Harmonic functions

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1 Review: completing the example from last lecture

1.1 Statement

We defined the unit disk above the imaginary axis as D^+. We defined the function

$$f(z) = \frac{z-1}{z+1}$$

Our goal is to find $f(D^+)$.

1.2 Follow the boundary

1.2.1 Real line

Let $x \in \mathbb{R}$.

$$f(x) = \frac{x-1}{x+1} = 1 - \frac{2}{x+1}$$

The derivative is

$$f'(x) = \frac{2}{(x+1)^2} > 0$$

Thus f is a strictly increasing function on the real axis.

We also have

$$f(x) \to -\infty \text{ as } x \to -1^+$$

$x = 1$ maps to 0 and $x = -1$ maps to $-\infty$. Thus the image of the boundary of D^+ on the real axis is onto the negative real axis.

1.2.2 Top of the unit circle

The upper half of the unit circle is a part of a clircle. Since f is an LFT, the circular boundary of D^+ must map to part of a clircle. The circular boundary of D^+ contains the point $x = -1$, which maps to ∞, so the image of the circular boundary must be a line segment.

The boundaries of D^+ meet at right angles. Since LFTs preserve angles by definition, the image of the circular part of the boundary must be a line through the origin at a right angle to the real axis. The only such line is the imaginary axis.

1.2.3 Pick a point

By the arguments in the last lecture, we have immediately that $f(D^+)$ is either

- the second quadrant or
- the compliment of the second quadrant

There are two ways to determine which is the case.

Easy Pick a point in D^+ and find its image. Taking $f\left(\frac{i}{2}\right)$ gives a point in the second quadrant, so f must map D^+ bijectively onto the second quadrant.

Easier Consider the angle of the vector from the point $x = 1$ to any point in the interior of D^+. This vector has a positive imaginary component. Since f is an LFT and preserves angles, any such point in D^+ must have a positive imaginary part relative to $f(1) = 0$. Hence $f(D^+)$ must lie in the upper half plane.

1.3 General method

The properties of LFTs presented in the last lecture can greatly simplify calculations about mapping between domains. However, the method only works for using LFTs to map between domains whose boundaries are unions of clircles.

1.4 Riemann mapping theorem

Previously, we defined conformal mappings.

Definition (Conformal mapping)**.** A conformal mapping between domains D_1 and D_2 is a bijective map $f : D_1 \leftrightarrow D_2$ such that f and f^{-1} are differentiable.

It is often useful to use conformal maps to map a complicated domain to a simpler one to solve a problem. It turns out that it is possible, in theory, to map any complicated domain to the unit disk.

Theorem (Riemann mapping theorem)**.** *Let $D \subset \mathbb{C}$, $D \neq \mathbb{C}$ be a nonempty* simply connected *domain. Then $\exists$ a conformal mapping*

$$f : D \leftrightarrow \mathbb{D} = \{ z : |z| < 1 \}$$

Note.

- This is a complex version of a general change of variables theorem.
- This theorem says that, up to a change of variables, there are exactly two simply connected domain. Every simply connected domain not equal to $\mathbb{C}$ is equivalent to the unit disk.
- The proof does not say how to find f. Finding f in many special cases is a topic of mathematical research.
- We will not use or prove this theorem. The proof is beyond the scope of this class.

2 Harmonic functions

2.1 Definition

Definition (Harmonic functions)**.** Let $D \subset \mathbb{C}$ be a domain. A function $U : D \to \mathbb{R}$ is *harmonic* if it is twice continuously differentiable on D, and

$$u_{xx} + u_{yy} \equiv 0$$

Note.

- The first condition implies that the partial derivatives u_x, u_y, u_{xx}, u_{xy}, u_{yx}, u_{yy} exist and are continuous on D.

- The second-order partial differential equation defining the condition on u is called the Laplace condition. It can also be written

$$\Delta u = 0$$

where Δ stands for the set of partial derivatives above.

2.2 Applications

The equations describing many physical situations satisfy the Laplace equation. Examples include

- heat transfer
- electrostatic potential
- gravitational potential
- fluid flow

The Laplace equation is prominent in physical situations because

- it is rotationally symmetric. Physical laws are usually rotationally symmetric, so the equations describing those laws must also be symmetric.
- second derivatives are common in physics because of Newton's second law.

2.3 Harmonic functions from complex differentiable functions

Theorem. *If $f : D \to \mathbb{C}$ is differentiable and $f = u + \imath v$, then $\Delta u = 0$ (and $\Delta v \equiv 0$).*

Proof. Write the Laplace equation as

$$\Delta u = (u_x)_x + (u_y)_y$$

Since f is differentiable by assumption, the Cauchy-Riemann equations apply,

$$\begin{aligned} \Delta u &= (v_x)_x + (-v_x)_y \\ &= v_{xy} - v_{yx} \end{aligned}$$

It is a basic fact from real analysis that if v has continuous second partial derivatives, then these second order partial derivatives commute.

$$v_{xy} = v_{yx}$$

$\square$

Note. We will later prove a near converse to this theorem, which says that harmonic functions are essentially the real parts of complex differentiable functions.

2.4 Harmonic conjugates

2.4.1 Basic facts

Definition (Harmonic conjugate)**.** Let u by harmonic in D where $u : D \to \mathbb{R}$. Then $v : D \to \mathbb{R}$ is a harmonic *conjugate* of u in D if

- v is harmonic
- $u + \imath v$ is differentiable in D in the complex sense

Note. The point of this definition is that any harmonic function u is only half of a logically complete whole, namely the differentiable function $f = u + \imath v$.

Fact. If D is *connected*, then any two harmonic conjugates of u in D differ only by a constant.

Proof. If $u + \imath v$ and $u + \imath w$ are differentiable, then

$$\imath(v - w) \begin{cases} \text{is differentiable} \\ \text{is } \imath\mathbb{R}\text{-valued} \end{cases}$$

We showed previously that any function that is purely imaginary or purely real and differentiable is constant. Thus $v - w$ is constant in D $\square$

Note. The relationship between u and v is *not* symmetric. If v is a harmonic conjugate of u, then $-u$ is a harmonic conjugate of v.

2.4.2 Examples

- $$u = x \qquad v = y \qquad x + \imath y = z$$

- $$u = \frac{x}{x^2+y^2} \qquad v = \frac{-y}{x^2+y^2} \qquad (u+\imath v)(z) = \frac{x-\imath y}{x^2+y^2} = \frac{1}{z}$$

- $D = \mathbb{C}_\pi$

$$\begin{aligned} u(x+\imath y) = u(z) &= \ln|z| \\ v(z) &= \mathrm{Arg} z \\ (u+\imath v)(z) &= \ln|z| + \imath \mathrm{Arg}(z) = \mathrm{Log} z \end{aligned}$$

In all three examples, the function obtained by taking $u + \imath v$ is complex differentiable, as required by the definition of v as the harmonic conjugate of u.

2.5 Theorem and corollaries

2.5.1 Existence of harmonic conjugates

Theorem. *If D is a* simply connected *domain, then any harmonic $u : D \to \mathbb{R}$ has a harmonic conjugate $v : D \to \mathbb{R}$*

Proof. One of the most basic techniques we have for finding a function is to integrate. To do so, fix $z_0 \in D$. Write

$$f(z) = u(z_0) + \int_{\gamma(z_0,z)} g$$

where γ is any contour from z_0 to z in D. We do not know yet what g is, but we want it to be some differentiable function.

If we want to get f, the logical choice of integrand is f'. However, since we don't know f, we don't know f'. Our task is to find an expression for f' given the information in the statement of the theorem, namely u.

Since the derivative of a complex function must be the same along any direction,

$$f' = u_x + \imath v_x$$

We can use the Cauchy-Riemann equations to get rid of v_x. We have then

$$f' = u_x - \imath u_y$$

We can now define g in terms of quantities that appear in the statement of the problem,

$$g = u_x - \imath u_y$$

We conclude that

$$\int_{\gamma(z_0,z)} g$$

is independent of $\gamma(z_0, z)$ and f is differentiable with $f' = u_x - \imath u_y$ provided the following conditions hold:

- the domain D is simply-connected. We have assumed this in the statement.
- g is differentiable.

We need to show that $u_x - \imath u_y$ is differentiable. The proof has thus far not used the assumption that u is harmonic. To check that $u_x - \imath u_y$ is differentiable, it suffices to verify the Cauchy-Riemann equations directly. We need

$$\begin{aligned} (u_x)_x &= (-u_y)_y \\ (u_x)_y &= -(-u_y)_x \end{aligned}$$

The first comes from LaPlace's equation. The second is true because of the general fact that partial derivatives commute.

The final step is to show that the solution for f is unique. We must show that if $f = U + \imath V$ is another choice for f, then $\nabla U = \nabla u$. Since these two functions have the same derivative, they are the same function.

□

2.5.2 Harmonic functions are the real part of differentiable functions

Corollary. *If u is harmonic in some open set D and $z_0 \in D$, then $\exists$ a differentiable function f and a neighborhood $N \subset D$ of z_0 such that $u = Re(f)$ in N.*

Note. This theorem says that we can always transfer questions about harmonic functions to questions about differentiable functions considered in the proper domains.

Example.

$$D = \mathbb{C} \setminus \{ 0 \} \qquad u = \ln |z|$$

u has *no* harmonic conjugate in D. This is not a contradiction, since D is not simply connected.

Proof of assertion in example. Suppose v is a harmonic conjugate of u. v exists by the previous theorem. Then by a previous example today, $v - \operatorname{Arg} z$ must be constant in $\mathbb{C}_\pi$.

Consider any point on the negative real axis. As the point is approached separately from above and below, there is a discontinuity in $\operatorname{Arg} z$, but v is continuos, so it has no discontinuity. It is impossible to have a discontinuous jump in a constant function, so this is a contradiction.

□

2.5.3 Composition of harmonic functions

Corollary. *If $u : D \to \mathbb{R}$ is harmonic and $g : \tilde{D} \to D$ is differentiable, then $u \circ g : \tilde{D} \to \mathbb{R}$ is* harmonic*.*

Proof. Write $u = \operatorname{Re}(f)$, where f is differentiable. Then $u \circ g = \operatorname{Re}(f \circ g)$, and $f \circ g$ is differentiable, so $\Delta (u \circ g) = 0$. □

2.5.4 Evaluating harmonic functions using average values

Corollary. *Assume*

- $u : D \to \mathbb{R}$ *is harmonic*
- $r > 0$

- $z_0 \in D$
- $|w - z_0| \leq r \Rightarrow w \in D$

Then

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u\left(z_0 + re^{\imath\theta}\right) d\theta$$

Note.

- This theorem says that the value of the harmonic function at the center of the a disk is equal to the average value of the integral of the function along the boundary of the disk.
- Physically, this theorem expresses a conservation law of harmonic functions.

Proof. Draw a slightly larger disk encompassing the one we are interested in. We can then use a previous corollary to write $u = \mathrm{Re}(f)$. We have previously proven a theorem for complex differentiable functions,

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{\imath\theta}) d\theta$$

Note. This proof demonstrates the usual procedure for dealing with harmonic functions: find a region in which $u = \mathrm{Re} f$, and then use facts about differentiable functions.

□