King Saud University
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Department of Mathematics

254 Math Exercises
System of Linear Equations
Ch. (3)

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Gaussian Elimination Method

Example: Use simple Gaussian Elimination Method to find the values for α for which the following linear system is consistent

$$x_1 + 2x_2 - 3x_3 = 4$$

$$3x_1 - x_2 + 5x_3 = 2$$

$$4x_1 + x_2 + (\alpha^2 - 14)x_3 = \alpha + 2$$

Also write the solution for α =1

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & \alpha^2-14 & \alpha+2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & \alpha^2-2 & \alpha-14 \end{array}\right]$$

$-3R_1+R_2, -4R_1+R_3$

$$\xrightarrow[-R_2+R_3]{} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & \alpha^2-16 & \alpha-4 \end{array}\right]$$

$\Rightarrow \alpha^2 - 16 = 0$

$\Rightarrow \alpha = \pm 4 \qquad \alpha = 4$

$\alpha = 4 \Rightarrow$ infinitly many solution $\quad 0\ 0\ 0 \mid 0$

$\alpha = -4 \Rightarrow$ no solution $\quad 0\ 0\ 0 \mid -8$

$\Rightarrow \alpha \neq \pm 4$ the system has unique solution

$\Rightarrow$ If $\alpha \neq -4$ the system is consistent

Now If $\alpha = 1$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & 0-15 & -3 \end{array}\right]$$

$-15 x_3 = -3 \Rightarrow x_3 = \frac{1}{5}$

$-7 x_2 = -10 - \frac{14}{5} = +\frac{64}{5}$

$x_1 = 4 + \frac{3}{5} - \frac{128}{35} = \frac{33}{35}$

Example: Use simple Gaussian Elimination Method to find the values for α such that the linear system

$$x_1 + x_2 + \alpha x_3 = 1$$

$$x_1 + \alpha x_2 + x_3 = 1$$

$$\alpha x_1 + x_2 + 5x_3 = -3$$

Is singular, use the smallest positive integer value of constant α to find the solution.

Solution:

$$\left[\begin{array}{ccc|c} 1 & 1 & \alpha & 1 \\ 1 & \alpha & 1 & 1 \\ \alpha & 1 & 5 & -3 \end{array}\right] \xrightarrow{-R_1+R_2;\ -\alpha R_1+R_3} \left[\begin{array}{ccc|c} 1 & 1 & \alpha & 1 \\ 0 & \alpha-1 & 1-\alpha & 0 \\ 0 & 1-\alpha & 5-\alpha^2 & -3-\alpha \end{array}\right]$$

$$R_2+R_3 \left[\begin{array}{ccc|c} 1 & 1 & \alpha & 1 \\ 0 & \alpha-1 & 1-\alpha & 0 \\ 0 & 0 & -\alpha^2-\alpha+6 & -3-\alpha \end{array}\right]$$

R_2: $\alpha = 1 \Rightarrow R_2$ is Zero's $\quad 0\ 0\ 0 \mid 0$

the system is singular (has infinitly many solution)

R_3: $-\alpha^2-\alpha+6=0 \Rightarrow (\alpha+3)(\alpha-2)=0$

$\alpha = -3$ or $\alpha = 2$

If $\alpha = -3$ singular (infinitly many solution) $0\ 0\ 0 \mid 0$

$\alpha = 2$ singular (no solution) $0\ 0\ 0 \mid -5$

Now the smallest positive integer is $\alpha = 3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & -6 & -6 \end{array}\right]$$

$-6x_3 = -6 \Rightarrow x_3 = 1$

$2x_2 = 2 \Rightarrow x_2 = 1$

$x_1 = 1 - 1 - 3 \Rightarrow x = -3$

Example: Use simple Gaussian Elimination Method to find all values of a and b for which the following linear system is consistent or inconsistent. Find the solution when the system is consistent.

$$x_1 - 2x_2 + 3x_3 = 4$$
$$2x_1 - 3x_2 + ax_3 = 5$$
$$3x_1 - 4x_2 + 5x_3 = b$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 4 \\ 2 & -3 & a & 5 \\ 3 & -4 & 5 & b \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & 4 \\ 0 & 1 & a-6 & -3 \\ 0 & 2 & -4 & b-12 \end{array}\right]$$

$-2R_1+R_2, -3R_1+R_3$

$$\xrightarrow[-2R_2+R_3]{} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 4 \\ 0 & 1 & a-6 & -3 \\ 0 & 0 & 8-2a & b-6 \end{array}\right]$$

- inconsistent if: $8-2a=0$ and $b-6\neq 0$

$\Rightarrow a=4$ and $b\neq 6$

- consistent if

$8-2a=0$ and $b-6=0$

$a=4$ and $b=0$

this case given infinitly many solu.

and If $a \neq 4$ and $b \in \mathbb{R}$
this case given unique solution.

Now the solution If $a = 4$ and $b = 6$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 4 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

R_2: $x_2 - 2x_3 = 3 \Rightarrow x_2 = 2x_3 + 3$

R_1: $x_1 - 2x_2 + 3x_3 = 4$

$\Rightarrow x_1 - 4x_3 + 6 + 3x_3 = 4$

$x_1 = -2 + x_3$

Let $x_3 = t$; $t \in \mathbb{R}$

$x_1 = -2 + t$, $x_2 = 2t - 3$; $x_3 = t$; $t \in \mathbb{R}$

Example: Use simple Gaussian Elimination Method to find all values of α and β for which the following linear system is consistent or inconsistent. Find the solution when the system is consistent.

$$2x_1 - x_2 + 3x_3 = 1$$
$$4x_1 + 2x_2 + 2x_3 = 2\alpha$$
$$2x_1 + x_2 + x_3 = \beta$$

Solution:

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 4 & 2 & 2 & 2\alpha \\ 2 & 1 & 1 & \beta \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 0 & 4 & -4 & 2\alpha-2 \\ 0 & 2 & -2 & \beta-1 \end{array}\right]$$

$\rightarrow -2R_1+R_2 ; -R_1+R_3$

$$-\tfrac{1}{2}R_2+R_3 \left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 0 & 4 & -4 & 2\alpha-2 \\ 0 & 0 & 0 & \beta-\alpha \end{array}\right]$$

R_3 Zero's

$\Rightarrow$ inconsistent If $\beta-\alpha \neq 0 \Rightarrow \beta \neq \alpha$

$\Rightarrow$ consistent If $\beta-\alpha = 0 \Rightarrow \beta = \alpha$

Now we find the solution If $\beta = \alpha$

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 0 & 4 & -4 & 2\alpha-2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$4x_2 = 2\alpha - 2 + 4x_3$

$x_2 = \frac{1}{2}\alpha - \frac{1}{2} + x_3$

$2x_1 = 1 - 3x_3 + \frac{1}{2}\alpha - \frac{1}{2} + x_3$

$= \frac{1}{2} + \frac{1}{2}\alpha - 2x_3$

$x_1 = \frac{1}{4} + \frac{1}{4}\alpha - x_3$

let $x_3 = t$; $t \in \mathbb{R}$

$x_1 = \frac{1}{4} + \frac{1}{4}\alpha - t$, $x_2 = \frac{1}{2}\alpha - \frac{1}{2} + t$

Example: Solve the following linear system using the Gaussian Elimination Method with partial pivoting

$$x_1 + 3x_2 - x_3 = 4$$
$$5x_1 - 2x_2 - x_3 = -2$$
$$2x_1 + 2x_2 + x_3 = 9$$

Solution:

نعيد ترتيب النظام بحيث تكون المعادلة التي فيها اكبر معامل مطلق لـ x_1 هي المعادلة الاولى

$$5x_1 - 2x_2 - x_3 = -2$$
$$x_1 + 3x_2 - x_3 = 4 \quad \rightarrow$$
$$2x_1 + 2x_2 + x_3 = 9$$

$$\left[\begin{array}{ccc|c} 5 & -2 & -1 & -2 \\ 1 & 3 & -1 & 4 \\ 2 & 2 & 1 & 9 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 5 & -2 & -1 & -2 \\ 0 & 17 & -4 & 22 \\ 0 & 14 & 7 & 49 \end{array}\right]$$

$-R_1 + 5R_2$, $-2R_1 + 5R_2 \rightarrow$

$\xrightarrow{-14/17 R_2 + R_3}$ $\left[\begin{array}{ccc|c} 5 & -2 & -1 & -2 \\ 0 & 17 & -4 & 22 \\ 0 & 0 & \frac{175}{17} & \frac{525}{17} \end{array}\right]$

$$\frac{175}{17}x_3 = \frac{525}{17} \Rightarrow x_3 = \frac{525}{175} = \boxed{3 = x_3}$$

$$17x_2 = 22 + 4(3)$$

$$x_2 = \frac{34}{17} = \boxed{2 = x_2}$$

$$5x_1 = -2 + 4 + 3$$

$$\boxed{x_1 = 1}$$

Example: compute the inverse A^{-1},where

$$A=\begin{bmatrix}2 & 1 & 2\\ 1 & 2 & 3\\ 4 & 1 & 2\end{bmatrix}$$

By solving the system $AB = I$, using Gauss elimination by partial pivoting where $B = A^{-1}$.then use it to solve the system $Ax = [1,1,2]^T$

Solution:

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LU Decomposition Method

$$AX = b \quad \text{linear system}$$

$$A = L.U\ ,$$

L: lower matrix U: upper matrix

$$AX = b \Rightarrow\ L.U.X = b$$

$$UX = Y \Rightarrow \qquad LY = b$$

$$UX = Y$$

Doolittle's Method:

$$L=\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \qquad U=\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Crout's Method:

$$L=\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \qquad U=\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

Example: consider the linear system $Ax = b$ where

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & 1 \\ 2 & -2 & -1 \end{bmatrix}, \quad b = [2,4,3]^T$$

Find the L.U decomposition of matrix A using Doolittle's method ($l_{ii} = 1$) and then solve the system

Solution:

$A = L.U$

طريقة حساب L و U

١- اختزال المصفوفة A (تصفير ما تحت القطر)

بأستخدام عملية الضرب والجمع فقط

٢- المصفوفة الناتجة تكون هي U

٣- المصفوفة L اغلب عناصرها معلومة و المتبقي نحسبه بوضع عكس اشارة عملية الضرب

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & 1 \\ 2 & -2 & -1 \end{bmatrix}$$

$-2R_1 + R_2, \; -2R_1 + R_3$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow U = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Now to find the solution of system

$$Ly = b \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 2 & 1 & 0 & 4 \\ 2 & 0 & 1 & 3 \end{array}\right]$$

$\boxed{y_1 = 2}$ and $2y_1 + y_2 = 4 \Rightarrow \boxed{y_2 = 0}$

$2y_1 + y_3 = 3 \Rightarrow y_3 = 3 - 4$

$\boxed{y_3 = -1}$

$\Rightarrow Ux = y$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{array}\right]$$

$\boxed{x_3 = 1}$

$x_2 = -x_3 \Rightarrow \boxed{x_2 = -1}$

$x_1 = 2 + x_2 \Rightarrow \boxed{x_1 = 1}$

Example: Use L.U decomposition of matrix A using Doolittle's method ($l_{ii} = 1$) to find the values of α for which the following system has unique solution, and infinitely many solution. Write down the solution for both cases.

$$x_1 + 0.5x_2 + \alpha x_3 = 0.5$$
$$2x_1 - 3x_2 + x_3 = -1$$
$$-x_1 - 1.5x_2 + 2.5x_3 = -1$$

Solution:

$A = L.U$

$$\begin{bmatrix} 1 & 0.5 & \alpha \\ 2 & -3 & 1 \\ -1 & -1.5 & 2.5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0.5 & \alpha \\ 0 & -4 & 1-2\alpha \\ 0 & -1 & 2.5+\alpha \end{bmatrix}$$

$-2R_1 + R_2, R_1 + R_3$

$$\xrightarrow[-\frac{1}{4}R_2 + R_3]{} \begin{bmatrix} 1 & 0.5 & \alpha \\ 0 & -4 & 1-2\alpha \\ 0 & 0 & 2.25+1.5\alpha \end{bmatrix}$$

$$\Rightarrow U = \begin{bmatrix} 1 & 0.5 & \alpha \\ 0 & -4 & 1-2\alpha \\ 0 & 0 & 2.25+1.5\alpha \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0.25 & 1 \end{bmatrix}$$

Now

$$Ly = b \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0.5 \\ 2 & 1 & 0 & -1 \\ -1 & 0.25 & 1 & -1 \end{array}\right]$$

$y_1 = 0.5$

$y_2 = -1 - 1 \Rightarrow y_2 = -2$

$y_3 = 0$

$$\Rightarrow Ux = y \Rightarrow \left[\begin{array}{ccc|c} 1 & 0.5 & \alpha & 0.5 \\ 0 & -4 & 1-2\alpha & -2 \\ 0 & 0 & 2.25+1.5\alpha & 0 \end{array}\right]$$

R_3: $2.25 + 1.5\alpha = 0 \Rightarrow \alpha = -1.5$

If $\alpha = -1.5$ the system has infinitly many sola.

$$\left[\begin{array}{ccc|c} 1 & 0.5 & 1.5 & 0.5 \\ 0 & -4 & 4 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_2 = 0.5 + x_3 \Rightarrow x_3 = m$

$x_1 = 0.25 - 2x_3$

$\Rightarrow x_2 = 0.5 + m,\ x_1 = 0.25 - 2m,\ m \in \mathbb{R}$

If $\alpha \neq -1.5$ the system has unique solu.

$\Rightarrow x_3 = 0$, $x_2 = 0.5$ $x_1 = 0.25$

Example: Find α which the following matrix A singular using the LU decomposition method by Doolittle's method

$$A=\begin{pmatrix}1 & 2 & \alpha\\ 2 & 7 & 3\alpha\\ \alpha & 3\alpha & 4\end{pmatrix}$$

Use the smallest positive integer value of α to find the solution of the system $Ax = [1,0,3]^T$ using Doolittle's method

Solution:

$$A = L.U$$

$$A=\begin{bmatrix}1 & 2 & \alpha\\ 2 & 7 & 3\alpha\\ \alpha & 3\alpha & 4\end{bmatrix}\longrightarrow\begin{bmatrix}1 & 2 & \alpha\\ 0 & 3 & \alpha\\ 0 & \alpha & 4-\alpha^2\end{bmatrix}$$

$$-2R_1+R_2,\ -\alpha R_1+R_3$$

$$\xrightarrow[-\alpha/3\,R_2+R_3]{}\begin{bmatrix}1 & 2 & \alpha\\ 0 & 3 & \alpha\\ 0 & 0 & 4-\frac{4\alpha^2}{3}\end{bmatrix}$$

$$U=\begin{bmatrix}1 & 2 & \alpha\\ 0 & 3 & \alpha\\ 0 & 0 & 4-\frac{4}{3}\alpha^2\end{bmatrix},\ L=\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ \alpha & \alpha/3 & 1\end{bmatrix}$$

Now $A = L.U$ and mean that A singular

$\Rightarrow det(A) = 0$

$\Rightarrow det(A) = det(L.U) = det(L) \cdot det(U)$

$\Rightarrow det(L) = 1 \; ; \; det(U) = 1 \times 3 \times (4 - 4/3\alpha^2)$

$\Rightarrow det(A) = 3(4 - 4/3\,\alpha^2) = 12 - 4\alpha^2$

$12 - 4\alpha^2 = 0 \Rightarrow \alpha^2 = 3$

$\alpha = \pm\sqrt{3}$

Now smallest positive integer is $\alpha = 1$

$$Ly = b \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 1/3 & 1 & 3 \end{array}\right] \quad \begin{array}{l} y_1 = 1 \\ y_2 = -2 \\ y_3 = 8/3 \end{array}$$

$$Ux = y \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 8/3 & 8/3 \end{array}\right] \quad \begin{array}{l} x_3 = 1 \\ x_2 = -1 \\ x_3 = 1 \end{array}$$

Example: Find all values of α that make the following matrix singular using the LU decomposition method by Doolittle's method

$$A=\begin{pmatrix} 1 & 2 & \alpha \\ 2 & 8 & 2\alpha \\ \alpha & 2\alpha & 9 \end{pmatrix}$$

Use the smallest positive integer value of α to find the unique solution of the system $Ax = [1,2,3]^T$ using Doolittle's method

Solution:

Example: consider the linear system $Ax = b$ where

$$A = \begin{bmatrix} -5 & 2 & 1 \\ 1 & -10 & 1 \\ 1 & 1 & -4 \end{bmatrix}, \quad b = [-3,27,4]^T$$

Solve the linear system by L.U decomposition method using crout's method ($u_{ii} = 1$)

Solution:

١- طريقة الاختزال

نجعل المصفوفة A مصفوفة علوية بالشروط التاليه

١- نجعل عناصر القطر تساوي ١ (ضرب بعدد ثابت)

٢- تصفير عناصر ما تحت القطر (الضرب والجمع معاً)

ل U هي المصفوفة العلوية الناتجه

عناصر L كتالي

١- عناصر القطر مقلوب الضرب

٢- بقية العناصر معكوس الجمع والضرب

$$\xrightarrow{-\frac{1}{5}R_1} \begin{bmatrix} 1 & -2/5 & -1/5 \\ 1 & -10 & 1 \\ 1 & 1 & -4 \end{bmatrix} \xrightarrow[-R_1+R_3]{-R_1+R_2} \begin{bmatrix} 1 & -2/5 & -1/5 \\ 0 & -48/5 & 6/5 \\ 0 & 7/5 & -19/5 \end{bmatrix}$$

$$\xrightarrow{-5/48\,R_2} \begin{bmatrix} 1 & -2/5 & -1/5 \\ 0 & 1 & -1/8 \\ 0 & 7/5 & -19/5 \end{bmatrix} \xrightarrow{-7/5\,R_2+R_3} \begin{bmatrix} 1 & -2/5 & -1/5 \\ 0 & 1 & -1/8 \\ 0 & 0 & -29/8 \end{bmatrix}$$

$$\rightarrow -\tfrac{8}{29}R_3 \begin{bmatrix} 1 & -2/5 & -1/5 \\ 0 & 1 & -1/8 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$$\text{Now } L = \begin{bmatrix} -5 & 0 & 0 \\ 1 & -48/5 & 0 \\ 1 & 7/5 & -29/8 \end{bmatrix}$$

$A = L \cdot U$ طريقة الضرب

$$= \begin{bmatrix} l_{11} & 0 & 0 \\ l_{12} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \cdot \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} \end{bmatrix}$$

$$A = \begin{bmatrix} -5 & 2 & 1 \\ 1 & -10 & 1 \\ 1 & 1 & -4 \end{bmatrix} \Rightarrow l_{11} = -5$$

$l_{21} = 1$, $l_{31} = 1$

$-5u_{12} = 2 \Rightarrow \boxed{u_{12} = -2/5}$

$-5\,u_{13} = 1 \Rightarrow \boxed{u_{13} = -1/5}$

$-2/5 + l_{22} = -10 \Rightarrow \boxed{l_{22} = -48/5}$

$-1/5 + \frac{-48}{5}\,u_{23} = 1 \Rightarrow \boxed{u_{23} = -1/8}$

$-2/5 + l_{32} = 1 \Rightarrow \boxed{l_{32} = 7/5}$

$-1/5 + 7/5 \cdot -1/8 + l_{33} = -4 \Rightarrow l_{33} = -29/8$

$$\Rightarrow U = \begin{bmatrix} 1 & -2/3 & -1/5 \\ 0 & 1 & -1/8 \\ 0 & 0 & 1 \end{bmatrix} \; ; \; L = \begin{bmatrix} -5 & 0 & 0 \\ 1 & -48/5 & 0 \\ 1 & 7/5 & -29/8 \end{bmatrix}$$

$$\text{Now } Ly = b \Rightarrow \left[\begin{array}{ccc|c} -5 & 0 & 0 & -3 \\ 1 & -48/5 & 0 & 27 \\ 1 & 7/5 & -29/8 & 4 \end{array}\right]$$

$\Rightarrow y_1 = \frac{3}{5} \; ; \; y_2 = -11/4 \; ; \; y_3 = -2$

$$Ux = y \Rightarrow \left[\begin{array}{ccc|c} 1 & -2/3 & -1/5 & 3/5 \\ 0 & 1 & -1/8 & -11/4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$\Rightarrow x_3 = -2 \; , \; x_2 = -3 \; ; \; x_1 = -1$

Example: for what value b the following matrix is singular using crout's method

$$A = \begin{bmatrix} 2 & -4 & b \\ 2 & 4 & 3 \\ 4 & -2 & 5 \end{bmatrix}$$

Solution:

اولا نجيب المصفوفتين L و U

* طريقة الاقتزال

$$\xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & -2 & b/2 \\ 2 & 4 & 3 \\ 4 & -2 & 5 \end{bmatrix}$$

$$\xrightarrow[-4R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & -2 & b/2 \\ 0 & 8 & 3-b \\ 0 & 6 & 5-2b \end{bmatrix}$$

$$\xrightarrow{\frac{1}{8}R_2} \begin{bmatrix} 1 & -2 & b/2 \\ 0 & 1 & 3/8 - b/8 \\ 0 & 6 & 5-2b \end{bmatrix} \xrightarrow[-6R_2+R_3]{} \begin{bmatrix} 1 & -2 & b/2 \\ 0 & 1 & 3/8 - b/8 \\ 0 & 0 & \frac{11}{4} - \frac{5b}{4} \end{bmatrix}$$

$$\xrightarrow{\frac{4}{11-5b}R_3} \begin{bmatrix} 1 & -2 & b/2 \\ 0 & 1 & 3/8 - b/8 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 8 & 0 \\ 4 & 6 & 11/4 - 5b/4 \end{bmatrix}$$

* طريقة الضرب :

$$A = L \cdot U = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

Now

$$A = \begin{bmatrix} 2 & -4 & b \\ 2 & 4 & 3 \\ 4 & -2 & 5 \end{bmatrix}$$

$\Rightarrow$

$\underline{l_{11} = 2}$ $\quad$ $\underline{l_{21} = 2}$ $\quad$ $\underline{l_{31} = 4}$

$2u_{12} = -4 \Rightarrow \underline{u_{12} = -2}$

$2u_{13} = b \Rightarrow u_{13} = b/2$

$-4 + l_{22} = 4 \Rightarrow l_{22} = 8$

$b + 8u_{23} = 3 \Rightarrow u_{23} = 3/8 - b/8$

$-8 + l_{32} = -2 \Rightarrow l_{32} = 6$

$2b + \frac{18}{8} - \frac{6b}{8} + l_{33} = 5 \Rightarrow l_{33} = \frac{11}{4} - \frac{5b}{4}$

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 8 & 0 \\ 4 & 6 & \frac{11}{4} - \frac{5b}{4} \end{bmatrix}, U = \begin{bmatrix} 1 & -2 & b/2 \\ 0 & 1 & 3/8 - b/8 \\ 0 & 0 & 1 \end{bmatrix}$$

Now A singular if $\det(A) = 0$

$$\det(A) = \det(L \cdot U) = \det(L) \cdot \det(U) =$$

$$= 2 \times 8 \times \frac{11 - 5b}{4} \times 1 = 0$$

$$\Rightarrow 16\left(\frac{11 - 5b}{4}\right) = 0$$

$$\Rightarrow 11 - 5b = 0 \Rightarrow b = \frac{11}{5}$$

Example: Find determinant of the coefficient matrix A of the following linear system using L.U decomposition by crout's method. If $\det(A) = |A| = -1$, then find the unique solution of the following system.

$$x_1 \qquad\qquad - x_3 = 1$$
$$x_2 + x_3 = 1$$
$$-x_1 + x_2 + \alpha x_3 = -1$$

Solution:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & \alpha \end{bmatrix}$$

طريقة الضرب

$A = L \cdot U$

$$= \begin{bmatrix} \ell_{11} & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} \cdot \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \ell_{11} & \ell_{11}u_{12} & \ell_{11}u_{13} \\ \ell_{21} & \ell_{21}u_{12} + \ell_{22} & \ell_{21}u_{13} + \ell_{22}u_{23} \\ \ell_{31} & \ell_{31}u_{12} + \ell_{32} & \ell_{31}u_{13} + \ell_{32}u_{23} + \ell_{33} \end{bmatrix}$$

$\ell_{11} = 1 \qquad \ell_{21} = 0 \qquad \ell_{31} = -1$

$u_{12} = 0 \qquad\qquad u_{13} = -1$

$\ell_{22} = 1 \qquad u_{23} = 1 \qquad ; \ell_{32} = 1$

$1 + 1 + \ell_{33} = \alpha \Rightarrow \boxed{\ell_{33} = \alpha - 2}$

$$\Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & \alpha-2 \end{bmatrix}, U = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

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$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & \alpha \end{bmatrix}$$

$$R_1 + R_3 \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & \alpha-1 \end{bmatrix} \to$$

$$\to -R_2 + R_3 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & \alpha-2 \end{bmatrix} \xrightarrow{\frac{1}{\alpha-2} R_3} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & \alpha-2 \end{bmatrix}$$

$A = L.U \Rightarrow det(A) = det(L.U) = det(L) \cdot det(U)$

$\Rightarrow det(L) \cdot det(U) = -1$

$\Rightarrow (\alpha - 2) \cdot 1 = -1 \Rightarrow \alpha - 2 = -1$

$\Rightarrow \alpha = 1$

Now put $\alpha = 1$ and solve the linear system

$$Ly = b \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ -1 & 1 & -1 & -1 \end{array}\right]$$

$\Rightarrow y_1 = 1$, $y_2 = 1$ and $y_3 = 1$

$$Ux = y \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$\Rightarrow x_3 = 1$

$x_2 + 1 = 1 \Rightarrow x_2 = 0$

$x_1 - 1 = 1 \Rightarrow x_1 = 2$

Iterative Method

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$Ax = b, where\ A = L + D + U$$

Jacobi Iterative Method:

$$T_j = -D^{-1}(L + U), convergent\ if\ \|T_j\| < 1$$

$$x_1^{(k+1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)}\right]$$

$$x_2^{(k+1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(k)} - a_{23}x_3^{(k)}\right]$$

$$x_3^{(k+1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(k)} - a_{32}x_2^{(k)}\right]$$

$$x^{(k+1)} = T_j x^{(k)} + c_j, c_j = D^{-1}b$$

Gauss – seidel Iterative Method:

$$T_g = -(D+L)^{-1}U, convergent\ if\ \|T_g\| < 1$$

$$x_1^{(k+1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)}\right]$$

$$x_2^{(k+1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)}\right]$$

$$x_3^{(k+1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(k+1)} - a_{32}x_2^{(k+1)}\right]$$

$$x^{(k+1)} = T_g x^{(k)} + c_g, c_g = (D+L)^{-1}b$$

Error Bound

$$\|x - x^k\| \leq \frac{\|T\|^k}{1-\|T\|}\|x^1 - x^0\|$$

Note: If matrix A is ***Strictly Diagonally Dominant*** (SDD) the iterative method is convergent.

Example: Show that for the $Ax = b$ where,

$$A=\begin{bmatrix} -5 & 1 & 1 \\ 1 & -10 & 1 \\ 1 & 1 & -3 \end{bmatrix}$$

The Gauss –seidel method convergent faster than the Jacobi.

Solution:

$$T_J = -D^{-1}[L+U]$$

$$A = L + D + U = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} -5 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$D^{-1} = \begin{bmatrix} -1/5 & 0 & 0 \\ 0 & -1/10 & 0 \\ 0 & 0 & -1/3 \end{bmatrix}$$

$$T_J = \begin{bmatrix} -1/5 & 0 & 0 \\ 0 & -1/10 & 0 \\ 0 & 0 & -1/3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/5 & 1/5 \\ 1/10 & 0 & 1/10 \\ 1/3 & 1/3 & 0 \end{bmatrix}$$

$$\|T_J\|_\infty = \max\left\{\frac{2}{5}, \frac{2}{10}, \frac{2}{3}\right\} = \frac{2}{3} = 0.66 < 1 \text{ conv.}$$

$$T_g = -(D+L)^{-1} U$$

$$= -\begin{bmatrix} -5 & 0 & 0 \\ 1 & -10 & 0 \\ 1 & 1 & -3 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} -0.2 & 0 & 0 \\ -0.02 & -0.1 & 0 \\ -0.073 & -0.033 & -0.33 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0.2 & 0.2 \\ 0 & 0.02 & 0.12 \\ 0 & 0.073 & 0.106 \end{bmatrix}$$

$\|T_g\| = \max\{0.4, 0.14, 0.179\} = 0.4 < 1$ conv.

$\|T_g\| < \|T_J\|$

$\Rightarrow$ Gauss faster than Jacobi.

Example: Find the value of a such that the convergent of Jacobi method for the following system is

$$2x_1 + ax_2 + ax_3 = 1$$
$$ax_1 + 2x_2 + ax_3 = -1$$
$$ax_1 + ax_2 + 2x_3 = 1$$

If $a = \frac{1}{4}$, and $x^{(0)} = [-1,2,3]^T$ then compute the number of iterative needed to get accuracy within 10^{-4}.

Solution:

$$A = \begin{bmatrix} 2 & a & a \\ a & 2 & a \\ a & a & 2 \end{bmatrix}$$

$$T_J = -D^{-1}(L+U)$$

$$= -\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 0 & a & a \\ a & 0 & a \\ a & a & 0 \end{bmatrix} = \begin{bmatrix} 0 & a/2 & a/2 \\ a/2 & 0 & a/2 \\ a/2 & a/2 & 0 \end{bmatrix}$$

$$\|T_J\| = \max\{a, a, a\} = |a|$$

convergent $\Leftrightarrow \|T_J\| < 1$

$\Rightarrow \quad |a| < 1 \quad \Rightarrow \quad -1 < a < 1$

Now If $a = 1/4$ and $x^{(0)} = [-1, 2, 3]^T$

$$\|x - x^{(k)}\| \leq \frac{\|T\|^k}{1 - \|T\|} \|x^{(1)} - x^{(0)}\| \leq 10^{-4}$$

$$x_1^{(1)} = 1/2 \left[1 - 1/4(2) - 1/4(3)\right] = -1/8$$
$$x_2^{(1)} = 1/2 \left[-1 - 1/4(-1) - 1/4(3)\right] = -3/4$$
$$x_3^{(1)} = 1/2 \left[1 - 1/4(-1) - 1/4(2)\right] = 3/8$$

$$\|x^{(1)} - x^{(0)}\| = \|[-9/8, -11/4, -21/8]^T\| = 11/4$$

Now $a = 1/4 = 0.25$

$$\Rightarrow \frac{(0.25)^k}{1 - 0.25} \cdot \frac{11}{4} \leq 10^{-4}$$

$$(0.25)^k \leq 4/11 \cdot 3/4 \cdot 10^{-4}$$

$$k \ln(0.25) \leq \ln(3/11 \cdot 10^{-4})$$

$$k \geq \frac{\ln(3/11 \cdot 10^{-4})}{\ln(0.25)} = 7.85$$

$$k = 8$$

Example: Rearrange the linear system such that convergent of Jacobi method guaranteed

$$x_1 + 6x_2 - 3x_3 = 4$$
$$2x_1 + 2x_2 + 6x_3 = 7$$
$$5x_1 + 2x_2 - x_3 = 6$$

Use the Jacobi method to find the first two iteration using $x^{(0)} = [0,0,0]^T$, compute the error bound.

Solution:

Rearrange $Ax = b \rightarrow$ S.D.D

$\Rightarrow$

$5x_1 + 2x_2 - x_3 = 6$

$x_1 + 6x_2 - 3x_3 = 4$ S.D.D $\Rightarrow$ Conv.

$2x_1 + 2x_2 + 6x_3 = 7$

Now

$$x_1^{(k+1)} = \frac{1}{5}\left[6 - 2x_2^{(k)} + x_3^{(k)}\right]$$

$$x_2^{(k+1)} = \frac{1}{6}\left[4 - x_1^{(k)} + 3x_3^{(k)}\right]$$

$$x_3^{(k+1)} = \frac{1}{6}\left[7 - 2x_1^{(k)} - 2x_2^{(k)}\right]$$

$\Rightarrow$

$$x_1^{(1)} = \frac{1}{5}[6 - 0 - 0] \quad 6/5$$

$$x_2^{(1)} = 1/6\,[4 - 0 + 0] = 4/6 = 2/3$$

$$x_3^{(1)} = \frac{1}{6}\left[7 - 0 - 0\right] = \frac{7}{6}$$

$$x_1^{(2)} = \frac{1}{5}\left[6 - 2\cdot\frac{2}{3} + \frac{7}{6}\right] = 1.166\,7$$

$$x_2^{(2)} = \frac{1}{6}\left[4 - \frac{6}{5} + 3\cdot\frac{7}{6}\right] = 1.05$$

$$x_3^{(2)} = \frac{1}{6}\left[7 - 2\cdot\frac{6}{5} - 2\cdot\frac{2}{3}\right] = 0.544$$

Now

$$\|x - x_2\| \leq \frac{\|T\|^k}{1 - \|T\|}\,\|x^{(1)} - x^{(0)}\|$$

$$T_J = -D^{-1}(L+U) = -\begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & 1/6 \end{bmatrix}\begin{bmatrix} 0 & 2 & -1 \\ 1 & 0 & -3 \\ 2 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2/5 & -1/5 \\ 1/6 & 0 & -3/6 \\ 2/6 & 2/6 & 0 \end{bmatrix}$$

$$\|T_J\| = \max\left\{\frac{3}{5}, \frac{4}{6}, \frac{4}{6}\right\} = 0.66$$

$$\|x^{(1)} - x^{(0)}\| = \|[1.2,\ 0.66,\ 1.166\,7]^T\| = 1.2$$

$$\Rightarrow \|x - x_2\| \leq \frac{(0.66)^2}{1 - 0.66}(1.2) = 1.537\,4$$

Example: Consider the following nonhomogeneous linear system $Ax = b$, where

$$A = \begin{bmatrix} 0 & -1 & 4 \\ 5 & 0 & -1 \\ -1 & 3 & 0 \end{bmatrix} \text{ and } b = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

Rearrangement the given system such that the convergent of Gauss-Seidel iterative method is guaranteed. Then find upper bound for the error $\|x - x^{(5)}\|$ using Gauss-Seidel iterative method if $x^{(0)} = [0.3, 0.5, 1.0]^T$

Solution:

Rearrangement $Ax = b$

$$A = \begin{bmatrix} 5 & 0 & -1 \\ -1 & 3 & 0 \\ 0 & -1 & 4 \end{bmatrix} \quad , \quad b = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$$T_g = -(D+L)^{-1} U = -\begin{bmatrix} 5 & 0 & 0 \\ -1 & 3 & 0 \\ 0 & -1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 1/5 & 0 & 0 \\ 1/15 & 1/3 & 0 \\ 1/60 & 1/15 & 1/4 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1/5 \\ 0 & 0 & 1/15 \\ 0 & 0 & 1/60 \end{bmatrix}$$

Now

$$x_1^{(k+1)} = \frac{1}{5}\left[1 + x_3^{(k)}\right]$$

$$x_2^{(k+1)} = \frac{1}{3}\left[2 + x_1^{(k+1)}\right]$$

$$x_3^{(k+1)} = \frac{1}{4}\left[4 + x_2^{(k+1)}\right]$$

$k = 0$

$$x_1^{(1)} = \frac{1}{5}[1+1] = \frac{2}{5} = 0.4$$

$$x_2^{(1)} = \frac{1}{3}[2 + 0.4] = 0.8$$

$$x_3^{(1)} = \frac{1}{4}[4 + 0.8] = 1.2$$

Now $\|T_g\| = \max\{1/5, 1/15, 1/60\} = 1/5 = 0.2$

$$\|x^{(1)} - x^{(0)}\| = \|[0.1, 0.3, 0.2]^T\| = 0.3$$

$$\Rightarrow \|x - x^5\| \leq \frac{(0.2)^5}{1 - 0.2}(0.3) = 1.2 \times 10^{-4}$$

Example: Find the matrix form of the Jacobi method $x^{(k+1)} = T_j x^{(k)} + c_j, k \geq 0$ of the following system

$$6x_1 - 3x_2 + x_3 = 11$$

$$x_1 - 7x_2 + x_3 = 10$$

$$2x_1 + x_2 - 8x_3 = -15$$

Then use it to find the second approximation $x^{(2)}$ of the solution x using the initial approximation $x^{(0)} = [0,0,0]^T$.Compute an error bound for the error $\|x - x^{(10)}\|$.

Solution:

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Relative Error and Residual Vector

$$Ax = b$$

$$condition\ number\ k(A) = \|A\|\|A\|^{-1}$$

$$residual\ vector\ r = b - Ax^*, x^* approximation\ solution$$

$$relative\ error\ \frac{\|x - x^*\|}{\|x\|} \leq k(A)\frac{\|r\|}{\|b\|}$$

$$well\ condtion\ k(A) > 1 \quad (k(A) \leq 10)$$

$$ill\ condtion\ k(A) >>> 1 \quad (k(A) \geq 10)$$

Example: Consider the following matrix and inverse as follows:

$$A = \begin{bmatrix} 10 & -1 & 0 \\ -1 & 10 & -2 \\ 0 & -2 & 10 \end{bmatrix} \text{ and } A^{-1} = \begin{bmatrix} 48/475 & 1/95 & 1/475 \\ 1/95 & 2/19 & 2/95 \\ 1/475 & 2/95 & 99/950 \end{bmatrix}$$

Show that Gauss-Seidel method convergent faster than Jacobi method for linear system $Ax = [9,7,6]^T$ .If an approximate solution for system is $x^* = [0.97,0.91,0.74]^T$, then find the relative error.

Solution:

$$T_J = -D^{-1}(L+U) = -\begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -2 \\ 0 & -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1/10 & 0 \\ 1/10 & 0 & 2/10 \\ 0 & 2/10 & 0 \end{bmatrix} \Rightarrow \|T_J\| = \max\{1/10, 3/10, 2/10\} = 3/10 = 0.3 < 1$$

$$T_G = -(D+L)^{-1}U = \begin{bmatrix} 10 & 0 & 0 \\ -1 & 10 & 0 \\ 0 & -2 & 10 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/10 & 0 & 0 \\ 1/100 & 1/10 & 0 \\ 1/500 & 1/50 & 1/10 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/10 & 0 \\ 0 & 1/100 & 1/5 \\ 0 & 1/500 & 1/25 \end{bmatrix}$$

$\|T_G\| = \max\{1/10, 21/100, 26/500\} = 21/100 = 0.21 < 1$

since $\|T_g\| < \|T_J\| \Rightarrow$ Gauss conv. faster Jacobi

Now $\|A\| = 13$ and $\|A^{-1}\| = \max\{54/475, 13/95, 121/950\}$

$\Rightarrow \|A^{-1}\| = 13/95$

$K(A) = \|A\| \|A^{-1}\| = 13(13/95) = 1.7789$

$$r = b - Ax^* = \begin{bmatrix} 9 \\ 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 10 & -1 & 0 \\ -1 & 10 & -2 \\ 0 & -2 & 10 \end{bmatrix} \begin{bmatrix} 0.97 \\ 0.91 \\ 0.74 \end{bmatrix} = \begin{bmatrix} 0.21 \\ 0.35 \\ 0.42 \end{bmatrix}$$

$\|r\| = 0.42$, $\|b\| = 9$

$\Rightarrow$ The relative error

$$\frac{\|x - x^*\|}{\|x\|} \leq K(A) \cdot \frac{\|r\|}{\|b\|}$$

$$\leq (1.7789) \cdot \frac{(0.42)}{9} = 0.0830$$

Example: Find the condition number of the matrix

$A_n = \begin{bmatrix} 1 & 1 \\ 1 & 1-\frac{1}{n} \end{bmatrix}$, n=1,2,3,…

If n=2 and $x^* = [-1.99, 2.99]^T$ be approximation solution of linear system $Ax = [1, -0.5]^T$, find the relative error

Solution:

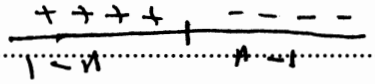
$K(A) = \|A\| \cdot \|A^{-1}\|$

$\|A\| = \max\{2, 2-1/n\} = 2 \quad ; n=1,2,3,\dots$

$A^{-1} = \begin{bmatrix} 1-1/n & -1 \\ -1 & 1 \end{bmatrix} \cdot \frac{1}{1-1/n-1} = -n \begin{bmatrix} 1-1/n & -1 \\ -1 & 1 \end{bmatrix}$

$\Rightarrow A^{-1} = \begin{bmatrix} -n+1 & n \\ n & -n \end{bmatrix}$

$-n+1=0 \Rightarrow n=1$



$\|A^{-1}\| = \max\{|-n+1|+n, 2n\}$

$= \max\{1-n+n, 2n\} = 2n$

$K(A) = 2 \cdot 2n = 4n$

Now if $n=2$

$A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0.5 \end{bmatrix}$ and $x^* = \begin{bmatrix} -1.99 \\ 2.99 \end{bmatrix}$; $b = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$

$$\frac{\|x - x^*\|}{\|x\|} \leq K(A) \cdot \frac{\|r\|}{\|b\|}$$

$$r = b - A_2 x = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} - \begin{bmatrix} 1 \\ -0.495 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.005 \end{bmatrix}$$

$\|r\| = 0.005$; $\|b\| = 1$, $K(A_2) = 8$ well cond.

$$\Rightarrow \frac{\|x - x^*\|}{\|x\|} \leq 8 \cdot \frac{0.005}{1} = 0.04$$

Example: Discuss the condition number of linear system

$$\frac{1}{2}x_1 + \frac{1}{3}x_2 = \frac{1}{63}$$

$$\frac{1}{3}x_1 + \frac{1}{4}x_2 = \frac{1}{168}$$

If $x^* = [0.142, -0.166]^T$ is an approximation solution of system then estimate the relative error

Solution:

$$A = \begin{bmatrix} 1/2 & 1/3 \\ 1/3 & 1/4 \end{bmatrix} ; b = \begin{bmatrix} 1/63 \\ 1/168 \end{bmatrix}$$

$$\|A\| = \max\{0.83, 0.58\} = 0.83$$

$$A^{-1} = \begin{bmatrix} 1/4 & -1/3 \\ -1/3 & 1/2 \end{bmatrix} \times \frac{1}{\frac{1}{8} - \frac{1}{9}} = 72 \begin{bmatrix} 1/4 & -1/3 \\ -1/3 & 1/2 \end{bmatrix}$$

$$\Rightarrow \|A^{-1}\| = 4\max\{41.76, 59.76\}$$

$$K(A) = \|A\| \cdot \|A^{-1}\| = 0.83 \times 59.76 = 49.6008 \gg 1$$

ill condition.

Now

$$r = b - A\tilde{x}$$

$$\Rightarrow r = \begin{bmatrix} 1/63 \\ 1/168 \end{bmatrix} - \begin{bmatrix} 0.071 + 0.0547 \\ 0.04686 - 0.415 \end{bmatrix}$$

$$= \begin{bmatrix} -4.268 \times 10^{-4} \\ 5.923 \times 10^{-4} \end{bmatrix}$$

$$\Rightarrow \|r\| = 5.923 \times 10^{-4} \; ; \quad \|b\| = \frac{1}{63} = 15.87 \times 10^{-3}$$

$$\Rightarrow \frac{\|x - x^*\|}{\|x\|} \leq K(A) \frac{\|r\|}{\|b\|}$$

$$\leq 49.6008 \times \frac{5.923 \times 10^{-4}}{15.87 \times 10^{-3}} = 1.8512$$