King Saud University: Mathematics Department Math-254
Second Semester 1439-40 H Final Examination
Maximum Marks = 40 Time: 180 mins.

Name of the Student:———————— **I.D. No.** ————————-

Name of the Teacher:———————— **Section No.** ————————

Note: Check the total number of pages are Six (6).

(**10 Multiple choice questions and Three (3) Full questions**)

The Answer Tables for Q.1 to Q.10 : Marks: 2 for each one ($2 \times 10 = 20$)

Ps. : Mark {a, b, c or d} for the correct answer in the box.

Q. No.	1	2	3	4	5	6	7	8	9	10
a,b,c,d										

Quest. No.	Marks
Q. 1 to Q. 10	
Q. 11	
Q. 12	
Q. 13	
Total	

Question 1: The number of bisections required to solve the equation $x^3 + x = 1$ in $[0, 1]$ accurate to within 10^{-3} is:

(a) 8 **(b)** 10 **(c)** 9 **(d)** 7

Question 2: Given $x_0 = 0$ and $x_1 = 1$, then the next approximation x_2 of the solution of the equation $x^4 + 2x = 1$ using the Secant method is:

(a) 0.333 **(b)** 0.500 **(c)** 0.250 **(d)** 0.225

Question 3: The order of multiplicity of the root $\alpha = 0$ of the equation $e^x - \dfrac{x^2}{2} = x + 1$ is:

(a) 2 **(b)** 1 **(c)** 3 **(d)** 4

Question 4: The goal of forward elimination steps in simple Gauss elimination is to reduce the coefficient matrix to a matrix of the form:

(a) Upper-triangular **(b)** Lower-triangular **(c)** Identity **(d)** Diagonal

Question 5: Let $A = \begin{pmatrix} \alpha & 0 \\ 1 & 1 \end{pmatrix}$ and $\alpha > 2$. If the condition number $k(A)$ of the matrix A is 6, then α equals to:

(a) 3 **(b)** 5 **(c)** 4 **(d)** 6

Question 6: If $f(x) = xe^x$, then $f[0, 1, 0]$ equals to:

(a) $e + 2$ **(b)** $e + 1$ **(c)** $e - 2$ **(d)** $e - 1$

Question 7: Using data points: $(0.0, -2.0), (0.1, -1.0), (0.15, 1.0), (0.2, 2.0), (0.3, 3.0)$, the best approximate value of the function $f(0.11)$ by a linear Lagrange polynomial is:

(a) -0.4 **(b)** -0.5 **(c)** -0.6 **(d)** -0.3

Question 8: If $f(0) = 3, f(1) = \dfrac{\alpha}{2}, f(2) = \alpha$, and Simpson's rule for $\displaystyle\int_0^2 f(x)\ dx$ gives 2, then the value of α is:

(a) 1.0 **(b)** 2.0 **(c)** 0.5 **(d)** 3.0

Question 9: When using the two-point formula (forward) with $h = 0.2$ for approximating the value of $f'(1)$, where $f(x) = \ln(x + 1)$, we have the computed approximation (accurate to 4 decimal places):

(a) 0.4766 **(b)** 0.4966 **(c)** 0.4666 **(d)** 0.4866

Question 10: Given $xy' + y = 1$, $y(1) = 0$, the approximate value of $y(2)$ using Euler's method when $n = 1$ is:

(a) 0.0 **(b)** 1.0 **(c)** 2.0 **(d)** 3.0

Solution of the Final Examination

The Answer Tables for Q.1 to Q.10 : Math

Check the correct answer in the box.

Q. No.	1	2	3	4	5	6	7	8	9	10
a,b,c,d	b	a	c	a	b	d	c	a	a	b

The Answer Tables for Q.1 to Q.10 : MAth

Check the correct answer in the box.

Q. No.	1	2	3	4	5	6	7	8	9	10
a,b,c,d	c	d	a	b	d	c	a	b	d	a

The Answer Tables for Q.1 to Q.10 : MATh

Check the correct answer in the box.

Q. No.	1	2	3	4	5	6	7	8	9	10
a,b,c,d	a	b	d	c	c	b	d	c	b	d

Question 11: Consider the following nonhomogeneous linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 5 & 0 & -1 \\ -1 & 3 & 0 \\ 0 & -1 & 4 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}.$$

Find the matrix form of Jacobi and Gauss-Seidel iterative methods and show that Gauss-Seidel method converges faster than Jacobi method for the given system. [7 points]

Solution. Here we will show that the l_∞-norm of the Gauss-Seidel iteration matrix T_G is less than the l_∞-norm of the Jacobi iteration matrix T_J, that is

$$\|T_G\|_\infty < \|T_J\|_\infty.$$

The Jacobi iteration matrix T_J can be obtained from the given matrix A as follows

$$T_J = -D^{-1}(L+U) = -\begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1/5 \\ 1/3 & 0 & 0 \\ 0 & 1/4 & 0 \end{pmatrix}.$$

Thus the matrix form of Jacobi iterative method is

$$\mathbf{x}^{(\mathbf{k+1})} = \begin{pmatrix} 0 & 0 & 1/5 \\ 1/3 & 0 & 0 \\ 0 & 1/4 & 0 \end{pmatrix} \mathbf{x}^{(\mathbf{k})} + \begin{pmatrix} 1/5 \\ 2/3 \\ 1 \end{pmatrix}, \quad k \geq 0.$$

Similarly, Gauss-Seidel iteration matrix T_G is defined as

$$T_G = -(D+L)^{-1}U = -\begin{pmatrix} 5 & 0 & 0 \\ -1 & 3 & 0 \\ 0 & -1 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and it gives

$$T_G = -\begin{pmatrix} 1/5 & 0 & 0 \\ 1/15 & 1/3 & 0 \\ 1/60 & 1/15 & 1/4 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1/5 \\ 0 & 0 & 1/15 \\ 0 & 0 & 1/60 \end{pmatrix}.$$

So the matrix form of Gauss-Seidel iterative method is

$$\mathbf{x}^{(\mathbf{k+1})} = \begin{pmatrix} 0 & 0 & 1/5 \\ 0 & 0 & 1/15 \\ 0 & 0 & 1/60 \end{pmatrix} \mathbf{x}^{(\mathbf{k})} + \begin{pmatrix} 1/5 \\ 11/15 \\ 71/60 \end{pmatrix}, \quad k \geq 0.$$

Since the l_∞-norm of the matrix T_J is

$$\|T_J\|_\infty = \max\left\{\frac{1}{5}, \frac{1}{3}, \frac{1}{4}\right\} = \frac{1}{3} = 0.3333 < 1,$$

and the l_∞-norm of the matrix T_G is

$$\|T_G\|_\infty = \max\left\{\frac{1}{5}, \frac{1}{15}, \frac{1}{60}\right\} = \frac{1}{5} = 0.2000 < 1.$$

Since $\|T_G\|_\infty < \|T_J\|_\infty$, which shows that Gauss-Seidel method will converge faster than Jacobi method for the given linear system. •

Question 12: Given $f(x) = x^{1/3}$, and $x_0 = 0, x_1 = 1, x_2 = 27, x_3 = 64$. Construct the divided differences table for the function. Find the linear splines which interpolate this data. Find the best approximation of $f(8)$ and the absolute error. [7 points]

Solution: The divided differences table for the given function $f(x) = x^{1/3}$, and at the points $x_0 = 0, x_1 = 1, x_2 = 27, x_3 = 64$ is as follows:

k	x_k	Zeroth Divided Difference	First Divided Difference	Second Divided Difference	Third Divided Difference
0	$x_0 = 0$	$f[x_0] = 0$			
1	$x_1 = 1$	$f[x_1] = 1$	$f[x_0, x_1] = 1$		
2	$x_2 = 27$	$f[x_2] = 3$	$f[x_1, x_2] = 0.0769$	$f[x_0, x_1, x_2] = -0.0342$	
3	$x_3 = 64$	$f[x_3] = 4$	$f[x_2, x_3] = 0.0270$	$f[x_1, x_2, x_3] = -0.0008$	$f[x_0, x_1, x_2, x_3] = 0.0005$

Linear spline:

$$s_k(x) = A_k + B_k(x - x_k)$$

where the values of the coefficients A_k and B_k are given as

$$A_k = y_k \quad \text{and} \quad B_k = f[x_k, x_{k+1}] = \frac{(y_{k+1} - y_k)}{(x_{k+1} - x_k)}$$

Given $x_0 = 0, x_1 = 1, x_2 = 27, x_3 = 64$, then we have

$$\begin{aligned} A_0 &= y_0 = 0 \\ A_1 &= y_1 = 1 \\ A_2 &= y_2 = 3 \\ A_3 &= y_3 = 4 \end{aligned}$$

and

$$\begin{aligned} B_0 &= f[x_0, x_1] = 1 \\ B_1 &= f[x_1, x_2] = 0.0769 \\ B_2 &= f[x_2, x_3] = 0.0270 \end{aligned}$$

Now the linear spline for three subintervals are define as

$$s(x) = \begin{cases} s_0(x) = A_0 + B_0(x - x_0) = 0 + 1(x - 0) = x, & 0 \leq x \leq 1 \\ s_1(x) = A_1 + B_1(x - x_1) = 1 + 0.0769(x - 1) = 0.9231 + 0.0769x, & 1 \leq x \leq 27 \\ s_2(x) = A_2 + B_2(x - x_2) = 3 + 0.0270(x - 27) = 2.2710 + 0.0270x, & 27 \leq x \leq 64 \end{cases}$$

The value $x = 8$ lies in the interval $[1, 27]$, so

$$f(8) \approx s_1(8) = 0.9231 + 0.0769(8) = 1.5383$$

Absolute error $= |f(8) - s_1(8)| = |2 - 1.5383| = 0.4615$. •

Question 13: Find the largest value of the step size h (one decimal place) that can be used of to estimate the integral $\int_1^2 \frac{e^{-x}}{x}\, dx$ to an accuracy of 0.5×10^{-2} using the composite Trapezoidal rule. Then, find the corresponding approximate value of the integral. [6 points]

Solution. Given $f(x) = \frac{e^{-x}}{x}$ and its derivatives can be obtained as

$$f'(x) = -e^{-x}\left[\frac{1}{x} + \frac{1}{x^2}\right] \quad \text{and} \quad f''(x) = e^{-x}\left[\frac{1}{x} + \frac{2}{x^2} + \frac{2}{x^3}\right].$$

Since f'' is decreasing function on $[1, 2]$, it takes its largest value at $x = 1$ in the interval. Hence

$$|f''(x)| \leq e^{-1}(1 + 2 + 2) = 5e^{-1} \approx 1.84.$$

Thus using the error formula of (composite Trapezoidal rule

$$|E| \leq \frac{h^2}{12}(M), \qquad \text{where} \quad M = \max_{1 \leq x \leq 2} |f''(x)|,$$

we have

$$|E| \leq \frac{h^2}{12}(1.84), \qquad \frac{h^2}{12}(1.84) \leq 0.5 \times 10^{-2}, \quad \text{gives} \qquad h = 0.1806,$$

(rounded $h = 0.2$). Thus the Trapezoidal rule for six points or $n = 5$ has the form

$$\int_1^2 f(x)\, dx \approx T_5(f) = \frac{h}{2}\Big[f(x_0) + 2\Big[f(x_1) + f(x_2) + f(x_3) + f(x_4)\Big] + f(x_5)\Big],$$

which is equal to

$$\int_1^2 f(x)\, dx \approx 0.1\Big[0.3679 + 2(0.2510 + 0.1761 + 0.1262 + 0.0918) + 0.0677\Big] = 0.1726,$$

and gives the approximation of the exact value 0.1705 of the given integral. •