=u'$

$\Rightarrow u=c\int xe^{-x}dx=c(-xe^{-x}-e^{-x})$ $(c=1)$ (2)

Then $y_2=e^x(-xe^{-x}-e^{-x})=-x-1$

$y^{(4)} - y = e^x + xe^{-x} + 2\sin x + x^3\cos x$

Charact Eq: $m^4 - 1 = 0 \ (*) \Rightarrow (m^2-1)(m^2+1) = 0$ ②

$\Longrightarrow m_1 = 1,\ m_2 = -1,\ m_3 = i;\ m_4 = -i$

Let $f_1(x) = e^x$, $f_2(x) = xe^{-x}$, $f_3(x) = 2\sin x + x^3\cos x$

Then

$y_{p_1} = Axe^x$ since $r_1 = 1$ is a simple root of $(*)$

$y_{p_2} = (ax+b)e^{-x}x$ since $r_2 = -1$ " " "

⑤

$y_{p_3} = \left[(A_3x^3 + B_2x^2 + C_1x + C_0)\cos x + (A_3^*x^3 + B_2^*x^2 + C_1^*x + C_0^*)\sin x\right]x$

Since $\alpha \pm i\beta = \pm i$ is a root of $(*)$

$y_p = y_{p_1} + y_{p_2} + y_{p_3}$

Q_3: $x^2y'' - 2xy' + 2y = 2x^3 \ln x$, $x > 0$

Homog Eq: $x^2y'' - 2xy' + 2y = 0$

$y = x^m$,

Charact Eq: $m^2 - 3m + 2 = 0 \implies m_1 = 1,\ m_2 = 2$

$$y_{gh} = C_1 x + C_2 x^2$$

By using the method of Variations of parameters:

$$y_p = C_1(x) y_1 + C_2(x) y_2$$

$$\begin{cases} C_1'(x) y_1 + C_2'(x) y_2 = 0 \\ C_1'(x) y_1' + C_2'(x) y_2' = 2x \ln x \end{cases}$$

That is
$$\begin{cases} C_1'(x) x + C_2'(x) x^2 = 0 \\ C_1'(x) + 2 C_2'(x) x = 2x \ln x \end{cases}$$

$$D = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2$$

$$C_1'(x) = \frac{\begin{vmatrix} 0 & x^2 \\ 2x \ln x & 2x \end{vmatrix}}{x^2} = -\frac{2x^3 \ln x}{x^2} = -2x \ln x$$

$$\implies C_1(x) = -2 \int x \ln x \, dx = -2\left[\frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} dx\right]$$
$$= -x^2 \ln x + \frac{x^2}{2}$$

$$C_2'(x) = \frac{\begin{vmatrix} x & 0 \\ 1 & 2x \ln x \end{vmatrix}}{x^2} = \frac{2x^2 \ln x}{x^2} = 2 \ln x$$

$$\implies C_2(x) = 2 \int \ln x \, dx = 2x \ln x - 2x$$

Thus $y_p = C_1(x)\, x + C_2(x)\, x^2 = \left(\frac{x^2}{2} - x^2 \ln x\right)x + (2x \ln x - 2x)x^2$

$$= x^3\left(\ln x - \frac{3}{2}\right)$$

Then $y = y_p + y_h$.

Q_4: $\begin{cases} \dfrac{dx}{dt} = 2 + 4y \\ \dfrac{dy}{dt} = 1 + x \end{cases}$

operator form: $\begin{cases} D[x] - 4y = 2 \longrightarrow (1) \\ D[y] - x = 1 \longrightarrow (2) \end{cases}$

Apply D to (2) and sum with (1), we get

$D^2[y] - 4y = 2 \Leftrightarrow y'' - 4y = 2$

$y_y = y_{gh} + y_p$

Ch Eq: $m^2 - 4 = 0 \Rightarrow m_1 = 2,\ m_2 = -2$

$y_{gh} = c_1 e^{2t} + c_2 e^{-2t}$

$y_p = A \Rightarrow y_p' = 0,\ y_p'' = 0$

Then we have $-4A = 2 \Rightarrow A = -\frac{1}{2}$,

and $y_g = y_{gh} + y_p = c_1 e^{2t} + c_2 e^{-2t} - \frac{1}{2}$

From eq (2), $x(t) = y' - 1 = 2c_1 e^{2t} - 2c_2 e^{-2t} - 1$

$x(t) = 2c_1 e^{2t} - 2c_2 e^{-2t} - 1$